# Traveling waves in lattice models of multidimensional and multicomponent media. II. Ergodic properties and dimension

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We demonstrate a spatio–temporal chaos in lattice models of multidimensional and multicomponent media on the set of traveling waves solutions running with large enough velocities. We describe stability properties of such solutions, construct invariant measures with "good" ergodic properties concentrated on the above set and study different types of dimensions including the correlation dimension.



## I. INTRODUCTION

How can a finite-dimensional dynamics appear in extended systems with dissipation and energy pumping? This problem has recently attracted the attention of physicists and mathematicians. The two cases of bounded and unbounded media should be treated separately. In the case of bounded media the explanation seems to be more or less clear: the linearized monodromy operator has a discrete spectrum and modes with big wave numbers are dumped due to dissipation. In the case of unbounded media the situation is not well understood. One of the possible mechanisms is that in some range of parameters there may exist subsets of solutions described by finite-dimensional dynamical systems. Moreover some of these sets turn out to be stable with respect to small perturbations of initial data. For example, let us consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t}=f(u)+\varkappa A\Delta u,$$

where $u\in\mathbb{R}^d$,

$$\Delta u=\left(\frac{\partial^2 u_1}{\partial x^2},\ldots,\frac{\partial^2 u_d}{\partial x^2}\right),$$

$A$ is the matrix of coupling, $\varkappa$ is the diffusion coefficient, $f:\mathbb{R}^d\to\mathbb{R}^d$ is the nonlinear term. To illustrate the type of nonlinearity let us mention here the following examples:

one-dimensional, one-component media:

$f(u)=u(1-u)$ is the quadratic polynomial—for the Kolmogorov–Petrovsky–Peskunov equation ($a$ is a parameter);

$f(u)=\alpha u(1-u)(u-a)$ is the cubic polynomial—for the Huxley equation ($\alpha,a$ are parameters);

one-dimensional, two-component media:

$f(u,v)=(\alpha\varphi(u)-\alpha v,\beta u-\gamma v)$

for Fitz-Hugh Nagumo equation ($\alpha$, $\beta$, $\gamma$ are parameters, $\varphi$ is a function).

For the diffusion equation the following sets of solutions are known:

(a) static solutions described by the system with $d$ degrees of freedom:

$$\varkappa A\frac{d^2u}{dx^2}+f(u)=0;$$

(b) spatially homogeneous solutions described by the ordinary differential equation

$$\frac{du}{dt}=f(u);$$

(c) traveling waves; they are solutions of the form $u(x,t)=w(\xi)$ where $\xi=x-ct$ is the traveling coordinate ($c$ is the velocity of the wave) and $w(\xi)$ is the solution of the differential equation with dissipation

$$-c\frac{dw}{d\xi}=f(w)+\varkappa A\frac{d^2w}{d\xi^2}.$$

If these sets are stable in the above sense the corresponding finite-dimensional dynamics is said to be "physically realizable." In this case the original infinite-dimensional system can display a very complicated behavior both in time and in space. For example, it is known (cf. Refs. 1–4) that the existence of traveling waves with chaotic profiles can cause the appearance of space–time chaos in the medium. It is due to the perturbations arising in different parts of space in an unpredictable way and oscillating in time like a realization of a random process. The type of chaos that usually occurs for such systems, is, as a rule, not so developed as one observed and studied in Refs. 5 and 6 because it can be found only for special sets of solutions.

Partial differential equations are still very difficult for complete analysis of chaotic behavior. In this paper we consider their space–time discrete versions called lattice models. From the physical viewpoint lattice models can play an independent role and serve as phenomenological models describing the interaction of particlelike localized structures in dissipative nonequilibrium media (cf. Ref. 7). From a mathematical viewpoint they form a broad and interesting class of infinite-dimensional systems displaying nontrivial dynamics which can be studied by applying some methods of the theory of finite-dimensional dynamical systems. In this paper we demonstrate the spatio–

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temporal chaos in lattice models of unbounded media based on the set of traveling waves running with a big velocity.

