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Travelling waves in lattice models of multi-dimensional and multi-component media: I. General hyperbolic properties

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Abstract. We study the stability of motion in the form of travelling waves in lattice models of unbounded multi-dimensional and multi-component media with a nonlinear prime term and small coupling depending on a finite number of space coordinates. Under certain conditions on the nonlinear term we show that the set of travelling waves running with the same sufficiently large velocity forms a finite-dimensional submanifold in infinite-dimensional phase space endowed with a special metric with weights. It is 'almost' stable and contains a finite-dimensional strongly hyperbolic subset invariant under both evolution operator and space translations.

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Introduction

Recently many chain and lattice models of non-equilibrium media with dissipation and energy pumping have been of great interest (cf for example [1, 2]). There is a deep physical reason for this. Some experimental works (cf experiments with convection in [3, 4], Taylor–Dean flow [5], Faraday turbulent ripples [6]) showed that particle-like localized structures can arise in a medium if the energy pumping is large enough. These structures have individual degrees of freedom and each of them can be described by means of finite-dimensional dynamical systems, thus dynamics of the medium is treated as a result of an interaction of these subsystems. This is the way to build a phenomenological lattice model. For example, it is known that some types of motions of non-equilibrium media can be described by a space- and time-discrete version of the Ginzburg–Landau equation. In the case of one-component and one-dimensional media this leads to the equation [1, 2, 7, 8]

$$u_j(n+1) = u_j(n) - (1 - \mathrm{i}\beta)u_j(n)\,|u_j(n)|^2 + \varkappa(u_{j-1}(n) - 2u_j(n) + u_{j+1}(n))$$
$$u_j(n) \in \mathbb{C}, n, j \in \mathbb{Z}$$

where β and $\varkappa$ are real numbers. In the more general case when the medium is one-dimensional but multi-component we have

$$u_j(n+1) = F_j(u_j(n), \beta) + \gamma(u_j(n) - u_{j-1}(n)) + \varkappa(u_{j-1}(n) - 2u_j(n) + u_{j+1}(n))$$

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where $u_j(n) \in \mathbb{R}^n$ or $u_j(n) \in \mathbb{C}^n$. Here the equation $u_j(n+1) = F_j(u_j(n), \beta)$ describes the dynamics of an individual system and $\gamma, \varkappa$ characterize the connection between them (γ reflects the non-mutual coupling between elements and $\varkappa$ is the coefficient (or the matrix) of diffusion and is usually sufficiently small). For simplicity assume that the medium is homogenous. Therefore the function F_j does not depend on j. The case of inhomogeneous media is still very difficult for rigorous investigation.

Physicists also consider a more general case (cf for instance, [1, 2, 9–11])

$$u_j(n+1) = f(u_j(n)) + \varkappa g(\{u_i(n)\}_{i=j-s}^{j+s}) \qquad n, j \in \mathbb{Z}$$

where f is a C^2-diffeomorphism, g is C^2-map and $u_j(n) \in \mathbb{R}^d$. This equation is called the evolution equation. Our aim is to study stability of motions in lattice models of the above type for unbounded media and therefore with infinite-dimensional phase space of states. Let us note that these models are used in physics to describe media with the behaviour in the inner part of spatial region independent of the behaviour on the boundary. For such media (and such models) the physical reasons for the appearance of finite-dimensional limit sets are not well understood. On the other hand, some computer simulations of the Ginzburg–Landau equation display finite-dimensional attractors.

In this paper we are interested in the solutions of the evolution equation in the form of travelling waves

$$u_j(n) = \psi(lj + mn)$$

where m/l is the 'velocity' of the wave and $\psi = \{\psi(k)\}$ is a function ($k = lj + mn + m$). We describe the behaviour and stability of travelling waves in the case when

$$m > ls + 1.$$

From a physical viewpoint this case corresponds to travelling waves with 'large' velocity and as experiments show they appear to be stable in a large domain of parameter space.

The evolution equation can be solved if boundary conditions are fixed. For space-unbounded models this can be done by fixing the growth rate of solutions at infinity along spatial coordinates. In other words we will consider only the solutions $u(n) = (u_j(n))$ which satisfy

$$\|u(n)\|_{q_1,q_2} = \sum_{j \geq 0} \frac{\|u_j(n)\|}{q_1^j} + \sum_{j<0} \frac{\|u_j(n)\|}{q_2^{-j}} < \infty$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^d$ and $q_1 > 1$, $q_2 > 1$. If $u(n) = (u_j(n))$ is a travelling wave then the growth of perturbations along this solution can have different rates in forward and backward directions. That is why, in order to analyse the stability of such a solution, we work with different q_1 and q_2.

As soon as the boundary conditions are fixed the evolution equation can be treated as an infinite-dimensional dynamical system with the phase space

$$\mathcal{M}_{q_1,q_2} = \{u = (u_j) : \|u\|_{q_1,q_2} < \infty\}$$

and the evolution operator

$$\Phi_\varkappa(u) = (f(u_j) + \varkappa g(\{u_i\}_{i=j-s}^{j+s})) \qquad u = (u_j).$$

$\mathcal{M}_{q_1,q_2}$ is a Banach space with the metric $\|\cdot\|_{q_1,q_2}$. One of the important features of dynamics of unbounded media is that the evolution operator $\Phi_\varkappa$ is not differentiable in the sense of Freshet but only in the sense of Gautex (i.e. it is differential only along directions corresponding to finite-dimensional perturbations). It may lead to an instability along some infinite-dimensional directions. Moreover, it may also happen that a strongly linearly stable stationary solution is not isolated.

Usually a travelling wave is a bounded solution of the evolution equation. Nevertheless, its small perturbations running with the same velocity can spread out along spatial coordinates with an exponential rate. If a travelling wave is stable a small perturbation tends to zero in each coordinate when time tends to infinity. However, this convergence to zero is not uniform over coordinates. The metrics with weights $\|\cdot\|_{q_1,q_2}$ allow one to take this phenomenon into account. Such metrics were introduced in [16] to study hyperbolic properties of spatially homogeneous solutions and to establish the existence of finite-dimensional attractors in the systems with drifted coupling ('drift' systems).

Let $u(n)=(u_j(n))$ be a solution of the evolution equation representing the travelling wave $u_j(n)=\psi(lj+mn)$. It is easy to see that the function ψ should satisfy the following 'travelling wave equation'

$$\psi(k)=f(\psi(k-m))+\varkappa g(\{\psi(k-m+lj)\}_{j=-s}^{s})$$

where $k=lj+mn+m$ is the 'travelling coordinate'. We first shall study hyperbolic properties of the solutions of this equation. For $q_1>1$, $q_2>1$ denote by $\Psi_{\varkappa,q_1,q_2}\subset\mathcal{M}_{q_1,q_2}$ the set of solutions of the travelling wave equation.

Let us introduce the shift $S_\varkappa:\Psi_{\varkappa,q_1,q_2}\to\Psi_{\varkappa,q_1,q_2}$, $(S_\varkappa\psi)(k)=\psi(k+1)$. Obviously, it acts as the nonlinear operator given by the right-hand side of the travelling wave equation.

If q_1 and q_2 are large enough (where q_1 does not depend on $\varkappa$ but q_2 does) we will show that there exists a smooth imbedding

$$\chi_\varkappa:\mathbb{R}^{d(ls+m)}\to\mathcal{M}_{q_1,q_2}$$

with $\chi_\varkappa(\mathbb{R}^{d(ls+m)})=\Psi_{\varkappa,q_1,q_2}$. This means that $\Psi_{\varkappa,q_1,q_2}$ is a smooth $d(ls+m)$-dimensional submanifold in $\mathcal{M}_{q_1,q_2}$. Moreover, we will show that the travelling wave equation generates a map

$$F_\varkappa:\mathbb{R}^{d(ls+m)}\to\mathbb{R}^{d(ls+m)}$$

such that the diagram

$$\begin{array}{ccc}\mathbb{R}^{d(ls+m)} & \xrightarrow{\chi_\varkappa} & \Psi_{\varkappa,q_1,q_2}\subset\mathcal{M}_{q_1,q_2}\\ {\scriptstyle F_\varkappa}\downarrow & & \downarrow{\scriptstyle S_\varkappa}\\ \mathbb{R}^{d(ls+m)} & \xrightarrow{\chi_\varkappa} & \Psi_{\varkappa,q_1,q_2}\subset\mathcal{M}_{q_1,q_2}\end{array}$$

is commutative. The map $F_\varkappa$ is a multi-dimensional version of the famous Hénon map. For $\varkappa\neq 0$ it is a C^2-diffeomorphism while F_0 is not invertible.

Let us now assume that the prime nonlinear term in the evolution equation is a map possessing a hyperbolic locally maximal closed invariant set $\Lambda\subset\mathbb{R}^d$ (cf definition below). We will show that in this case the map $F_\varkappa$ for all sufficiently small $\varkappa$ also possesses a locally maximal closed invariant hyperbolic set $\Lambda_\varkappa$. The problem of the existence of $\Lambda_\varkappa$ for $F_\varkappa$ is a classical problem of small perturbations of the smooth map F_0 with the hyperbolic locally maximal invariant closed set Λ_0. However, as we mentioned above, F_0 is not invertible. Such a situation was considered in [20, 21], but for the sake of the reader's convenience we present a method of solving this problem in the appendix.

It is easy to see now that the set

$$\Lambda_{\varkappa,q_1,q_2} = \chi_\varkappa(\Lambda_\varkappa) \subset \Psi_{\varkappa,q_1,q_2}$$

is locally maximal closed invariant hyperbolic for the map $S_\varkappa$. This describes the hyperbolic property of the restriction $S_\varkappa \mid \Psi_{\varkappa,q_1,q_2}$.

In fact for arbitrary $q_1 > 1$, $q_2 > 1$ one can still construct the map $\chi_\varkappa$ which is correctly defined on $\Lambda_\varkappa$ and is a homeomorphism onto its image. In this case $\Lambda_{\varkappa,q_1,q_2} \subset \mathcal{M}_{q_1,q_2}$ is 'topologically hyperbolic' in the sense that one can construct 'traces' of stable and unstable local manifolds which lie in $\Lambda_{\varkappa,q_1,q_2}$ (cf theorem 2 below). However, in general, $\chi_\varkappa$ cannot be extended to $\mathbb{R}^{d(ls+m)}$ such that $\Lambda_{\varkappa,q_1,q_2}$ is not contained in any even topological finite-dimensional submanifold in $\mathcal{M}_{q_1,q_2}$.

The next step in our study is to describe stability of a travelling wave $\psi = \{\psi(k)\}$ in directions transversal to $\Psi_{\varkappa,q_1,q_2}$. Let us introduce the 'travelling evolution operator' $Q_\varkappa$ in $\mathcal{M}_{q_1,q_2}$ acting by the formula

$$(Q_\varkappa w)(k) = f(w_{k-m}) + \varkappa g(\{w_{k-m+li}\}_{i=-s}^{s})$$

where $w = (w_k) \in \mathcal{M}_{q_1,q_2}$. It is easy to see that

$$Q_\varkappa \mid \Psi_{\varkappa,q_1,q_2} = Id \mid \Psi_{\varkappa,q_1,q_2}.$$

This means that solutions of the travelling wave equations are the fixed points for $Q_\varkappa$.

We construct a filtration of affine subspaces

$$V^{s,0}_{\varkappa,q_1,q_2} \subset V^{s,-1}_{\varkappa,q_1,q_2} \subset V^{s,-2}_{\varkappa,q_1,q_2} \subset \ldots \subset \mathcal{M}_{q_1,q_2}$$

such that $Q_\varkappa$ is contracting along each $V^{s,j}_{\chi,q_1,q_2}$ with parameters of contraction $(C(k), \gamma)$, $C(k) > 0, 0 < \gamma < 1$. The union

$$V^{s,\infty}_{\varkappa,q_1,q_2} = \bigcup_{j \leqslant 0} V^{s,j}_{\varkappa,q_1,q_2}$$

is everywhere dense in $\mathcal{M}_{q_1,q_2}$ and is a non-uniformly stable affine subspace for $Q_\varkappa$ at the point ψ. Let us note that the above filtration can be constructed for arbitrary $q_2 > 1$, $q_1 > q_1^{(0)}$, where $q_1^{(0)} > 1$ is a constant independent of $\varkappa$. In particular, the situation can occur when the set $\Lambda_{\varkappa,q_1,q_2}$ is only topologically hyperbolic while the map $q_\varkappa$ is differentiable and exponentially stable along each $V^{s,j}_{\varkappa,q_1,q_2}$.

In order to complete our consideration we have to pass from the travelling wave equation to the evolution equation. Assuming again that q_1 and q_2 are sufficiently large we associate with each $\psi \in \Psi_{\varkappa,q_1,q_2}$ the solution $u(\psi)$ of the evolution equation such that $u(\psi)(j) = \psi(lj)$. This induces the map $\alpha_\varkappa$ from $\Psi_{\varkappa,q_1,q_2}$ onto the set

$$\mathcal{A}_{\varkappa,q_1,q_2} = \alpha_\varkappa(\Psi_{\varkappa,q_1,q_2})$$

such that $\alpha_\varkappa(\psi) = u(\psi)$. $\mathcal{A}_{\varkappa,q_1,q_2}$ is the set of solutions of the evolution equation in the form of travelling waves. It is crucial here that $u(\psi)$ no longer belongs to $\mathcal{M}_{q_1,q_2}$, but to $\mathcal{M}_{q_1',q_2'}$. It is also worthwhile emphasizing that, as we show, the map $\alpha_\varkappa$ is injective (this is due to our assumption that the velocity of the wave is given by relatively prime numbers m, l). This implies that the evolution operator $\Phi_\varkappa$ is invertable on $\mathcal{A}_{\varkappa,q_1,q_2}$. Besides, this set is a smooth $d(ls+m)$-dimensional submanifold in $\mathcal{M}_{q_1',q_2'}$. We will also show (cf proposition 4 below) that the diagram

$$\begin{array}{ccc} \Psi_{\varkappa,q_1,q_2} & \xrightarrow{\alpha_\varkappa} & \mathcal{A}_{\varkappa,q_1,q_2} \subset \mathcal{M}_{q_1',q_2'} \\ {\scriptstyle S_\varkappa^m}\downarrow & & \downarrow{\scriptstyle \Phi_\varkappa} \\ \Psi_{\varkappa,q_1,q_2} & \xrightarrow{\alpha_\varkappa} & \mathcal{A}_{\varkappa,q_1,q_2} \subset \mathcal{M}_{q_1',q_2'} \end{array}$$

is commutative. The set

$$\mathcal{L}_{\varkappa,q_1,q_2} = \alpha_\varkappa(\Lambda_{\varkappa,q_1,q_2}) \subset \mathcal{A}_{\varkappa,q_1,q_2}$$

is locally maximal hyperbolic closed invariant. For arbitrary $q_1 > 1$, $q_2 > 1$ this set can be still defined to be a topologically hyperbolic subset in $\mathcal{M}_{q_1',q_2'}$ in the above sense.