Let us illustrate the main results of the paper by considering the space–time discrete version of the one-dimensional nonlinear diffusion equation

$$u_j(n+1)=u_j(n)+\varphi(u_j(n))+\varkappa(u_{j-1}(n)-2u_j(n)+u_{j+1}(n))=h(u_j(n))+\varkappa(u_{j-1}(n)-2u_j(n)+u_{j+1}(n)). \quad (1.1)$$

Here $u=\{u_j(n)\}\in(\mathbb{R}^d)^{\mathbb{Z}}$, $d\geqslant 1$ is a characteristic of the medium; $j\in\mathbb{Z}$ is the discrete spatial coordinate; $n\in\mathbb{Z}^+$ is the discrete time; $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ introduces a nonlinearity, $h=I+\varphi$. If $d>1$ this represents the case of one-dimensional $d$-component media. In Sec. II we describe the case of multidimensional media. The equation (1.1) can be solved if a boundary condition at infinity is fixed. We consider such solutions which can grow with prescribed rates, namely, given $q_1>1$, $q_2>1$ the solution $u(n)=(u_j(n))$ should satisfy

$$\|u(n)\|^2_{q_1,q_2}=\sum_{j\geqslant 0}\frac{|u_j(n)|^2}{q_1^j}+\sum_{j<0}\frac{|u_j(n)|^2}{q_2^{-j}}<\infty.$$

The equation (1.1) with this boundary condition generates the infinite-dimensional nonlinear dynamical system $(\Phi_\varkappa,\mathscr{M}_{q_1,q_2})$, where $\mathscr{M}_{q_1,q_2}=\{u=(u_j):\|u\|_{q_1,q_2}<\infty\}$ is the infinite-dimensional Banach space and $\Phi_\varkappa$ is the nonlinear evolution operator given by the right-hand side of Eq. (1.1). The metric $\|\cdot\|_{q_1,q_2}$ is called metric with weights. Such metrics were introduced in Ref. 8 for the lattice models of drift-type systems and turned out to be very useful for studying stability properties. In Ref. 9 they were used to investigate hyperbolic properties of traveling wave solutions in lattice models of unbounded multidimensional and multicomponent media (cf. also below).

Let us emphasize that Eq. (1.1) admits the group of space translations $S^v$ given by $S^v(u)=u+v$, $u, v\in\mathscr{M}_{q_1,q_2}$, i.e., $S^v$ commutes with the evolution operator.

The solutions of Eq. (1.1) in the form $u_j(n)=\psi(lj+mn)$, $l, m\in\mathbb{Z}$ are said to be traveling waves. Here $m/l$ is the velocity of the wave and the function $\psi$ satisfies the "traveling wave equation"

$$\psi(k+m+l)=h(\psi(k+l))+\varkappa(\psi(k)-2\psi(k+l)+\psi(k+2l)), \quad (1.2)$$

where $k=lj+mn-l$ is the "traveling" coordinate. We consider only the case of "large" velocities when $l>0$ and $m\geqslant l+1$. Let us set

$$x_k^{(i)}=\psi(k+i-1),\quad i=1,\dots,l+m.$$

Equation (1.2) is now equivalent to the following system of equations

$$x_{k+1}^{(1)}=x_k^{(2)},\dots,x_{k+1}^{(l+m-1)}=x_k^{(l+m)},$$

$$x_{k+1}^{(l+m)}=h(x_k^{(l+1)})+\varkappa(x_k^{(1)}-2x_k^{(l+1)}+x_k^{(2l+1)}). \quad (1.3)$$

The equations (1.3) introduce a finite-dimensional dynamical system acting on $d(l+m)$-dimensional phase space for which the variable $k$ is the discrete time. In other words we have the map $F_\varkappa:\mathbb{R}^{d(l+m)}\to\mathbb{R}^{d(l+m)}$ such that $F_\varkappa(\{x_k^{(i)}\}_{i=1,\dots,l+m})=\{x_{k+1}^{(i)}\}_{i=1,\dots,l+m}$. This map can be interpreted as a multidimensional version of the famous Henon map (cf. Ref. 10). Any bounded trajectory of this map corresponds to a bounded solution of Eq. (1.2), i.e., to a stationary traveling wave of the equation (1.1).

For small enough $\varkappa$ the hyperbolic properties of the map $F_\varkappa$ depend very much on the hyperbolic properties of the map $h$. For example, it was shown in Ref. 9 that if $h$ has an invariant closed hyperbolic set $\Lambda\subset\mathbb{R}^d$ then $F_\varkappa$ also possesses an invariant closed hyperbolic set $\Lambda_\varkappa$ (cf. definitions below). Moreover, it was proved in Ref. 9 that

(1) the set of traveling wave solutions of Eq. (1.1) forms a finite-dimensional smooth submanifold $\mathscr{A}_\varkappa$ in the infinite-dimensional phase space invariant under both evolution operator and space translations;

(2) $\mathscr{A}_\varkappa$ is stable in a weak sense with respect to the perturbations in directions transversal to it;

(3) there exists a hyperbolic set $\mathscr{L}_\varkappa\subset\mathscr{A}_\varkappa$ which is closed and invariant with respect to both evolution operator and space translations;

(4) the evolution operator is invertible on $\mathscr{A}_\varkappa$; moreover it is a diffeomorphism.

The situation described above can occur for Fitz-Hugh Nagamo equation in some range of parameters $\alpha$, $\beta$, $\gamma$. However, the map $F_\varkappa$ can possess strong hyperbolic properties even when the map $h$ does not. This may happen, for example, when $h$ is a one-dimensional smooth piecewise monotonic map. In general, one can expect the may $F_\varkappa$ to inherit the same (or even better) type of hyperbolic behavior as the map $h$ has. In the other cases when $h$ has a strange attractor (of type of Lorenz or Lozi) $F_\varkappa$ is expected to have a strange attractor too.

In this paper we are interested in chaotic behavior of traveling wave solutions of Eq. (1.1). It is worthwhile to emphasize that in lattice models we deal with two types of dynamical systems: time evolution and space translations. The latter is a dynamical system with multidimensional time, or an action of the lattice $\mathbb{Z}^t$, $t\geqslant 1$. The chaotic regime is irregular in time with respect to the evolution operator while the space translations describe a spatial distribution of chaotic patterns.

The type of chaotic behavior of the evolution operator $\Phi_\varkappa$ on the set of traveling wave solutions $\mathscr{A}_\varkappa$ is mainly determined by the map $F_\varkappa$. Namely, we shall show that any finite measure $\mu$ invariant under $F_\varkappa$ induces a measure $\mu_{q_1,q_2}$ on $\mathscr{A}_\varkappa$ invariant under $\Phi_\varkappa$. This measure is also invariant under space translations. Moreover, if $\mu$ is mixing, the measure $\mu_{q_1,q_2}$ is also mixing with respect to both the evolution operator and space translations. According to Refs. 5 and 6 one can say that the system (1.1) displays a space–time chaos on the set of traveling wave solutions (with respect to the measure $\mu_{q_1,q_2}$). We consider also some special type of invariant measures on $\mathscr{L}_\varkappa$ playing an important role in applications.

In general, $\mathscr{L}_\varkappa$ is a hyperbolic set. There is a well-developed ergodic theory for the map $\Phi_\varkappa|\mathscr{L}_\varkappa$. Namely, using results in Ref. 11 for any Hölder continuous function $\varphi$ on $\mathscr{L}_\varkappa$ we shall construct an ergodic measure $\mu_{\varkappa,\varphi}$ (called Gibbs measure) on $\mathscr{L}_\varkappa$ invariant under both evolution operator and space translation; it also has many "good" ergodic properties.