Now let us describe the stability of travelling waves in directions transversal to $\mathcal{A}_{\varkappa,q_1,q_2}$. Usually one can expect stability in the sense of damping of perturbations running together with the travelling wave. This means that stability is non-stationary. In addition it is, as a rule, non-uniform along spatial coordinates. This leads to the following definition.

Let us fix $u \in \mathcal{M}_{q_1,q_2}$, $q_2 > 1$, $q_2 > 1$ and consider its semi-trajectory $u(n) = \Phi_\varkappa^n(u)$, $n > 0$. We call a two-parameter family of subspaces $\mathcal{E}_n^k \subset T_{u(n)}\mathcal{M}_{q_1,q_2}$, $k \in \mathbb{Z}$, $n > 0$, the non-stationary invariant filtration if

(1) they form a filtration: $\mathcal{E}_n^k \supset \mathcal{E}_n^{k+1}$;

(2) they are invariant: there exists $r > 0$ such that $d\Phi_\varkappa \mathcal{E}_n^k \subset \mathcal{E}_{n+1}^{k-r}$.

The non-stationary invariant filtration is referred to as infinitesimally stable if there exist $a > 0$, $0 < \gamma < 1$, such that for any $k > 0$, $p > 0$, $n > 0$ and any $\eta \in \mathcal{E}_p^{k+an}$

$$\|d\Phi_\varkappa^n \eta\|_{q_1,q_2} \leqslant C_{k,p}\gamma^n \,\|\eta\|_{q_1,q_2}$$

where $C_{k,p} > 0$ is a constant independent of n.

The above definition expresses stability with respect to the differential of the evolution operator. We now consider local stability with respect to the evolution operator itself. We call a two-parameter family of submanifolds $\mathcal{V}_n^k \subset \mathcal{M}_{q_1,q_2}$, $u(n) \in \mathcal{V}_n^k$ the non-stationary invariant filtration if

(1) they form a filtration: $\mathcal{V}_n^k \supset \mathcal{V}_n^{k+1}$;

(2) they are invariant: there exists $r > 0$ such that $\Phi_\varkappa(\mathcal{V}_n^k) \subset \mathcal{V}_{n+1}^{k-r}$.

The non-stationary invariant filtration is said to be locally stable if there exist $a > 0$, $0 < \gamma < 1$, such that for any $k > 0$, $p > 0$, $n > 0$ and any $v \in \mathcal{V}_p^{k+an}$

$$\|\Phi_\varkappa^n(v) - u(n)\|_{q_1,q_2} \leqslant C_{k,p}\gamma^n \cdot \|v - u(0)\|_{q_1,q_2}$$

where $C_{k,p} > 0$ is a constant independent of n.

Let us now define the extension $\tilde{\alpha}_\varkappa$ of $\alpha_\varkappa$ onto $\mathcal{M}_{q_1,q_2}$ by setting

$$\tilde{\alpha}_\varkappa(w) = (w_{li}) \qquad w = (w_i) \in \mathcal{M}_{q_1,q_2}$$

as well as the extension $\tilde{S}_\varkappa$ of $S_\varkappa$ by setting

$$\tilde{S}_\varkappa(w) = (w_{k+1}) \qquad w = (w_k) \in \mathcal{M}_{q_1,q_2}.$$

We shall show (cf proposition 6 below) that the diagram

$$\begin{array}{ccc} \mathcal{M}_{q_1,q_2} & \xrightarrow{\tilde{\alpha}_\varkappa} & \mathcal{M}_{q_1',q_2'} \\ {\scriptstyle Q_\varkappa \circ S_\varkappa^m}\downarrow & & \downarrow{\scriptstyle \Phi_\varkappa} \\ \mathcal{M}_{q_1,q_2} & \xrightarrow{\tilde{\alpha}_\varkappa} & \mathcal{M}_{q_1',q_2'} \end{array}$$

is commutative (let us note again that $Q_\varkappa \mid \Psi_{\varkappa,q_1,q_2} = Id$, $\tilde{S}_\varkappa \mid \Psi_{\varkappa,q_1,q_2} = S_\varkappa$). Using this we shall prove that

$$\mathcal{V}_n^k = \mathcal{V}^{s,k}_{\varkappa,q_1,q_2}(u(n)) = \tilde{\alpha}_\varkappa(V^{s,lk}_{\varkappa,q_1,q_2}(w(n)))$$

where $u(n) = \tilde{\alpha}_\varkappa(w(n))$, $u(n) = \Phi_\varkappa^n(u(0))$, $k \in \mathbb{Z}$, $n > 0$ form a non-stationary invariant filtration which is locally stable with respect to $\Phi_\varkappa$. Moreover, the set

$$\mathcal{V}^{s,\infty}_{\varkappa,q_1,q_2}(u) = \bigcup_{k\in\mathbb{Z}} \mathcal{V}^{s,k}_{\varkappa,q_1,q_2}(u)$$

is everywhere dense in $\mathcal{M}_{q_1',q_2'}$.

Now we can describe the entire picture of the hyperbolic behaviour of travelling waves. Fix any $q_1 > q_1^{(0)} > 1$, $q_2 > 1$ where $q_1^{(0)}$ is a constant independent of $\varkappa$. Take any $l, m > ls + 1$ and choose $q_i(l) > 1$ such that $q_i(l)^l = q_i$, $i = 1, 2$. The solutions of the evolution equation in the form of travelling waves moving with the velocity m/l form a set $\mathcal{A}^{l,m}_{\varkappa,q_1(l),q_2(l)} \subset \mathcal{M}_{q_1,q_2}$. It contains a finite-dimensional subset $\mathcal{L}^{l,m}_{\varkappa,q_1(l),q_2(l)}$ which is invariant under the evolution operator and topologically hyperbolic with respect to the metric $\|\cdot\|_{q_1,q_2}$ in $\mathcal{M}_{q_1,q_2}$. It can be constructed as pointed out above. If $q_1(l)$ and $q_2(l)$ turn out to be large enough (such that $q_1(l)$ does not depend on $\varkappa$ but $q_2(l)$ does) then $\mathcal{A}^{l,m}_{\varkappa,q_1(l),q_2(l)}$ is a smooth $d(ls+m)$-dimensional submanifold in $\mathcal{M}_{q_1,q_2}$ and $\mathcal{L}^{l,m}_{\varkappa,q_1(l),q_2(l)}$ becomes a hyperbolic subset in the usual sense. If $q_1(l) > q_1^{(0)}$ then any travelling wave lying in an affine subspace $\mathcal{V}^{s,k}_{\varkappa,q_1(l),q_2(l)}(u) \subset \mathcal{M}_{q_1,q_2}$, converges to $\mathcal{A}^{l,m}_{\varkappa,q_1(l),q_2(l)}$ and the union of these affine subspaces is everywhere dense. If, in addition to that $q_2(l)$ is large enough (depending again on $\varkappa$) then $\mathcal{V}^{s,k}_{\varkappa,q_1(l),q_2(l)}(u)$ is transversal to the tangent space to $\mathcal{A}^{l,m}_{\varkappa,q_1(l),q_2(l)}$ at u. Let us notice that for any distinct pairs (l_1, m_1), (l_2, m_2) the sets $\mathcal{L}^{l_1 m_1}_{\varkappa,q_1(l),q_2(l)}$ and $\mathcal{L}^{l_2 m_2}_{\varkappa,q_1(l),q_2(l)}$ are disjoint and therefore the corresponding spaces $\mathcal{V}^{s,\infty}_{\varkappa,q_1(l),q_2(l)}(u)$ are also disjoint.

As we have seen above, parameters q_1, q_2 play an important role in our construction and deeply reflect a smooth topological structure of hyperbolic sets for the evolution operator. When these parameters increase one can observe two critical phenomena. The first is due to the transition from $q_1 < q_1^{(0)}$ to $q_1 > q_1^{(0)}$ (with any $q_2 > 1$), while the second corresponds to the transition over $q_2 = q_2^{(0)}(\varkappa)$. For $1 < q_1 < q_1^{(0)}$ and $q_2 > 1$ the set $\mathcal{L}_{\varkappa,q_1,q_2}$ is topologically hyperbolic as a subset in $\mathcal{A}_{\varkappa,q_1,q_2}$. But its stability in 'transversal' directions is not clear yet. When $q_1 > q_1^{(0)}$ the set $\mathcal{A}_{\varkappa,q_1,q_2}$ becomes non-uniformly stable along the filtration $\mathcal{V}^{s,k}_{\varkappa,q_1,q_2}$, $k \in \mathbb{Z}$. In fact, given $u \in \mathcal{A}_{\varkappa,q_1,q_2}$, this filtration appears when q_1 exceeds some critical value $q_1^{(0)}(u)$ (such that $q_1^{(0)} = \sup_{u\in\mathcal{A}_{\varkappa,q_1,q_2}} q_1^{(0)}(u)$). If q_2 exceeds $q_2^{(0)}(\varkappa)$ then $\mathcal{A}_{\varkappa,q_1,q_2}$ becomes a smooth submanifold and $\mathcal{L}_{\varkappa,q_1,q_2}$ is a hyperbolic subset in the usual sense.

Let us emphasize that $q_1^{(0)}$ and $q_2^{(0)}(\varkappa)$ depend on the velocities of travelling waves but they are uniformly bounded from above.

So far we have considered only one-dimensional multi-component models. However the multi-dimensional models are also of great interest; for example, spacetime-discrete versions of the generalized Ginzburg–Landau equation, the

Kuramoto–Sivashinsky equation, reaction–diffusion equation and the Huxley equation are of this type. They describe the appearance and interaction of structures in non-equilibrium media and have recently become the subject of study. For instance, the discrete version of the two-dimensional Huxley equation has the form

$$\begin{aligned} u_{j_1,j_2}(n+1) = u_{j_1,j_2}(n) + \alpha f(u_{j_1,j_2}(n)) \\ + \varkappa(u_{j_1-1,j_2}(n) - 4u_{j_1,j_2}(n) + u_{j_1+1,j_2}(n) + u_{j_1,j_2-1}(n) \\ + u_{j_1,j_2+1}(n)). \end{aligned}$$

It was considered in [22].

The methods of study presented in this paper work in the multi-dimensional case as well as in the one-dimensional one. The main reason for this is that the 'travelling wave equation' is still one-dimensional with respect to the travelling variable while the dimension of the domain of values of function ψ depends on the dimension of the medium. That is why the solutions of the travelling wave equation inherit the same properties and their hyperbolic behaviour in the space $\mathcal{M}_{q_1,q_2}$ can be described by the imbedding map $\chi_\varkappa$. The next transition to the travelling waves solution of the original evolution equation is given by another imbedding $\alpha_\varkappa$. It is defined as $\alpha_\varkappa(\psi(\bar{j})) = \psi((\bar{l},\bar{j}))$, where $\bar{l} = (l_1, \ldots, l_t)$, $l_i \geq 0$, $i = 1, \ldots, t$ is the 'wavevector', $\bar{j} = (j_1, \ldots, j_t) \in \mathbb{Z}^t$ and t is the dimension of the medium (here $(\bar{l},\bar{j})$, is the usual scalar product). The map $\alpha_\varkappa$ is well-defined as a map into a space $\mathcal{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$, where $\bar{q}_1(\bar{l}) = (q_1^{l_i})$, $\bar{q}_2(l) = (q_2^{l_i})$, $i = 1, \ldots, t$ and $\mathcal{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ consists of all the functions $u: \mathbb{Z}^t \to \mathbb{R}^d$, $u = (u(\bar{j}))$ for which

$$\|u\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} = \sum_{j_1} \cdots \sum_{j_t} \frac{\|u(j_1, \ldots, j_t)\|}{q(j_1) \cdot \ldots \cdot q(j_t)} < \infty$$

(here $q(j_i) = q_1^{l_i j_i}$ if $j_i \geq 0$ and $q(j_i) = q_2^{-l_i j_i}$ if $j_i < 0$).

As we can see, despite the multi-dimensionality of the medium, the type of metric we use to express the hyperbolic behaviour depends on two parameters q_1, q_2 in the one-dimensional case. After that the picture of hyperbolic behaviour described above in the one-dimensional case can be reproduced (with few and almost trivial modifications) in the multi-dimensional case.

Although we consider only multi-component media (with at least two components) our methods may also work in the one-component case. The reason is that we use essentially information about the hyperbolic behaviour of the map $F_\varkappa$. It can be studied, and turns out to be sufficiently strongly hyperbolic even if the map f does not have a non-trivial hyperbolic set or is not invertible. This can often occur when f is one-dimensional [19].

1. Definitions and general properties of travelling waves

Given $q_1 > 1$, $q_2 > 1$ let us consider the set

$$\mathcal{M}_{q_1,q_2} = \left\{ u: \mathbb{Z} \to \mathbb{R}^d:\ u = (u_j)_{j \in \mathbb{Z}},\ \|u\|_{q_1,q_2} = \sum_{j \geq 0} \frac{\|u_j\|}{q_1^j} + \sum_{j<0} \frac{\|u_j\|}{q_2^{-j}} < \infty \right\}$$

(here $\|\cdot\|$ is a norm in the Euclidean space $\mathbb{R}^d$). It is easy to see that $\mathcal{M}_{q_1,q_2}$ is the Banach space with the norm $\|\cdot\|_{q_1,q_2}$.

Let $f:\mathbb{R}^d\to\mathbb{R}^d$ be a diffeomorphism of class C^2, $g:(\mathbb{R}^d)^{2s+1}\to\mathbb{R}^d$ a map of class $C^2(s\geqslant 0)$ and $\varkappa>0$ a real number. Define the map $\Phi_\varkappa:\mathcal{M}_{q_1,q_2}\to(\mathbb{R}^d)^{\mathbb{Z}}$ by the formula

$$\Phi_\varkappa(u)=(f(u_j)+\varkappa g(\{u_i\}_{i=j-s}^{j+s}))\qquad u=(u_j)\in\mathcal{M}_{q_1,q_2}.$$

Proposition 1. Assume that there exists $M>0$ such that for $l=1,2$

$$\sup_{x\in\mathbb{R}^d}\|d^lf_x\|\leqslant M\qquad \sup_{1\leqslant i\leqslant 2s+1}\sup_{x_i\in\mathbb{R}^d}\left\|\frac{\partial^l g}{\partial x_i^l}(x_1,\dots,x_{2s+1})\right\|\leqslant M. \tag{1}$$

Then for any $q_1>1$, $q_2>1$

1. $\Phi_\varkappa$ is a map from $\mathcal{M}_{q_1,q_2}$ into itself;
2. $\Phi_\varkappa$ is differentiable in the sense of Gautex and its Gautex differential at a point $u=(u_j)$ is given by the linear map

$$(d\Phi_\varkappa\eta)(j)=f'(u_j)\cdot\eta_j+\sum_{i=j-s}^{j+s}a_i\eta_i$$

where

$$a_i=\frac{\partial g}{\partial u_i}(\{u_i\}_{i=j-s}^{j+s}).$$

Proof. We have under these assumptions that

$$\begin{aligned}
\|\Phi_\varkappa u\|_{q_1,q_2}&\leqslant\sum_{j\geqslant 0}\left[\frac{\|f(u_j)\|}{q_1^j}+\varkappa\frac{\|g(\{u_i\}_{i=j-s}^{j+s})\|}{q_1^j}\right]+\sum_{j<0}\left[\frac{\|f(u_j)\|}{q_2^{-j}}+\varkappa\frac{\|g(\{u_i\}_{i=j-s}^{j+s})\|}{q_2^{-j}}\right]\\
&\leqslant\sum_{j\geqslant 0}\left[\frac{\|f(u_j)-f(0)\|}{q_1^j}+\varkappa\cdot\frac{\|g(\{u_i\}_{i=j-s}^{j+s})-g(0)\|}{q_1^j}+\frac{\|f(0)\|+\varkappa\|g(0)\|}{q_1^j}\right]\\
&\quad+\sum_{j<0}\left[\frac{\|f(u_j)-f(0)\|}{q_2^{-j}}+\varkappa\cdot\frac{\|g(\{u_i\}_{i=j-s}^{j+s})-g(0)\|}{q_2^{-j}}+\frac{\|f(0)\|+\varkappa\|g(0)\|}{q_2^{-j}}\right]\\
&\leqslant M\left(\sum_{j=0}^{\infty}\frac{\|u_j\|}{q_1^j}+\varkappa\sum_{j=s+1}^{\infty}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_1^j}\right)+(\|f(0)\|+\varkappa\|g(0)\|)\cdot\sum_{j=0}^{\infty}1/q_1^j\\
&\quad+M\left(\sum_{j=-\infty}^{-1}\frac{\|u_j\|}{q_2^{-j}}+\varkappa\sum_{j=-\infty}^{-s-1}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_2^{-j}}\right)+(\|f(0)\|+\varkappa\|g(0)\|)\sum_{j=-\infty}^{-1}1/q_2^{-j}\\
&\quad+\varkappa M\sum_{j=0}^{s}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_2^j}+\varkappa M\sum_{j=-s}^{-1}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_2^{-j}}\leqslant M\|u\|_{q_1,q_2}\\
&\quad+\varkappa M(2s+1)\cdot q^s\cdot\|u\|_{q_1,q_2}+(\|f(0)\|+\varkappa\|g(0)\|)\left(\frac{q_1}{q_1-1}+\frac{1}{q_2-1}\right)\\
&\quad+\varkappa M\sum_{j=0}^{s}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_1^j}+\sum_{j=-s}^{-1}\sum_{i=j-s}^{j+s}\frac{\|u_i\|}{q_2^{-j}}<\infty
\end{aligned}$$

where $q=\max\{q_1,q_2\}$. This proves the first statement. In order to prove the second one we have to estimate the norm of the vector $(\Phi_\varkappa(u+\xi)-\Phi_\varkappa(u)-d\Phi_\varkappa\cdot\xi)_j$ in

$\mathbb{R}^d$. Evidently, it is less than $M \cdot \sup_{|i-j|\leqslant s} \|\xi_i\|^2$. The second statement follows from here. □

Fix $u \in \mathcal{M}_{q_1,q_2}$ and consider the sequence of points $u(n) = \Phi_\varkappa^n(u)$ such that $u(0) = u$. We have

$$u_j(n+1) = f(u_j(n)) + \varkappa g(\{u_i(n)\}_{i=j-s}^{j+s}) \qquad j \in \mathbb{Z}, n \geqslant 0. \tag{2}$$

Let us fix $m, l \in \mathbb{Z}^+$ that are relatively prime numbers such that

$$m \geqslant ls + 1. \tag{3}$$

We are looking for a special trajectory $\{u(n)\}$ of the form

$$u_j(n) = \psi(lj + mn) = \psi(k - m) \qquad k = lj + mn + m. \tag{4}$$

Here $\psi: \mathbb{Z} \to \mathbb{R}^d$ is a function (let us note that for any k there exist j, n such that $lj + mn = k$). Trajectory (4) represents a 'travelling wave' with 'velocity' m/l. It follows from (2) that ψ should satisfy the relation

$$\psi(k) = f(\psi(k-m)) + \varkappa g(\{\psi(k - m + li)\}_{i=-s}^{s}). \tag{5}$$

We shall show (cf below, proposition 2) that the function ψ is uniquely defined by (5) if we know the values

$$x_p = \psi(-m - ls + p + 1) \qquad p = 1, \ldots, ls + m. \tag{6}$$

In order to describe the set of all the travelling waves let us start by considering the map

$$F_\varkappa : (\mathbb{R}^d)^{ls+m} \to (\mathbb{R}^d)^{ls+m}, F_\varkappa(x_1, \ldots, x_{ls+m}) = (\bar{x}_1, \ldots, \bar{x}_{ls+m})$$

where

$$\begin{aligned} &\bar{x}_1 = x_2, \bar{x}_2 = x_3, \ldots, \bar{x}_{ls+m-1} = x_{ls+m}, \\ &\bar{x}_{ls+m} = f(x_{ls+1}) + \varkappa g(\{x_{p(i)}\}_{i=-s}^{s}), \; p(i) = l(s+i) + 1. \end{aligned} \tag{7}$$

First we shall study hyperbolic properties of the map $F_\varkappa$ for small enough $\varkappa$. It is easy to see that $|\det(dF_\varkappa(x))| = \varkappa \left|\det \dfrac{\partial g}{\partial x_1}\right|$ for any $x = (x_i) \in (\mathbb{R}^d)^{ls+m}$, $i = 1, \ldots, ls + m$. We shall assume that

$$\det \frac{\partial g(x)}{\partial x_1} \neq 0 \qquad \text{for any } x = (x_i) \in (\mathbb{R}^d)^{2s+1}. \tag{8}$$

It follows from (8) that $F_\varkappa$ is a diffeomorphism of class C^2 for any $\varkappa \neq 0$.

Consider the map f. Assume that it has a hyperbolic locally maximal set Λ. This means [15]

(a) Λ is a closed invariant subset in $\mathbb{R}^d$;

(b) there exists a bounded open subset $V \subset \mathbb{R}^d$ such that $\Lambda \subset V$,

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(V); \tag{9}$$

(c) Λ is a hyperbolic subset for f; for $x \in \Lambda$ we denote by $E^{s,u}(x)$ the stable and unstable subspaces at x; they have properties that

$$E^s(x) \oplus E^u(x) = \mathbb{R}^d \qquad dfE^{s,u}(x) = E^{s,u}(f(x))$$

and for any $n \geqslant 0$

$$\|df_x^n v\| \leqslant c\lambda^n \|v\| \qquad v \in E^s(x)$$

$$\|df_x^n v\| \geqslant c^{-1}\lambda^{-n} \|v\| \qquad v \in E^u(x).$$

where $c > 0, 0 < \lambda < 1$ do not depend on n, x and v.

In fact one can construct families of stable and unstable cones $C^s(x, \alpha), C^u(x, \alpha) \subset \mathbb{R}^d$ (cf [15]; α is the angle of the cone at $x \in V$) having the properties:

$$C^s(x, \alpha) \cap C^u(x, \alpha) = 0$$

$$dfC^u(x, \alpha) \subset C^u(f(x), \alpha) \qquad dfC^s(x, \alpha) \supset C^s(f(x), \alpha)$$

and for any $n \geqslant 0$ for which $f^k(x) \in V$, $k = 0, \ldots, n$

$$\|df_x^n v\| \leqslant c_0\lambda^n \|v\| \qquad v \in C^s(x, \alpha)$$

$$\|df_x^n v\| \geqslant c_0^{-1}\lambda_0^{-n} \|v\| \qquad v \in C^u(x, \alpha)$$

where $c_0 > 0$ is a constant independent of n, x and v. Conversely, if we have two families of cones $C^{s,u}(x, \alpha)$ satisfying the above properties one can introduce stable and unstable subspaces at any $x \in \Lambda$ by setting

$$E^s(x) = \bigcap_{n \geqslant 0} df^{-n}C^s(f^n(x), \alpha), E^u(x) = \bigcap_{n \geqslant 0} df^n C^u(f^{-n}(x), \alpha).$$

Put

$$\tilde{V} = \bigoplus_{i=1}^{ls+m} V_i \qquad V_i \equiv V \qquad \tilde{C}^{s,u}(x, \alpha) = \bigoplus_{i=1}^{ls+m} C^{s,u}(x_i, \alpha) \qquad x = (x_i).$$

The cones $\tilde{C}^{s,u}(x, \alpha)$ are respectively stable and unstable and introduce the hyperbolic structure on $\tilde{V}$, i.e.

$$dF_0(x)\tilde{C}^u(x, \alpha) \subset \tilde{C}^u(F_0(x), \alpha) \qquad dF_0\tilde{C}^s(x, \alpha) \supset \tilde{C}^s(F_0(x), \alpha)$$

and for any $n \geqslant 0$ for which $F_0^{(ls+m)k}(x) \in \tilde{V}$, $k = 0, 1, \ldots, n$

$$\begin{aligned} \|dF_0^{(ls+m)n} v\| &\leqslant C_1\lambda_1^n \|v\| \qquad v \in \tilde{C}^s(x, \alpha) \\ \|dF_0^{(ls+m)n} v\| &\geqslant C_1^{-1}\lambda_1^{-n} \|v\| \qquad v \in \tilde{C}^u(x, \alpha) \end{aligned} \tag{10}$$

where C_1, $0 < \lambda_1 < 1$ are constants independent of x, v, n.

Moreover, let us consider the set

$$\tilde{\Lambda}_0 = \bigoplus_{i=1}^{ls+m} \Lambda_i \qquad \Lambda_i \equiv \Lambda.$$

For any $x \in \tilde{\Lambda}_0$, $x = (x_1, \ldots, x_{ls+m})$, $x_i \in \mathbb{R}^d$ one can define dF_0-invariant stable and unstable subspaces at x by setting

$$E_0^{s,u}(x) = \bigoplus_{i=1}^{ls+m} E^{s,u}(x_i).$$

Theorem 1. There exists $\varkappa_0 > 0$ such that for any $\varkappa$, $|\varkappa| \leq \varkappa_0$

1. the map F_0 has an invariant set $\Lambda_0 \subset \tilde{V}$, $F_0(\Lambda_0) = \Lambda_0$ which is locally maximal and hyperbolic;

2. if $\varkappa \neq 0$, the map $F_\varkappa$ has a hyperbolic locally maximal set $\Lambda_\varkappa \subset \tilde{V}$.

The fact that the second statement follows from the first one is essentially proved in [20, 21]. However for the sake of completeness and reader's convenience we present the full proof of theorem 1 in the appendix.

For each $y \in \Lambda_\varkappa$ one can construct stable and unstable smooth local submanifolds which we denote by $V_\varkappa^{s,u}(y)$. They have the following properties: for any $y \in \Lambda_\varkappa$

(1) $y \in V_\varkappa^{s,u}(y)$:
(2) $T_y V_\varkappa^{s,u}(y) = E_\varkappa^{s,u}(y)$:
(3) $F_\varkappa(V_\varkappa^s(y)) \subset V_\varkappa^s(F_\varkappa(y))$, $F_\varkappa^{-1}(V_\varkappa^u(y)) \subset V_\varkappa^u(F_\varkappa^{-1}(y))$;
(4) for any $n \geq 0$

$$\rho(F_\varkappa^n(y'), F_\varkappa^n(y)) \leq C_\varkappa^{(1)} \mu_\varkappa^n \rho(y', y) \qquad \text{if } y' \in V_\varkappa^s(y)$$

$$\rho(F_\varkappa^{-n}(y'), F_\varkappa^{-n}(y)) \geq C_\varkappa^{(1)} \mu_\varkappa^n \rho(y', y) \qquad \text{if } y' \in V_\varkappa^u(x)$$

where $C_\varkappa^{(1)} > 0$, $0 < \lambda_\varkappa < \mu_\varkappa < 1$ are some constants and $\rho(y', y)$ is the distance between $y', y \in (\mathbb{R}^d)^{ls+m}$.