We also consider the special case when $\mathscr{L}_\varkappa$ is a hyperbolic attractor. This means that there is an open subset $\mathscr{B}_\varkappa \subset \mathscr{A}_\varkappa$ such that $\Phi_\varkappa(\mathscr{B}_\varkappa) \subset \mathscr{B}_\varkappa$ and

$$\mathscr{L}_\varkappa = \bigcap_{n\geqslant 0} \Phi_\varkappa^n(\mathscr{B}_\varkappa).$$

We shall construct the Bowen–Ruelle–Sinai measure $\mu_\varkappa$ concentrated on $\mathscr{L}_\varkappa$ and invariant under both evolution operator and space translations. This measure plays a special role from physical point of view: It is the limit distribution for the evolution of a smooth initial distribution given in $\mathscr{B}_\varkappa$. This measure has some "nice" ergodic properties: Under certain assumptions on the nonlinear term $h$ it is mixing with respect to both time evolution and space translations.

Another characteristic of chaotic behavior which has recently become of a great interest is the fractal dimension of the set $\mathscr{L}_\varkappa$. There are many different characteristics of dimension type that are used to describe the fractal structure of $\mathscr{L}_\varkappa$. Among them there are Hausdorff dimension, information dimension, correlation dimension, etc. The most general way to introduce many of them is given by the generalized spectrum for dimension, i.e., a one-parameter family of characteristics of dimension type (cf. Ref. 12). For the appropriate value of the parameter one can obtain the correlation dimension—one of the most popular characteristic of dimension type which is often used for numerical calculation of fractal dimension, cf. Ref. 13. We apply results from this paper to our case and study the correlation dimension as well as other characteristics of dimension type with respect to the evolution operator and an invariant ergodic measure. Our main result is that all these characteristics coincide; thus the common value represents the fractal dimension. We also consider the same problem concerning the correlation dimension and other characteristics of dimension type with respect to the group of space translations. The first but sufficiently restricted result in this direction was established in Ref. 14. We give the full solution of this problem. We show that all the characteristics of dimension type calculated with respect to an invariant ergodic measure coincide and the common value is the same as the one calculated with respect to the evolution operator. This property can be considered the main feature of the space–time chaos. It is worthwhile to emphasize that the dimensionlike characteristics of the measure specified by the evolution operator or space translations are completely determined by the corresponding dimensionlike characteristics of an appropriate measure specified by the map $F_\varkappa$. Thus this map handles all the important features of the chaotic behavior of traveling wave solutions of the evolution equation (related to ergodicity and dimension). This reflects the finite-dimensional nature of space–time chaos in the set of traveling waves in lattice models. The study of ergodic properties and dimension of the map $F_\varkappa$ is a "pure finite-dimensional" problem and in many "structurally stable" cases is determined by the corresponding properties of the nonlinear map $h$.

## II. BASIC RESULTS

We start from a brief summary of the main results in Ref. 9 describing hyperbolic properties of traveling wave solutions.

Given $\bar{q}_1=(q_{1i})$, $\bar{q}_2=(q_{2i})$, $q_{1i}>1$, $q_{2i}>1$, $i=1,\ldots,t$ let us consider the set

$$\mathscr{M}_{\bar{q}_1,\bar{q}_2}=\Big\{u:\mathbb{Z}^t\to\mathbb{R}^d: u=(u(\bar{j})_{\bar{j}\in\mathbb{Z}^t}),$$

$$\|u\|^2_{\bar{q}_1,\bar{q}_2}=\sum_{j_1}\cdots\sum_{j_t}\frac{\|u(j_1,\ldots,j_t)\|^2}{q(j_1)\cdot\;\cdots\;\cdot q(j_t)}\Big\},$$

where $q(j_i)=q_{1i}^{j_i}$ if $j_i\geqslant 0$ and $q(j_i)=q_{2i}^{-j_i}$ if $j_i<0$ and $\|\cdot\|$ denotes some norm in $\mathbb{R}^d$.