2. Hyperbolic properties of travelling waves along tangent directions

We shall study equation (5) assuming that $\varkappa \neq 0$. Denote by $\Psi_\varkappa \subset (\mathbb{R}^d)^{\mathbb{Z}}$ the set of solutions of this equation. The relations (6) define the map from $(\mathbb{R}^d)^{ls+m}$ into $\Psi_\varkappa$ which we denote by $\chi_\varkappa$. Consider also the shift $S_\varkappa : \Psi_\varkappa \to \Psi_\varkappa$, $(S_\varkappa \psi)(k) = \psi(k+1)$, $\psi \in \Psi_\varkappa$.

Proposition 2. (1) $\chi_\varkappa$ is one-to-one onto $\Psi_\varkappa$ satisfying

$$\chi_\varkappa \circ F_\varkappa = S_\varkappa \circ \chi_\varkappa. \tag{11}$$

In other words the sequence $\chi_\varkappa(x)(k)$ satisfies equation (5).

(2) If $x \in (\mathbb{R}^d)^{ls+m}$ is a point such that the trajectory $\{F_\varkappa^n(x)\}$ remains in a bounded domain then $\|\chi_\varkappa(x)\|_{q_1,q_2} < \infty$ for any $q_1 > 1$, $q_2 > 1$.

Proof. The first statement immediately follows from (5) and the definitions. Given $x \in (\mathbb{R}^d)^{ls+m}$ we have that

$$\chi_\varkappa(x)(k) = \begin{cases} (x)_k & \text{if } -ls-m \leqslant k \leqslant -1 \\ (F_\varkappa^{k+1}(x))_{ls+m} & \text{if } k \geqslant 0 \\ ((F_\varkappa)^{k+ls+m}(x))_1 & \text{if } k \leqslant -ls-m-1 \end{cases}$$

where $(x)_k$ means the k-coordinate of x. This implies the second statement. □

Theorem 2. (1) For any $q_1 > 1$, $q_2 > 1$ the map $\chi_\varkappa \mid \Lambda_\varkappa$ is a continuous imbedding into $\mathcal{M}_{q_1,q_2}$.

(2) The set $\Lambda_{\varkappa,q_1,q_2} = \chi_\varkappa(\Lambda_\varkappa) \subset \mathcal{M}_{q_1,q_2} \cap \Psi_\varkappa \overset{\text{def}}{=} \Psi_{\varkappa,q_1,q_2}$ is closed and invariant under $S_\varkappa$; the map $S_\varkappa \mid \Psi_{\varkappa,q_1,q_2}$ is a homeomorphism.

(3) The sets

$$V^{s,u}_{\varkappa,q_1,q_2}(\psi) = \chi_\varkappa(V^{s,u}_\varkappa(x) \cap \Lambda_\varkappa),\ \psi = \chi_\varkappa(x)$$

are 'traces' of stable and unstable local topological submanifolds in $\mathcal{M}_{q_1,q_2}$; this means that they consist of points $v \in \Lambda_{\varkappa,q_1,q_2}$ lying in a small niehgbourhood of ψ in $\mathcal{M}_{q_1,q_2}$ such that

$$\lim_{n\to\infty} \|S_\varkappa^n(\psi) - S_\varkappa^n(v)\|_{q_1,q_2} = 0 \qquad \text{if } v \in V^s_{\varkappa,q_1,q_2}(\psi);$$

$$\lim_{n\to\infty} \|S_\varkappa^{-n}(\psi) - S_\varkappa^{-n}(v)\|_{q_1,q_2} = 0 \qquad \text{if } v \in V^u_{\varkappa,q_1,q_2}(\psi).$$

Proof. Given $q_1 > 1$, $q_2 > 1$, $\varepsilon > 0$ there exists $N > 0$ such that for any $x \in \Lambda_\varkappa$

$$\sum_{k\leqslant -N} \frac{\|\chi_\varkappa(x)(k)\|}{q_2^{-k}} + \sum_{k\geqslant N} \frac{\|\chi_\varkappa(x)(k)\|}{q_1^k} \leqslant \varepsilon.$$

This implies the first statement. The others follow from here and theorem 1. □

From now on we assume that in addition to (1) the function g satisfies the following condition: for any $x = (x_1, \ldots, x_{2s+1}) \in \mathbb{R}^{d(2s+1)}$ the matrix $\partial g/\partial x_1(x)$ is invertable and

$$\sup_{x\in\mathbb{R}^{d(2s+1)}} \left| \left(\frac{\partial g}{\partial x_1}(x) \right)^{-1} \right| < \infty. \tag{12}$$

Theorem 3. Assume that f and g satisfy conditions (1), (8) and (12). There exist $q_1^{(0)} > 1$, $q_2^{(0)}(\varkappa) > 1$ such that for any $q_1 \geqslant q_1^{(0)}$, $q_2 \geqslant q_2^{(0)}(\varkappa)$

(1) the map $\chi_\varkappa$ is a smooth imbedding of $(\mathbb{R}^d)^{ls+m}$ into $\mathcal{M}_{q_1,q_2}$; for $x \in (\mathbb{R}^d)^{ls+m}$ the differential $d\chi_\varkappa(x)$ is given as following: $d\chi_\varkappa(x)v = \eta$ where $v \in (\mathbb{R}^d)^{ls+m}$ and η satisfies the relationship

$$\eta(k) = df(\psi(k-m))\eta(k-m) + \varkappa \sum_{i=-s}^{s} a_i \eta(k-m+li) \tag{13}$$

with

$$a_i = \frac{\partial g}{\partial u_i}(\{\psi(k-m+li)\}_{i=-s}^{s})$$

(2) $\sup_{x\in\mathbb{R}^{d(ls+m)}} \|d\chi_\varkappa(x)\|_{q_1,q_2} < \infty$;

(3) the set $\Psi_{\varkappa,q_1,q_2} = \chi_\varkappa(\mathbb{R}^d)^{(ls+m)}$ is a finite-dimensional (of dimension $d(ls+m)$) smooth submanifold in $\mathcal{M}_{q_1,q_2}$;

(4) the set $\Lambda_{\varkappa,q_1,q_2}=\chi_\varkappa(\Lambda_\varkappa)\subset\Psi_{\varkappa,q_1,q_2}$ is closed and invariant under $S_\varkappa$.

(5) $\Lambda_{\varkappa,q_1,q_2}$ is a hyperbolic locally maximal set for $S_\varkappa$ (with respect to $\Psi_{\varkappa,q_1,q_2}$) with

$$E^{s,u}_{\varkappa,q_1,q_2}(\psi)=d\chi_\varkappa(E^{s,u}_\varkappa(x)),\ \psi=\chi_\varkappa(x)$$

as a stable and unstable subspaces at $\psi\in\Lambda_{\varkappa,q_1,q_2}$ and

$$V^{s,u}_{\varkappa,q_1,q_2}(\psi)=\chi_\varkappa(V^{s,u}_\varkappa(x)),\ \psi=\chi_\varkappa(x)$$

as stable and unstable local smooth submanifolds in $\Psi_{\varkappa,q_1,q_2}$ respectively.

Proof. For $x,v\in(\mathbb{R}^d)^{ls+m}$ define

$$\eta(x,v)=\begin{cases}(v)_k & \text{if } -ls-m\leqslant k\leqslant -1\\ (dF_\varkappa^{k+1}(x)v)_{ls+m} & \text{if } k\geqslant 0\\ ((dF_\varkappa)^{k+ls+m}(x)v)_1 & \text{if } k\leqslant -ls-m-1\end{cases}\tag{14}$$

where $(v)_k$ means the k-coordinate of vector v. It is easy to see that $\eta(x,v)(k)$ satisfies (13). The trivial calculation shows also that

$$dF_\varkappa^{-1}=\begin{pmatrix}0 & \dots & & & \frac{1}{\varkappa}\left(\frac{\partial g}{\partial x_1}\right)^{-1} & \\ E & 0 & \dots & & & \\ 0 & E & 0 & \dots & & \\ \cdot & \cdot & \cdot & \dots & \cdot & \\ 0 & 0 & 0 & \dots & E & 0\end{pmatrix}$$

where the entries in the first line are either equal to 0 or have the form $(1/\varkappa)(\partial g/\partial x_1)^{-1}\bar{g}$ with $\|\bar{g}\|\leqslant M$. This and assumptions (1) and (12) imply that

$$\|dF_\varkappa\|\leqslant M_1\qquad \|dF_\varkappa^{-1}\|\leqslant M_2(\varkappa)$$

where $M_1>0$ and $M_2(\varkappa)>0$ are constants. In particular, $\eta(x,v)\in\mathcal{M}_{q_1,q_2}$ for some $q_1>1$, $q_2>1$, q_1 does not depend on $\varkappa$ (but q_2 does; moreover $q_2\to\infty$ as $\varkappa\to 0$). It follows from (1) that

$$\sup_{x\in\mathbb{R}^{d(ls+m)}}\|\eta(x,v)\|_{q_1,q_2}\leqslant C\,\|v\|$$

where $C=\sup_{x\in\mathbb{R}^{d(ls+m)}}\|dF_\varkappa(x)\|$. Now using (1), (12), (14) it is not difficult to verify that

$$\|(\chi_\varkappa(x+v)-\chi_\varkappa(x)-\eta(x,v))(k)\|\leqslant\begin{cases}0 & \text{if } -ls-m\leqslant k\leqslant 1\\ C_1^{k+1}\,\|v\|^2 & \text{if } k>0\\ C_2^{-k-ls-m}\,\|v\|^2 & \text{if } k\leqslant -ls-m-1\end{cases}$$

where $C_1>0$, $C_2>0$ are constants independent of x,v,k (but C_2 depends on $\varkappa$). This means that $\eta(x,\cdot)=d\chi_\varkappa(x)$. The latter implies statements 1 and 2. The others follow from them, (11) and theorem 2. □

We shall study stability properties of solutions of equation (2) in form (4). Given $\psi\in\Psi_\varkappa$ define $u(\psi)=(u(\psi)_j)=(\psi(lj))$.

Proposition 3. If $\psi \in \mathcal{M}_{q_1,q_2}$ for some $q_1 > 1$, $q_2 > 1$ then $u(\psi) \in \mathcal{M}_{q_1^l,q_2^l}$.

Proof. We have that

$$\sum_{j\geq 0} \frac{\|u(\psi)_j\|}{q_1^{lj}} + \sum_{j<0} \frac{\|u(\psi)_j\|}{q_2^{-lj}}$$
$$= \sum_{j\geq 0} \frac{\|\psi(lj)\|}{q_1^{lj}} + \sum_{j<0} \frac{\|\psi(lj)\|}{q_2^{-lj}} \leqslant \sum_{k\geq 0} \frac{\|\psi(k)\|}{q_1^{k}} + \sum_{k<0} \frac{\|\psi(k)\|}{q_2^{+k}} = \|\psi\|_{q_1,q_2}. \qquad \square$$

Let us introduce the map $\alpha_\chi : \Psi_{\varkappa,q_1,q_2} \to \mathcal{M}_{q_1^l,q_2^l}$ by setting $\alpha_\varkappa \psi = u(\psi)$.

Proposition 4. (1) $\alpha_\varkappa \circ S_\varkappa^m = \Phi_\varkappa \circ \alpha_\varkappa$; (2) $\alpha_\varkappa$ is a one-to-one map on $\Psi_{\varkappa,q_1,q_2}$.

Proof. (1) It follows from (2), (15) that

$$\Phi_\varkappa(u(\psi)_j) = (f(u(\psi)_j) + \varkappa g(\{u(\psi)_i\}_{i=j-s}^{j+s})$$
$$= (f(\psi(lj)) + \varkappa g(\psi(li)_{i=j-s}^{j+s}) = \alpha_\varkappa(\psi(lj+m)) = \alpha_\varkappa \circ S_\varkappa^m(\psi).$$

(2) Assume in contrast that there exist two functions $\psi', \psi'' \in \Psi_{\varkappa,q_1,q_2}$ such that $u(\psi') = u(\psi'')$. This means that $\psi'(lj) = \psi''(lj)$ for any $j \in \mathbb{Z}$ but $\psi'(k_0) \neq \psi''(k_0)$ for some $k_0 \in \mathbb{Z}$. Set $k(n,j) = nm + lj$, $n \geqslant 0$. We will show by induction over n that $\psi'(k(n,j)) = \psi''(k(n,j))$ for all $n \geqslant 0$. For $n = 0$ it is obvious. Suppose that it holds for all $0 \leqslant n \leqslant N$. It follows from (5) and our induction assumption that

$$\psi'(k(N+1,j)) = \psi'(Nm + lj + m) = f(\psi'(Nm + lj))$$
$$+ \varkappa g(\{\psi'(Nm + lj + li)\}_{i=-s}^{s}) = f(\psi''(Nm + lj))$$
$$+ \varkappa g(\{\psi''(Nm + lj + li)\}_{i=-s}^{s}) = \psi''(k(N+1,j)).$$

Since m and l are relatively prime numbers for any $k \in \mathbb{Z}$ one can find $n, j, n \geqslant 0$ such that $mn + lj = k$. This implies the desired result. $\square$

Denote by

$$\mathcal{A}_{\varkappa,q_1,q_2} = \alpha_\varkappa(\Psi_{\varkappa,q_1,q_2})$$

the set of solutions of equation (2) in the form of travelling waves (4) lying in $\mathcal{M}_{q_1^l,q_2^l}$.

Corollary. The evolution map $\Phi_\varkappa$ is one-to-one on $\mathcal{A}_{\varkappa,q_1,q_2}$ and, hence, is a homeomorphism.

Proof. This follows directly from statement 2 of theorem 2 and statement 1 of proposition 4. $\square$

The next two theorems follow from propositions 3, 4, and the corollary.

Theorem 4

(1) The set $\mathcal{L}_{\varkappa,q_1,q_2} = \alpha_\varkappa(\Lambda_{\varkappa,q_1,q_2})$ is closed and invariant under $\Phi_\varkappa$.

(2) The sets

$$\mathscr{V}^{s,u}_{\varkappa,q_1,q_2}(u) = \alpha_\varkappa(V^{s,u}_{\varkappa,q_1,q_2}(\psi)), u = \alpha_\varkappa(\psi)$$

are 'traces' of stable and unstable local topological submanifolds in $\mathscr{M}_{q_1',q_2'}$ (cf theorem 2).

Theorem 5. Assume that f, g satisfy conditions (1), (8), (12) and $q_1 \geqslant q_1^{(0)}$, $q_2 \geqslant q_2^{(0)}(\varkappa)$ (cf theorem 3). Then

(1) the map $\alpha_\varkappa \mid \Psi_{\chi,q_1,q_2}$ is a smooth mapping into $\mathscr{M}_{q_1',q_2'}$; in particular the set $\mathscr{A}_{\varkappa,q_1,q_2}$ is a smooth finite-dimensional (of dimension $d(ls+m)$) submanifold in $\mathscr{M}_{q_1',q_2'}$;

(2) $\mathscr{L}_{\varkappa,q_1,q_2}$ is a hyperbolic locally maximal set for $\Phi_\varkappa$ (with respect to $\mathscr{A}_{\varkappa,q_1,q_2}$) with

$$\mathscr{E}^{s,u}_{\varkappa,q_1,q_2}(u) = d\alpha_\varkappa(E^{s,u}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \alpha_\varkappa(\psi)$$

as stable and unstable subspaces at $u \in \mathscr{L}_{\varkappa,q_1,q_2}$ and

$$\mathscr{V}^{s,u}_{\varkappa,q_1,q_2}(u) = \alpha_\varkappa(V^{s,u}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \alpha_\varkappa(\psi)$$

as stable and unstable local smooth submanifolds in $\mathscr{A}_{\varkappa,q_1,q_2}$ respectively.

3. Stability properties of travelling waves in transversal directions

We shall study stability properties of equation (5) in directions 'transversal' to the set $\Psi_{\varkappa,q_1,q_2}$, $q_1 > 1$, $q_2 > 1$ (see theorems 2 and 3). Let us introduce the operator $Q_\varkappa$ by setting for any $\{w_k\} \subset \mathscr{M}_{q_1,q_2}$

$$Q_\varkappa(\{w_k\}) = \{f(w_{k-m}) + \varkappa g(\{w_{k-m+li}\}_{i=-s}^{s})\}.$$

The trajectories $\{Q_\varkappa^n(\{w_k\})\}$, $n \in \mathbb{Z}$, correspond to waves running with velocity m/l. The solution of (5) represent those of them which are stationary. Others are non-stationary ones. It is clear that $Q_\varkappa|_{\Psi_{\varkappa,q_1,q_2}} = Id|_{\Psi_{\varkappa,q_1,q_2}}$. The nonlinear operator $Q_\varkappa$ has a special 'drift' form. Operators of such a type have been studied in [16].

Proposition 5. For any $q_1 > 1$, $q_2 > 1$

(1) $Q_\varkappa$ is a map from $\mathscr{M}_{q_1,q_2}$ into itself;

(2) $Q_\varkappa$ is differentiable in the sense of Gautex and its Gautex differential at a point $\{w_k\}$ is given by the linear map

$$(dQ_\varkappa\eta)(k) = f'(w_{k-m})\eta_{k-m} + \varkappa \sum_{i=-s}^{s} a_i\eta(k-m+li)$$

where $a_i = (\partial g/\partial w_{k-m+li})(\{w_{k-m+li}\}_{i=-s}^{s})$.

Proof. This is the same as for proposition 1. □

Let us fix $\psi \in \Psi_{\varkappa,q_1,q_2}$. We shall describe subspaces on which the action of $dQ_\varkappa$ is contracting. Given $k \in \mathbb{Z}^+$ define $E^{s,k}_{\varkappa,q_1,q_2}(\psi) = \{\eta \in \mathscr{M}_{q_1,q_2} : \eta = (\eta_j)$ where $\eta_j = 0$ for $j \leqslant k\}$. It is easy to see that these subspaces have the following properties:

(1) $E^{s,k}_{\varkappa,q_1,q_2}(\psi)$ form a filtration: $E^{s,k}_{\varkappa,q_1,q_2}(\psi) \supset E^{s,k+1}_{\varkappa,q_1,q_2}(\psi)$;

(2) if $E^{s,\infty}_{\varkappa,q_1,q_2}(\psi) = \bigcup_{k\in\mathbb{Z}} E^{s,k}_{\varkappa,q_1,q_2}(\psi)$ then $\overline{E^{s,\infty}_{\varkappa,q_1,q_2}}(\psi) = \mathcal{M}_{q_1,q_2}$;

(3) $dQ_\varkappa E^{s,k}_{\varkappa,q_1,q_2}(\psi) \subseteq E^{s,k+m-ls}_{\varkappa,q_1,q_2}(\psi)$.

The following result shows that the above subspaces are stable with respect to $dQ_\varkappa$.

Theorem 6. Assume that f and g satisfy conditions (1) and (8). There exists a constant $q_1^{(0)} > 1$ with the following property: for any $q_1 \geqslant q_1^{(0)}$, $q_2 > 1$ and $k \in \mathbb{Z}$ one can find a constant $C(k) = C(q_1, q_2, k) > 0$ such that for any $\psi \in \Psi_{\varkappa,q_1,q_2}$, $\eta \in E^{s,k}_{\varkappa,q_1,q_2}(\psi)$, $n \geqslant 0$

$$\|dQ^n_\varkappa(\psi)\eta\|_{q_1,q_2} \leqslant C(k)\cdot\gamma^n \|\eta\|_{q_1,q_2}$$

where $\gamma, 0 < \gamma < 1$ is a constant independent of k, n.

Proof. First of all we show that $dQ_\varkappa$ is a bounded linear operator. Our arguments generalize those we have used in the proof of proposition 1 (which is essentially the same if one assumes below $m = 0$). Let $\eta \in \mathcal{M}_{q_1,q_2}$. We have by virtue of (1)

$$\|dQ_\varkappa\eta\|_{q_1,q_2} = \sum_{j=-\infty}^{-1} \frac{\|(dQ_\varkappa\eta)_j\|}{q_2^{-j}} + \sum_{j=0}^{\infty} \frac{\|(dQ_\varkappa\eta)_j\|}{q_1^j} \leqslant M\left(\sum_{j=-\infty}^{-1} A_j^{(1)} + \sum_{j=0}^{m-1} A_j^{(2)} + \sum_{j=m}^{\infty} A_j^{(3)}\right)$$

where

$$A_j^{(1)} = \frac{\|\eta(j-m)\|}{q_2^{-m} q_2^{j-m}} + \varkappa \sum_{i=-s}^{s} \frac{\|\eta(j-m+li)\|}{q_2^{(j-m+li)}\cdot q_2^{-m+li}}$$

$$A_j^{(2)} = \frac{\|\eta(j-m)\|}{q_2^{|j-m|}\cdot q_1^j\cdot q_2^{-|j-m|}} + \varkappa\left(\sum_{\substack{i=-s\\ j-m+li<0}}^{s} \frac{\|\eta(j-m+li)\|}{q_2^{|j-m+li|}\cdot q_1^j\cdot q_2^{-|j-m+li|}} + \sum_{\substack{i=-s\\ j-m+li\geqslant 0}}^{s} \frac{\|\eta(j-m+li)\|}{q_1^{j-m+li}\cdot q_1^{m-li}}\right)$$

$$A_j^{(3)} = \frac{\|\eta(j-m)\|}{q_1^{j-m}\cdot q_1^m} + \varkappa \sum_{i=-s}^{s} \frac{\|\eta(j-m+li)\|}{q_1^{j-m+li}\cdot q_1^{m-li}}.$$

Let us now estimate these values. It is clear that

$$A_j^{(1)} \leqslant q_2^{m+ls}\cdot\left(\frac{\|\eta(j-m)\|}{q_2^{|j-m|}} + \varkappa\cdot\sum_{i=-s}^{s} \frac{\|\eta(j-m+li)\|}{q_2^{|j-m+li|}}\right)$$

$$A_j^{(2)} \leqslant q^{m+ls}\cdot\left(\frac{\|\eta(j-m)\|}{q_2^{|j-m|}} + \varkappa\cdot\sum_{\substack{i=-s\\ j-m+li<0}}^{s} \frac{\|\eta(j-m+li)\|}{q_2^{|j-m+li|}} + \varkappa\sum_{\substack{i=-s\\ j-m+li\geqslant 0}}^{s} \frac{\|\eta(j-m+li)\|}{q_1^{j-m+li}}\right)$$

where $q = \max\{q_1, q_2\}$, and

$$A_j^{(3)} \leqslant q_1^{m+ls}\cdot\left(\frac{\|\eta(j-m)\|}{q_1^{|j-m|}} + \varkappa\sum_{i=-s}^{s} \frac{\|\eta(j-m+li)\|}{q_1^{j-m+li}}\right).$$

Consequently,

$$\|dQ_\varkappa\eta\|_{q_1,q_2} \leqslant Mq^{m+ls}(1+\varkappa(2s+1))\,\|\eta\|_{q_1,q_2}.$$

It follows from here that

$$\|dQ_\varkappa\| \leqslant \tilde{C} \stackrel{\text{def}}{=} Mq^{m+ls}(1+\varkappa(2s+1)).$$

Let us fix first $k<0$. Define $n_0(k)=\left[\dfrac{m+ls-k}{m-ls+1}\right]+1$. One can verify that

$$dQ_\varkappa^{n_0(k)}E^{s,k}_{\varkappa,q_1,q_2}\subset E^{s,k_0}_{\varkappa,q_1,q_2}$$

for some $k_0>ls+m$. Besides, we have from above

$$\|dQ_\varkappa\eta\|_{q_1,q_2}\leqslant \tilde{C}^{n_0(k)}\cdot\|\eta\|_{q_1,q_2}. \tag{15}$$

Let us now take $n>n_0(k)$ and $\bar{\eta}=dQ_\varkappa^{n_0(k)}\eta$. We have that

$$dQ_\varkappa^n\eta = dQ_\varkappa^{n-n_0(k)}\cdot dQ_\varkappa^{n_0(k)}\eta = dQ_\varkappa^{n-n_0(k)}\bar{\eta}.$$

For any $\xi=(\xi_j)$ with $\xi_j=0$ if $j\geqslant m+ls$

$$\|dQ_\varkappa\xi\|_{q_1,q_2}=\sum_{j=m+ls+1}^{\infty}\frac{\|(dQ_\varkappa\xi)_j\|}{q_1^j}\leqslant M\cdot\sum_{j=m+ls+1}^{\infty}\left(\frac{\|\xi(j-m)\|}{q_1^{j-m}\cdot q_1^m}+\varkappa\sum_{i=-s}^{s}\frac{\|\xi(j-m+li)\|}{q_1^{j-m+li}\cdot q_1^{m-li}}\right)$$

$$\leqslant M\cdot\left(\frac{1}{q_1^m}+\frac{\varkappa}{q_1^{m-ls}}\cdot(2s+1)\right)\cdot\|\xi\|_{q_1,q_2}.$$

We assume that $1/q_1$ and $\varkappa$ are so small as to satisfy the inequality

$$\gamma\stackrel{\text{def}}{=}M\left(\frac{1}{q_1^m}+\frac{\varkappa}{q_1^{m-ls}}(2s+1)\right)<1. \tag{16}$$

From this and (15) we have

$$\|dQ_\varkappa^n\eta\|\leqslant\gamma^{n-n_0(k)}\cdot\tilde{C}^{n_0(k)}\,\|\eta\|_{q_1,q_2}.$$

Consequently, denoting $\gamma^{-n_0(k)}\cdot\tilde{C}^{n_0(k)}=C(k)$ the required inequality follows. The case $k>0$ can be established in the same way. □

Now we are able to formulate the final result describing linear stability properties of solutions $\psi\in\Psi_{\varkappa,q_1,q_2}$ for big enough q_1,q_2. It follows from theorems 3–6 and proposition 5.

Theorem 7. Assume that f and g satisfy conditions (1), (8), (12). Then for any small enough $\varkappa$ and q_1 satisfying (16) and also $q_1\geqslant q_1^0$, $q_2\geqslant q_2^0(\varkappa)$ (see theorem 3)

(1) the set $\Psi_{\varkappa,q_1,q_2}$ is a finite-dimensional smooth submanifold in $\mathcal{M}_{q_1,q_2}$;

(2) the map $Q_\varkappa$ possesses a closed invariant subset $\Lambda_{\varkappa,q_1,q_2}\subset\Psi_{\varkappa,q_1,q_2}$ which is a finite-dimensional hyperbolic locally maximal set with respect to $\Psi_{\varkappa,q_1,q_2}$;

(3) for any $\psi\in\Lambda_{\varkappa,q_1,q_2}$ there exists a finite-dimensional stable subspace $E^s_{\varkappa,q_1,q_2}(\psi)$, a finite-dimensional unstable subspace $E^u_{\varkappa,q_1,q_2}(\psi)$, and an infinite-dimensional stable everywhere dense subspace $E^{s,\infty}_{\varkappa,q_1,q_2}(\psi)$; the first two generate the tangent space $T_\psi\Psi_{\varkappa,q_q,q_2}$; the intersection of any two of them is 0.

We complete the above consideration by constructing a filtration of stable smooth manifolds corresponding to $E^{s,k}_{\varkappa,q_1,q_2}(\psi)$, $\psi \in \Psi_{\varkappa,q_1,q_2}$, $\psi = (\psi_j)$, $k \in \mathbb{Z}$. Denote

$$V^{s,k}_{\varkappa,q_1,q_2}(\psi) = \{w \in \mathcal{M}_{q_1,q_2} : w = (w_j), w_j = \psi_j \text{ for } j \leq k\}.$$

This is an affine space of infinite co-dimension passing through ψ. It has the following properties.

Theorem 8. For any $\psi \in \Psi_{\varkappa,q_1,q_2}$ with q_1 satisfying (16)

(1) $V^{s,k}_{\varkappa,q_1,q_2}(\psi)$ form a filtration: $V^{s,k}_{\varkappa,q_1,q_2}(\psi) \supset V^{s,k+1}_{\varkappa,q_1,q_2}(\psi)$;

(2) if $V^{s,\infty}_{\varkappa,q_1,q_2}(\psi) = \bigcup_{k \in \mathbb{Z}} V^{s,k}_{\varkappa,q_1,q_2}(\psi)$ then it is everywhere dense in $\mathcal{M}_{q_1,q_2}$,

(3) $V^{s,k}_{\varkappa,q_1,q_2}(\psi)$ is invariant under $Q_\varkappa$ in the following sense:

$$Q_\varkappa(V^{s,k}_{\varkappa,q_1,q_2}(\psi)) \subset V^{s,k+m-ls}_{\varkappa,q_1,q_2}(\psi).$$

(4) $V^{s,k}_{\varkappa,q_1,q_2}$ is stable: for any $w \in V^{s,k}_{\varkappa,q_1,q_2}$ and $n \geq 0$

$$\|Q^n_\varkappa(w) - Q^n_\varkappa(\psi)\|_{q_1,q_2} \leq C_1(k)\gamma^n \cdot \|w - \Psi\|_{q_1,q_2}$$

where $C_1(k) = C_1(q_1, q_2, k)$ is a constant and γ is a constant independent of k (cf (16));

(5) if, in addition, $q_1 \geq q_1^0$, $q_2 \geq q_2^0(\varkappa)$ then $V^{s,k}_{\varkappa,q_1,q_2}(\psi)$ is transversal to $\Psi_{\varkappa,q_1,q_2}$ and $V^{s,k}_{\varkappa,q_1,q_2}(\psi) \cap \Psi_{\varkappa,q_1,q_2} = \{\psi\}$.

Proof. In fact the only property which needs to be checked is the property (4). It can be proved in the same way as theorem 6 by using the property (3) with the following modification: instead of $dQ_\varkappa(\psi)$ we have to consider $dQ_\varkappa(\psi + tw)$ for some $0 \leq t \leq 1$. □

The previous results have an auxiliary character but we will use them to study stability properties of equation (2) along travelling waves in the space $\mathcal{M}_{q_1,q_2}$ in the direction transversal to $\mathcal{A}_{\varkappa,q_1,q_2}$.

We extend the map $\alpha_\varkappa$ from $\Psi_{\varkappa,q_1,q_2}$ to $\mathcal{M}_{q_1,q_2}$ by the following formula

$$\tilde{\alpha}_\varkappa w = u(w) = ((u(w))_j) \equiv (w_{lj})$$

for any $w \in \mathcal{M}_{q_1,q_2}$. Let us also consider the shift $\tilde{S}_\varkappa : \mathcal{M}_{q_1,q_2} \to \mathcal{M}_{q_1,q_2}$, $(\tilde{S}_\varkappa w)(k) = w(k+1)$.

It is easy to see that $\tilde{S}_\varkappa \mid \Psi_{\varkappa,q_1,q_2} = S_\varkappa$.

Proposition 6. (1) If $w \in \mathcal{M}_{q_1,q_2}$ then $u(w) \in \mathcal{M}_{q_1',q_2'}$; (2) the map $\tilde{\alpha}_\varkappa$ is a linear bounded operator; (3) $\tilde{\alpha}_\varkappa \circ Q_\varkappa \circ \tilde{S}^m_\varkappa = \Phi_\varkappa \circ \tilde{\alpha}_\varkappa$.

Proof. The first statement can be proved as in proposition 3. The second one is obvious. It follows from the definition of $\tilde{\alpha}_\varkappa$, $Q_\varkappa$ and $\tilde{S}_\varkappa$ that

$$\tilde{\alpha}_\varkappa \circ Q_\varkappa \circ \tilde{S}^m_\varkappa(w)_j = Q_\varkappa \circ \tilde{S}^m_\varkappa(w)_{lj} = f(w_{lj}) + \varkappa g(\{w_{lj+li}\}^s_{i=-s}).$$

On the other hand

$$\Phi_\varkappa \circ \tilde{\alpha}_\varkappa(\{w_j\}) = \Phi_\varkappa(\{w_{lj}\}) = \{f(w_{lj}) + \varkappa g(\{w_{lj+li}\}^s_{i=-s})\}.$$

This proves the third statement. □

Let us define for $u \in \mathcal{A}_{\varkappa,q_1,q_2}$, $u = \alpha_\varkappa(w)$, $w \in \Psi_\varkappa$:

(1) Subspaces

$$\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)=d\tilde{\alpha}_\varkappa E^{s,lk}_{\varkappa,q_1,q_2}(w);$$

(2) Submanifolds corresponding to $\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)$

$$\mathscr{V}^{s,k}_{\chi,q_1,q_2}(u)=\tilde{\alpha}_\varkappa(V^{s,lk}_{\varkappa,q_1,q_2}(w)).$$

The following results describes stability of the evolution operator $\Phi_\varkappa$ in directions transversal to $\mathscr{A}_{\varkappa,q_1,q_2}$.

Theorem 9. If (1), (8), (16) hold then for any $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ with q_1 satisfying (16).

1. the subspaces $\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)$ possess the following properties:

(a) they form the filtration:

$$\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)\supset\mathscr{E}^{s,k+1}_{\varkappa,q_1,q_2}(u);$$

(b) if $\mathscr{E}^{s,\infty}_{\varkappa,q_1,q_2}(u)=\bigcup_{k\in\mathbb{Z}}\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)$ then it is everywhere dense in $\mathscr{M}_{q_1',q_2'}$;

(c) $d\Phi_\varkappa\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u)\subset\mathscr{E}^{s,k-s}_{\varkappa,q_1,q_2}(\Phi_\varkappa(u))(d\Phi_\varkappa$ is the Gautex differential of $\Phi_\varkappa$; cf proposition 1);

(d) for any $u\in\mathscr{M}_{q_1',q_2'}$, $\eta\in\mathscr{E}^{s,k+mn}_{\varkappa,q_1,q_2}(u)$, $n\geqslant 0$

$$\|d\Phi^{ln}_\varkappa(u)\eta\|_{q_1',q_2'}\leqslant C(lk)\gamma^{ln}\,\|\eta\|_{q_1',q_2'}$$

where $C(lk)>0$, $0<\gamma<1$ are constants from theorem 6;

2. the manifolds $\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(\psi)$ for any $\psi\in\mathscr{A}_{\varkappa,q_1,q_2}$ have the following properties:

(a) they form a filtration

$$\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u)\supset\mathscr{V}^{s,k+1}_{\varkappa,q_1,q_2}(u);$$

(b) if $\mathscr{V}^{s,\infty}_{\varkappa,q_1,q_2}(u)=\bigcup_{k\in\mathbb{Z}}\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u)$ then it is everywhere dense in $\mathscr{M}_{q_1',q_2'}$;

(c) $\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u)$ is invariant under $\Phi_\varkappa$,

$$\Phi_\varkappa(\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u))\subset\mathscr{V}^{s,k-s}_{\varkappa,q_1,q_2}(\Phi_\varkappa(u));$$

(d) $\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u)$ is stable i.e. for any $\tilde{u}\in\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u)$, $n>0$

$$\|\Phi^{ln}_\varkappa(\tilde{u})-\Phi^{ln}_\varkappa(u)\|_{q_1',q_2'}\leqslant C_1(lk)\gamma^{ln}\,\|\tilde{u}-u\|_{q_1',q_2'}.$$

where $C_1(lk)>0$ is the constant from theorem 8.

Proof. The statements 1(a) and 1(b) are obvious. It follows from proposition 6 that

$$\begin{aligned}d\Phi_\varkappa\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}&=d\Phi_\varkappa\circ d\tilde{\alpha}_\varkappa E^{s,lk}_{\varkappa,q_1,q_2}=d\tilde{\alpha}_\varkappa\circ dQ_\varkappa\circ d\tilde{S}^m_\varkappa E^{s,lk}_{\varkappa,q_1,q_2}\\&=d\tilde{\alpha}_\varkappa\circ dQ_\varkappa E^{s,lk-m}_{\varkappa,q_1,q_2}=d\tilde{\alpha}_\varkappa E^{lk-ls}_{\varkappa,q_1,q_2}=\mathscr{E}^{s,k-s}_{\varkappa,q_1,q_2}.\end{aligned}$$

This proves the statement 1(c). To verify 1(d) let us first note that for any $n>0$ one can easily conclude from proposition 6 that

$$\tilde{\alpha}_{\varkappa}\circ(Q_{\varkappa}\circ\tilde{S}^m_{\varkappa})^n=\Phi^n_{\varkappa}\circ\tilde{\alpha}_{\varkappa}.$$

Since $Q_{\varkappa}$ and $\tilde{S}^m_{\varkappa}$ are commutative we have

$$(Q_{\varkappa}\circ\tilde{S}^m_{\varkappa})^n=Q^n_{\varkappa}\circ\tilde{S}^{mn}_{\varkappa}.$$

This implies that

$$\tilde{\alpha}_{\varkappa}\circ Q^n_{\varkappa}\circ\tilde{S}^{mn}_{\varkappa}=\Phi^n_{\varkappa}\circ\tilde{\alpha}_{\varkappa}.$$

It follows from here that

$$d\tilde{\alpha}_{\varkappa}\circ dQ^{ln}_{\varkappa}\mid E^{s,lk}_{\varkappa,q_1,q_2}=d\Phi^{ln}_{\varkappa}\circ d\tilde{\alpha}_{\varkappa}\mid E^{s,lk+lmn}_{\varkappa,q_1,q_2}=d\Phi^{ln}_{\varkappa}\mid \mathscr{E}^{s,k+mn}_{\varkappa,q_1,q_2}.$$

Let us note that obviously for any $\xi\in\mathscr{M}_{q_1,q_2}$

$$\|d\tilde{\alpha}_{\varkappa}\xi\|_{q_1',q_2'}\leqslant\|\xi\|_{q_1,q_2}.$$

Now the statement $1(d)$ follows from theorem 6. The statements $2(a)$–(d) can be proved in the same way. □

Remark. The statements $1(c)$, $2(c)$ show that a perturbation $h(n)=(h_j(n))$ of a travelling wave solution $u(n)=(u_j(n))=(\psi(lj+mn))$ moves under the evolution operator in the direction of this solution on the distance equals to the length of the interaction. The statements $1(d)$, $2(d)$ mean that those of the perturbations which run together with the travelling wave with the same velocity m/l (i.e. having the form $h_j(n)=\xi_{lj+mn}(n)$) are damped in time under the evolution operator.

Let us also note that the family of subspaces

$$\mathscr{E}^k_n\overset{\text{def}}{=}\mathscr{E}^{s,k}_{\varkappa,q_1,q_2}(u(n))\qquad u(n)=\Phi^n_{\varkappa}(u)$$

forms a non-stationary filtration invariant and infinitesimally stable with respect to $d\Phi_{\varkappa}$ and the family

$$\mathscr{V}^k_n\overset{\text{def}}{=}\mathscr{V}^{s,k}_{\varkappa,q_1,q_2}(u(n))$$

forms a non-stationary filtration invariant and stable with respect to $\Phi_{\varkappa}$.

We now formulate a result describing infinitesimal stability properties of the solutions of equation (2) in the form (4) of travelling waves (compare with theorem 7).

Theorem 10. Assume that f and g satisfy conditions (1), (8), (12). Then for any small enough $\varkappa$ and sufficiently large q_1, q_2 (q_2 depends on $\varkappa$ but q_1 does not; cf theorem 3)

(1) the set $\mathscr{A}_{\varkappa,q_1',q_2'}$ is a finite-dimensional smooth submanifold in $\mathscr{M}_{q_1',q_2'}$;

(2) the map $\Phi_{\varkappa}$ possesses a closed invariant set $\mathscr{L}_{\varkappa,q_1,q_2}\subset\mathscr{A}_{\varkappa,q_1',q_2'}$ which is a finite-dimensional hyperbolic set;

(3) for any $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ there exists a finite-dimensional stable subspace $\mathscr{E}^s_{\varkappa,q_1,q_2}(u)$, a finite-dimensional unstable subspace $\mathscr{E}^u_{\varkappa,q_1,q_2}$, and an infinite-dimensional everywhere dense subspace $\mathscr{E}^{s,\infty}_{\varkappa,q_1,q_2}(u)$; the first two generate the tangent space $T_u\mathscr{A}_{\varkappa,q_1,q_2}$; the intersection of any two of them is 0;

(4) the above subspaces are integrable: there exist finite-dimensional stable and unstable manifolds $\mathscr{V}^{s,u}_{\varkappa,q_1,q_2}(u)$ corresponding to $\mathscr{E}^{s,u}_{\varkappa,q_1,q_2}(u)$ and an infinite-dimensional manifold $\mathscr{V}^{s,\infty}_{\varkappa,q_1,q_2}(u)$ corresponding to $\mathscr{E}^{s,\infty}_{\varkappa,q_1,q_2}(u)$.

4. Travelling waves in multi-dimensional and multi-component media

In this section we shall show that our methods work in the case of multi-dimensional models. We give necessary definitions and formulate main results while the corresponding proofs can be conducted in the same way.

Given $\bar{q}_1 = (q_{1i})$, $\bar{q}_2 = (q_{2i})$, $q_{1i} > 1$, $q_{2i} > 1$, $i = 1, \ldots, t$ let us consider the set

$$\mathcal{M}_{\bar{q}_1,\bar{q}_2} = \left\{u : \mathbb{Z}^t \to \mathbb{R}^d : u = (u(\bar{j})_{\bar{j} \in \mathbb{Z}^t}), \|u\|_{\bar{q}_1,\bar{q}_2} = \sum_{j_1} \ldots \sum_{j_t} \frac{\|u(j_1, \ldots, j_t)\|}{q(j_1) \cdot \ldots \cdot q(j_t)}\right\}$$

where $q(j_i) = q_{1i}^{j_i}$ if $j_i \geq 0$ and $q(j_i) = q_{2i}^{-j_i}$ if $j_i < 0$.

It is easy to see that $\mathcal{M}_{\bar{q}_1,\bar{q}_2}$ is a Banach space with the norm $\|\cdot\|_{\bar{q}_1,\bar{q}_2}$.

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism of class C^2, $g : (\mathbb{R}^d)^{(2s+1)^t} \to \mathbb{R}^d$ a map of class $C^2 (s \geq 0)$ and $\varkappa > 0$ a real number. Define the map $\Phi_\varkappa : \mathcal{M}_{\bar{q}_1,\bar{q}_2} \to (\mathbb{R}^d)^{\mathbb{Z}^t}$ by the formula

$$\Phi_\varkappa(u) = (f(u(\bar{j})) + \varkappa g(\{u(\bar{i})\}_{|\bar{i}-\bar{j}| \leq s}))$$

where $u = (u(\bar{j}))$ and $|\bar{j}| = \sum_{i=1}^t |j_i|$.

Proposition 7. Assume that there exists $M > 0$ such that for $l = 1, 2$

$$\sup_{x \in \mathbb{R}^d} \|d^l f_x\| \leq M \qquad \left\| \frac{\partial^l g}{\partial u_{\bar{i}}^l}(\{u(\bar{j})\}) \right\| \leq M \tag{17}$$

for any $\bar{i}$, $|\bar{i}| \leq s$ and any point $u(\bar{j}) \in \mathbb{R}^{(2s+1)^t}$.

Then for any $\bar{q}_1 = (q_{1i})$, $\bar{q}_2 = (q_{2i})$, $q_{1i} > 1$, $q_{2i} > 1$, $i = 1, \ldots, t$

(1) $\Phi_\varkappa$ is a map from $\mathcal{M}_{\bar{q}_1,\bar{q}_2}$ into itself;

(2) $\Phi_\varkappa$ is differentiable in the sense of Gautex and its Gautex differential at a point $u = (u(\bar{j}))$ is given by the linear map

$$(d\Phi_\varkappa \eta)(\bar{j}) = f'(u(\bar{j}))\eta(\bar{j}) + \sum_{|\bar{i}-\bar{j}| \leq s} a(\bar{i})\eta(\bar{i})$$

where

$$a(\bar{i}) = \frac{\partial g}{\partial u_{\bar{i}}}(\{u(\bar{i})\}_{|\bar{i}-\bar{j}| \leq s}), \ \eta = (\eta(\bar{j})).$$

For any $u \in \mathcal{M}_{\bar{q}_1,\bar{q}_2}$ let us denote by $u(\bar{j}, n) = (\Phi_\varkappa^n u)(\bar{j})$. It satisfies

$$u(\bar{j}, n+1) = f(u(\bar{j}, n)) + \varkappa g(\{u(\bar{i}, n)\}_{|\bar{i}-\bar{j}| \leq s}) \tag{18}$$

$n \geq 0$, $u(\bar{j}, 0) = u(\bar{j})$.

Let us fix $m, l_1, \ldots, l_t \in \mathbb{Z}^+$ such that

(1) $m \geq s \sum_{i=1}^t l_i + 1$;

(2) the numbers $l_1, \ldots, l_t$ have the common divisor 1; in particular, the equation $l_1 x_1 + \ldots + l_t x_t = k$ has a solution in the set of integers for any $k \in \mathbb{Z}$.

The second condition implies that the numbers $l_1, \ldots, l_t, m$ have the common divisor 1 but is more strong than the condition we have used in the case $t = 1$.

The travelling wave is a solution of (18) in the form

$$u(\bar{j}, n) = \psi((\bar{l}, \bar{j}) + mn) = \psi(k - m)$$

where $(\bar{l}, \bar{j}) = \sum_{i=1}^t l_i j_i$, $k = (\bar{l}, \bar{j}) + mn + m$ and $\psi : \mathbb{Z} \to \mathbb{R}^d$ is a function. It satisfies the equation

$$\psi(k)=f(\psi(k-m))+\varkappa g(\{\psi(k-m+(\bar{l},\bar{i}))\}_{|\bar{i}|\leq s}). \tag{19}$$

The function ψ is uniquely defined by (19) if we know the values (6) where $l=\sum_{i=1}^{t} l_i$.

Let us introduce the 'travelling wave' map

$$F_\varkappa:(\mathbb{R}^d)^{ls+m}\to(\mathbb{R}^d)^{ls+m} \qquad F_\varkappa(x_1,\ldots,x_{ls+m})=(\bar{x}_1,\ldots,\bar{x}_{ls+m})$$

where

$$\begin{aligned}&\bar{x}_1=x_2,\ \bar{x}_2=x_3,\ldots,\bar{x}_{ls+m-1}=x_{ls+m},\\&\bar{x}_{ls+m}=f(x_{ls+1})+\varkappa g(\{x_{p(\bar{i})}\}_{|\bar{i}|\leq s}),\ p(\bar{i})=ls+(\bar{l},\bar{i})+1.\end{aligned} \tag{20}$$

We will assume that for any $x=(x_i)\in(\mathbb{R}^d)^{(2s+1)^t}$ the function g satisfies (8) and (12). The following result describes hyperbolic properties of the map $F_\varkappa$ given by (20).

Theorem 11. If f possesses a hyperbolic locally maximal set Λ (cf (9)) then the map $F_\varkappa$ satisfies the statements in theorem 1.

The hyperbolic properties of solutions of equation (19) in the space $\mathcal{M}_{q_1,q_2}$, $q_1>1$, $q_2>1$ can be studied in the way described above without any modifications. We formulate the final results. Denote by $\Psi_\varkappa$ the set of solutions of (19). The relations (20) define a map $\chi_\varkappa:(\mathbb{R}^d)^{ls+m}\to\Psi_\varkappa$. We also consider the shift $S_\varkappa:\Psi_\varkappa\to\Psi_\varkappa$, $(S_\varkappa\psi)(k)=\psi(k+1)$, $\psi\in\Psi_\varkappa$.

Theorem 12. (1) $\chi_\varkappa$ is one-to-one map onto $\Psi_\varkappa$ satisfying (11).

(2) For any $q_1>1$, $q_2>1$ the statements of theorem 2 hold.

(3) Assume that f and g satisfy conditions (17), (8) and (12). There exist $q_1^{(0)}>1$, $q_2^{(0)}(\varkappa)>1$ such that for any $q_1\geq q_1^{(0)}$, $q_2\geq q_2^{(0)}(\varkappa)$ the statements of theorem 3 hold.

We will use the previous notation $\Psi_{\varkappa,q_1,q_2}$ for the set of travelling waves from $\Psi_\varkappa$ belonging to $\mathcal{M}_{q_1,q_2}$ and $\Lambda_{\varkappa,q_1,q_2}=\chi_\varkappa(\Lambda_\varkappa)\subset\Psi_{\varkappa,q_1,q_2}$ for the hyperbolic set into $\Psi_{\varkappa,q_1,q_2}$.

Consider now the operator $Q_\varkappa$ given by the formula

$$Q_\varkappa(\{w_k\})=\{f(w_{k-m})+\varkappa g(\{w_{k-m+(\bar{l},\bar{i})}\}_{|\bar{i}|\leq s})\}$$

where $w=\{w_k\}\in\mathcal{M}_{q_1,q_2}$. This map satisfies statements of proposition 5 and theorems 6, 7 with respect to the filtration of subspaces $E^{s,k}_{\varkappa,q_1,q_2}(\psi)$, $\psi\in\Psi_{\varkappa,q_1,q_2}$. If f and g satisfy conditions (17), (8), (12) then theorem 8 holds with respect to the filtration of affine subspaces $V^{s,k}_{\varkappa,q_1,q_2}(\psi)$.

We shall study stability properties of solutions of (18) in the form of travelling waves. Given $\psi\in\Psi_\varkappa$ define $\alpha_\varkappa(\psi)=(u(\psi)(\bar{j}))=(\psi(\bar{l},\bar{j}))$.

Proposition 7. If $\psi\in\mathcal{M}_{q_1,q_2}$ for some $q_1>1$, $q_2>1$ then $u(\psi)\in\mathcal{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ where $\bar{q}_1(\bar{l})=(q_1^{l_i})$, $\bar{q}_2(\bar{l})=(q_2^{l_i})$, $i=1,\ldots,t$.

Proof. Let $\psi \in \mathcal{M}_{q_1,q_2}$. We have

$$\|u(\psi)\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} = \sum_{j_1} \dots \sum_{j_t} \frac{\|\psi((\bar{l},\bar{j}))\|}{q(j_1)\cdot \dots \cdot q(j_t)} = \sum_{k\in\mathbb{Z}} \sum_{(\bar{l},\bar{j})=k} \frac{\|\psi((\bar{l},\bar{j}))\|}{q(j_1)\cdot \dots \cdot q(j_t)}.$$

Consider some $\bar{j}=(j_1,\dots,j_t)$ such that $(\bar{l},\bar{j})=k$ and assume that $j_{\alpha_1}\geq 0,\dots,j_{\alpha_p}\geq 0$ while $j_{\beta_1}<0,\dots,j_{\beta_{t-p}}<0$. Then

$$Q = q(j_1)\cdot \dots \cdot q(j_t) = q_1^{l_{\alpha_1}j_{\alpha_1}+\dots+l_{\alpha_p}j_{\alpha_p}} q_2^{-(l_{\beta_1}j_{\beta_1}+\dots+l_{\beta_{t-p}}j_{\beta_{t-p}})}$$

Suppose first that $k \equiv l_{\alpha_1}j_{\alpha_1}+\dots+l_{\alpha_p}j_{\alpha_p}-(l_{\beta_1}j_{\beta_1}+\dots+l_{\beta_{t-p}}j_{\beta_{t-p}})\geq 0$. It is easy to see that

$$Q = q_2^k (q_1 q_2)^{l_{\alpha_1}j_{\alpha_1}+\dots+l_{\alpha_p}j_{\alpha_p}-k}$$

where the power $l_{\alpha_1}j_{\alpha_1}+\dots+l_{\alpha_p}j_{\alpha_p}-k = A(k,\bar{j})\geq 0$. If $k<0$ then

$$Q = q_2^{-k}(q_1q_2)^{k-(l_{\beta_1}j_{\beta_1}+\dots+l_{\beta_{t-p}}j_{\beta_{t-p}})}$$

where the power $k-(l_{\beta_1}j_{\beta_1}+\dots+l_{\beta_{t-p}}k_{\beta_{t-p}}) = B(k,\bar{j})>0$. It follows from what was said above that

$$\|u\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} = \sum_{k<0} \frac{\|\psi(k)\|}{q_2^{-k}} \sum_{(\bar{l},j)=k} \frac{1}{(q_1q_2)^{B(k,\bar{j})}} + \sum_{k\geq 0} \frac{\|\psi(k)\|}{q_1^k} \sum_{(\bar{l},\bar{j})=k} \frac{1}{(q_1q_2)^{A(k,\bar{j})}}.$$

One can show that

$$\max\left\{ \sum_{\substack{(\bar{l},\bar{j})=k\\ k\geq 0}} \frac{1}{(q_1q_2)^{A(k,\bar{j})}}, \sum_{\substack{(\bar{l},\bar{j})=k\\ k<0}} \frac{1}{(q_1q_2)^{B(k,\bar{j})}} \right\} \leq \text{constant}.$$

This implies the desired result. □

The next statement can be proved as proposition 4.

Proposition 8. (1) $\alpha_\varkappa \circ S_\varkappa^m = \Phi_\varkappa \circ \alpha_\varkappa$: (2) $\alpha_\varkappa$ is one-to-one map on $\Psi_{\varkappa,q_1,q_2}$.

Proof. The first statement follows from the definitions. To show the second, it is sufficient to repeat arguments in proposition 4 and to notice that according to our assumption for any $k\in\mathbb{Z}$ there exists $\bar{j}=(j_1,\dots,j_t)$ and $n>0$ such that $(\bar{l},\bar{j})+mn=k$. □

In particular, this result implies that the evolution operator $\Phi_\varkappa$ is one-to-one map on the set $\mathcal{A}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} = \alpha_\varkappa(\Psi_{\varkappa,q_1,q_2})$.

We extend the map $\alpha_\varkappa$ from $\Psi_{\varkappa,q_1,q_2}$ onto $\mathcal{M}_{q_1,q_2}$ by setting for any $w\in\mathcal{M}_{q_1,q_2}$

$$\tilde{\alpha}_\varkappa w = u(w) = ((u(w)(\bar{j}))) = (w(\bar{l},\bar{j})).$$

Proposition 9. (1) If $w\in\mathcal{M}_{q_1,q_2}$ then $u(w)\in\mathcal{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$. (2) The map $\tilde{\alpha}_\varkappa$ is a linear bounded operator. (3) $\tilde{\alpha}_\varkappa \circ Q_\varkappa \circ \tilde{S}_\varkappa^m = \Phi_\varkappa \circ \tilde{\alpha}_\varkappa$. (Recall that $\tilde{S}_\varkappa w(k) = w(k+1)$.)

Proof. We will prove only the third statement. One can see that

$$\tilde{\alpha}_\varkappa \circ Q_\varkappa \circ \tilde{S}_\varkappa^m(w)_{\bar{j}} = Q_\varkappa \circ \tilde{S}_\varkappa^m(w)_{(\bar{l},\bar{j})} = f(w_{(\bar{l},\bar{j})}) + \varkappa g(\{w_{(\bar{l},\bar{i})}\}_{|\bar{i}|\leq s}).$$

On the other hand

$$\Phi_\varkappa \circ \alpha_\varkappa(w_{\bar{j}}) = \Phi_\varkappa(w_{(\bar{l},\bar{j})}) = \{f(w_{(\bar{l},\bar{j})}) + \varkappa g(\{w_{(\bar{l},\bar{\imath})}\}_{|i|\leqslant s})\}. \qquad \square$$

Introduce now the set

$$\mathscr{L}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} = \alpha_\varkappa(\Lambda_{\varkappa,q_1,q_2})$$

the subspaces

$$\mathscr{E}^{s,u}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) = d\alpha_\varkappa(E^{s,u}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \alpha_\varkappa(\psi), \psi \in \Lambda_{\varkappa,q_1,q_2}$$

$$\mathscr{E}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) = d\tilde{\alpha}_\varkappa(E^{s,(\bar{l},\bar{k})}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \tilde{\alpha}_\varkappa(\psi), \psi \in \Psi_{\varkappa,q_1,q_2}$$

the local manifolds

$$\mathscr{V}^{s,u}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) = \alpha_\varkappa(V^{s,u}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \alpha_\varkappa(\psi), \psi \in \Lambda_{\varkappa,q_1,q_2}$$

and the affine subspaces

$$\mathscr{V}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) = \tilde{\alpha}_\varkappa(V^{s,(\bar{l},\bar{k})}_{\varkappa,q_1,q_2}(\psi)) \qquad u = \tilde{\alpha}_\varkappa(\psi), \psi \in \Psi_{\varkappa,q_1,q_2}.$$

$\mathscr{A}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ is a set of travelling wave solutions of equation (18), $\mathscr{L}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ is a locally maximal hyperbolic subset in $\mathscr{A}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$. The subspaces $\mathscr{E}^{s,u}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u)$ are stable and unstable in tangent directions at u. The local manifolds $\mathscr{V}^{s,u}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u)$ form stable and unstable sets at u in tangent direction. The subspaces $\mathscr{E}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u)$ form a non-stationary invariant filtration infinitisimally stable under $d\Phi_\varkappa$ and affine subspaces $\mathscr{V}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ form a non-stationary invariant filtration stable under $\Phi_\varkappa$. They also satisfy theorems 4, 5, 9, 10. In particular, we have

(1) $$\mathscr{E}^{s,\bar{k}_1}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) \subset \mathscr{E}^{s,\bar{k}_2}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$$

and

$$\mathscr{V}^{s,\bar{k}_1}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) \subset \mathscr{V}^{s,\bar{k}_2}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} \qquad \text{if } (l,\bar{k}_1) \geqslant (l,\bar{k}_2);$$

(2) $$d\Phi_\varkappa \mathscr{E}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u) \subset \mathscr{E}^{s,(\bar{l},\bar{k})-ls}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(\Phi_\varkappa(u))$$

and

$$\Phi_\varkappa(\mathscr{V}^{s,\bar{k}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u)) \subset \mathscr{V}^{s,(\bar{l},\bar{k})-ls}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(\Phi_\varkappa(u));$$

(3) for any $u \in \mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$, $\eta \in \mathscr{E}^{s,\bar{r}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$, $n \geqslant 0$

$$\|d\Phi^n_\varkappa(u)\eta\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} \leqslant C(\bar{k})\gamma^n \|\eta\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} \tag{21}$$

and for any $\tilde{u} \in \mathscr{V}^{s,\bar{r}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}(u)$, $n \geqslant 0$

$$\|\Phi^n_\varkappa(\tilde{u}) - \Phi^n_\varkappa(u)\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} \leqslant C_1(\bar{k})\gamma^n \|\tilde{u} - u\|_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})} \tag{22}$$

where $\bar{r} = n\bar{r}_0 - (n-1)\bar{k}$ and $\bar{r}_0$ is one of the integer solutions of the equation

$$(\bar{l}, \bar{r}_0) = (\bar{l}, \bar{k}) + m \tag{23}$$

(equation (23) has at least one integer solution due to our assumption about $l_1, \ldots, l_t$).

We shall explain (21) and (22). Let us first notice that by virtue of (23)

$(\bar{l}, \bar{k}) + mn = (\bar{l}, \bar{r})$. Therefore

$$d\tilde{\alpha}_{\varkappa} \circ dQ^n_{\varkappa} \mid E^{s,(\bar{l},\bar{k})}_{\varkappa,q_1,q_2} = d\Phi^n_{\varkappa} \circ d\tilde{\alpha}_{\varkappa} \circ dS^{-mn}_{\varkappa} \mid E^{s,(\bar{l},\bar{k})}_{\varkappa,q_1,q_2} = d\Phi^n_{\varkappa} \circ d\tilde{\alpha}_{\varkappa} \mid E^{s,(\bar{l},\bar{k})+mn}_{\varkappa,q_1,q_2}$$
$$= d\Phi^n_{\varkappa} \circ d\tilde{\alpha}_{\varkappa} \mid E^{s,(\bar{l},\bar{r})}_{\varkappa,q_1,q_2} = d\Phi^n_{\varkappa} \mid \mathscr{E}^{s,\bar{r}}_{\varkappa,\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}.$$

This immediately implies (21). The inequality (22) can be proved in the same way.

Remark. It is worthwhile to emphasize that, in the multidimensional case, due to our strong assumption about $l_1, \ldots, l_t$ the estimations of damping (21), (22) take place for any $n \geqslant 0$ while in the one-dimensional case they hold, in general, only for $n = ln'$, $n' \in \mathbb{Z}^+$ (cf theorem 9).

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Appendix. Proof of theorem 1.

Denote by $\Lambda_0 = \bigcap_{k \geqslant 0} F^k_0(\tilde{\Lambda}_0)$. First we prove the following statement.

Lemma 1. For any $y \in \Lambda_0$ there exists a sequence of points $y_k \in \Lambda_0$ such that $y_0 = y$, $F_0(y_k) = y_{k+1}$, $k \in \mathbb{Z}$.

Proof. It follows from the definition of Λ_0 that for any $y \in \Lambda_0$ and $k > 0$ there exists a point $y_k \in \tilde{\Lambda}_0$ such that $F^k_0(y_k) = y$. Given $m > 0$ consider a sequence of points $\{y^{(m)}_k = F^{k-m}_0(y_k)\}$. Let $m = 1$. One can find a subsequence $\{y^{(1)}_{k^{(1)}_j}\}$ converging to a point $y^{(-1)}$. It is easy to see that $F_0(y^{(-1)}) = y$. For $m = 2$ there exists a subsequence $k^{(2)}_j$ of the sequence $k^{(1)}_j$ such that $y^{(2)}_{k^{(2)}_j}$ tend to a point $y^{(-2)}$ and $F_0(y^{(2)}_{k^{(2)}_j}) = y^{(1)}_{k^{(2)}_j}$. This gives us that $F_0(y^{(-2)}) = y^{(-1)}$. Continuing this process one can construct a sequence of points $\{y^{(-m)}\}$ such that $F_0(y^{(-m)}) = y^{(-m+1)}$. In particular, this implies that $y^{(-m)} \in \Lambda_0$ for all $m > 0$. Define now $y_k = y^{(-k)}$ if $k < 0$, $y_0 = y$, $y_k = F^k_0(y)$ for $k > 0$.

Let us fix $y \in \Lambda_0$ and consider the sequence of points $\{y_k\}$ constructed in lemma 1. It is easy to see that there exists $r_0 > 0$ such that for any $k \in \mathbb{Z}$ the map F_0 can be represented in the form

$$F_0(\xi, \eta) = (A_k \xi + g^{(1)}_k(\xi, \eta), B_k \eta + g^{(2)}_k(\xi, \eta))$$

where $\xi \in E^s_0(y_k)$, $\eta \in E^u_0(y_k)$, $A_k = dF_0|_{E^s_0(y_k)}$, $B_k = dF_0|_{E^u_0(y_k)}$, $g_k(\xi, \eta) = (g^{(1)}_k(\xi, \eta), g^{(2)}_k(\xi, \eta))$ is well-defined when $\xi, \eta \in B(y_k, r_0)$ and

$$\left\| \prod_{k=1}^{(ls+m)n} A_k \right\| \leqslant C_1 \lambda_1^n \qquad \left\| \left(\prod_{k=1}^{(ls+m)n} B_k \right)^{-1} \right\| \leqslant C_1 \lambda_1^n \qquad g_k(0, = (0,0) \qquad dg_k(0,0) = (0,0).$$

Consider now the map $F_{\varkappa}$ for small enough $\varkappa$.

Lemma 2. There exist $\varkappa_0 > 0$ and $r, 0 < r \leq r_0$ such that for any $|\varkappa| \leq \varkappa_0$ the map $F_\varkappa$ can be written in the above local coordinates as

$$F_\varkappa(\xi, \eta) = (A_k\xi + g^{(1)}_{k,\varkappa}(\xi, \eta), B_k\eta + g^{(2)}_{k,\varkappa}(\xi, \eta))$$

where $\|g_{k,\varkappa} - g_k\|_{C^1} \leq \varepsilon(\varkappa)$ and $\varepsilon(\varkappa) \to 0$ as $\varkappa \to 0$.

If A_k were invertible with $\|A_k^{-1}\|$ uniformly bounded from above over k one could apply the well known approach due to Anosov and Alekseev [17, 18] to construct smooth stable and unstable local $F_\varkappa$-invariant manifolds in $B(y_k, r)$ for each $k \in \mathbb{Z}$ and $\varkappa, r$ to be small enough. The points of their intersection would form $F_\varkappa$-trajectory close to sequence $\{y_k\}$. In our case $\det A_k = 0$. However, it is still possible to construct unstable $F_\varkappa$-invariant local smooth manifolds $V_{\varkappa,k} \subset B(y_k, r)$, $k \in \mathbb{Z}$. Moreover, $V_{\varkappa,k}$ for each k has the following property: there exists a smooth function $\xi_{\varkappa,k}(\eta)$ defined in a ball $B^u(y_k, r_1)$ in the space $E^u_0(y_k)$ of radius $r_1, 0 < r_1 \leq r$ (r_1 does not depend on k) such that

$$V_{\varkappa,k} = \mathrm{Graph}\{\xi_{\varkappa,k}(\eta), \eta \in B^u(y_k, r_1)\}.$$

One can see that $F_\varkappa$ is invertable on $V_{\varkappa,k}, F_\varkappa^{-1}(V_{\varkappa,k}) \subset V_{\varkappa,k-1}$ and the intersection $\bigcap_{l\geq 0} F_\varkappa^{-l}(V_{\varkappa,k+l})$ consists of only one point $z_{\varkappa,k} \in B(y_k, r)$.

Consider the inverse limit (Ω, σ) of the endomorphism (Λ_0, F_0) where Ω consists of all sequences $\{y_k\}$ with $y_k \in \Lambda_0$ and $F_0(y_k) = y_{k+1}$, $k \in \mathbb{Z}$, and σ is the shift to the right: $\sigma(\{y_k\}) = \{y_{k+1}\}$. One can endow Ω with the metric $\rho(\{y'_k\}, \{y''_k\}) = \sum_{k=-\infty}^{\infty} |y'_k - y''_k|/2^{|k|}$.

Lemma 3. (1) There exists a continuous map $h_\varkappa : \Omega \to \tilde{V}$ such that $h_\varkappa \circ \sigma = F_\varkappa \circ h_\varkappa$; $h_\varkappa$ is a homeomorphism for $\varkappa \neq 0$. (2) The set $\Lambda_\varkappa = h_\varkappa(\Omega)$ is $F_\varkappa$-invariant closed locally maximal and hyperbolic.

Proof. (1). It is not difficult to verify that the map $h_\varkappa$ associating to each $\{y_k\} \in \Omega$ the point $z_{\varkappa,0} \in \tilde{V}$ has all the desired properties. In fact, continuity of $h_\varkappa$ can be proved by standard arguments [17]. To prove injectivity let us consider two sequences $\{y'_k\}$, $\{y''_k\}$ such that $y'_i \neq y''_i$ for some $i \in \mathbb{Z}$. If y'_i and y''_i belong to the same unstable leaf $V_{0,i}$ then the distance between $F_0^n(y'_i)$ and $F_0^n(y''_i)$ become big enough so that the corresponding points $z'_{\varkappa,i+n}$ and $z''_{\varkappa,i+n}$ do not coincide.

Assume now that y'_i and y''_i lie on different leaves but $z'_{\varkappa,i} = z''_{\varkappa,i}$. It is obvious that $\rho(y'_{i-p}, z'_{\varkappa,i-p}) \leq$ constant and $\rho(y''_{i-p}, z''_{\varkappa,i-p}) \leq$ constant uniformly over $p \geq 0$. Since $z'_{\varkappa,i-p} = z''_{\varkappa,i-p}$ this implies that $\rho(y'_{i-p}, y''_{i-p}) \leq$ constant. Therefore $\rho(V'_{0,i}, V''_{0,i}) \leq$ constant λ^p, $0 < \lambda < 1$. This means that y'_i and y''_i belong to the same leaf.

(2) First we will prove that Λ_0 is locally maximal. Namely we will show that

$$\Lambda_0 = \bigcap_{k\in\mathbb{Z}} F^k(\tilde{V}).$$

Indeed, if $y \in \bigcap_{k\in\mathbb{Z}} F^k(\tilde{V})$ then for any $j \geq 0$, $p \geq 0$ any point $y^j = (x_1, \ldots, x_{ls+m})$ from the set $F^{-j}(y)$ belongs to $\bigcap_{k\in\mathbb{Z}} F^{(ls+m)(k-p)}(\tilde{V})$. This and the definition of the map F_0 implies that $x_i \in \bigcap_{k\geq 0} F^{k-p}(\tilde{V})$. Since Λ is locally maximal (cf (9)) we can conclude from here that $x_i \in \Lambda$ for all $i = 1, \ldots, ls + m$, so $y_j \in \tilde{\Lambda}_0$. This means that $y \in \Lambda_0$.

Now we will prove that $\Lambda_\varkappa$ is locally maximal. Given $y \in \Lambda_0$ consider $\{y_k\} \in \Omega$, $y_0 = y$. For any $k \in \mathbb{Z}$ one can construct a stable local manifold $V^s(y_k)$. We show now that Λ_0 has a local product structure in the following sense. Take any $y, z \in \Lambda_0$ and any $\{y_k\}$, $\{z_k\} \in \Omega$, $y_0 = y$, $z_0 = z$. Denote $w_k = V^s(y_k) \cap V_{0k}(z_k)$. Since Λ_0 is locally maximal it is easy to see that $\{w_k\} \in \Omega$. In particular $V^s(y) \cap V_{00}(z) \in \Lambda_0$. It follows from the first statement that for small enough $\varkappa \neq 0$ the set $\Lambda_\varkappa$ has also the local product structure in the usual sense and, hence, is locally maximal. □

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