It is easy to see that $\mathscr{M}_{\bar{q}_1,\bar{q}_2}$ is a Banach space with the norm $\|\cdot\|_{\bar{q}_1,\bar{q}_2}$.

We work with different values $q_{j_i}$, $j=1, 2$, $i=1,\ldots,t$ because the growth of perturbations along a traveling wave has different rates in different directions.

Let $f:\mathbb{R}^d\to\mathbb{R}^d$ be a diffeomorphism of class $C^2$, $g:(\mathbb{R}^d)^{(2s+1)^t}\to\mathbb{R}^d$ a map of class $C^2(s\geqslant 0)$ and $\varkappa>0$ a real number. Define the map $\Phi_\varkappa:\mathscr{M}_{\bar{q}_1,\bar{q}_2}\to(\mathbb{R}^d)^{\mathbb{Z}^t}$ by the formula

$$\Phi_\varkappa(u)=(f(u(\bar{j}))+\varkappa g(\{u(i)\}_{|i-j|\leqslant s})),$$

where $u=(u(\bar{j}))$ and

$$|\bar{j}|=\sum_{i=1}^{t}|j_i|.$$

The map $\Phi_\varkappa$ determines an infinite-dimensional dynamical system which is the interaction of partial finite-dimensional dynamical systems given by $f$. We suppose that the interaction involves only a finite number of neighbors, that corresponds to the models of dissipative media with a finite size of interaction.

*Proposition 1: (Ref. 9)* Assume that there exists $M>0$ such that for $l=1, 2$

$$\sup_{x\in\mathbb{R}^d}\|d^l f_x\|\leqslant M,\left\|\frac{\partial^l g}{\partial u(l/i)}(\{u(\bar{j})\})\right\|\leqslant M \tag{2.1}$$

for any $\bar{i}$, $|\bar{i}|\leqslant s$ and any point $u(\bar{j})\in\mathbb{R}^{(2s+1)^t}$.

Then for any $\bar{q}_1=(q_{1i})$, $\bar{q}_2=(q_{2i})$, $q_{1i}>1$, $q_{2i}>1$, $i=1,\ldots,t$

(1) $\Phi_\varkappa$ is a map from $\mathscr{M}_{\bar{q}_1,\bar{q}_2}$ into itself;

(2) $\Phi_\varkappa$ is differentiable in the sense of Gâteaux and its Gâteaux differential at a point $u=(u(\bar{j}))$ is given by the linear map

$$(d\Phi_{\varkappa}\eta)(\bar{j})=f'(u(\bar{j}))\eta(\bar{j})+\sum_{|\bar{i}-\bar{j}|\leqslant s}a(\bar{i})\eta(\bar{i}), \qquad (2.2)$$

where

$$a(\bar{i})=\frac{\partial g}{\partial u_{\bar{i}}}(\{u(\bar{i})\}_{|\bar{i}-\bar{j}|\leqslant s}).$$

For any $u\in\mathscr{M}_{\bar{q}_1,\bar{q}_2}$ denote by $u(\bar{j},n)=(\Phi_{\varkappa}^{n}u)(\bar{j})$. It satisfies

$$u(\bar{j},n+1)=f(u(\bar{j},n))+\varkappa g(\{u(\bar{i},n)\}_{|\bar{i}-\bar{j}|\leqslant s}), \qquad (2.3)$$

$n\geqslant 0,\ u(\bar{j},0)=u(\bar{j})$.

Let us fix $m,\ l_1,...,l_t\in\mathbb{Z}^t (t\geqslant 2)$ such that

(1) $m\geqslant s\Sigma_{i=1}^{t} l_i+1$;

(2) the numbers $l_1,...,l_t$ have the common divisor 1; in particular, the equation $l_1x_1+\cdots+l_tx_t=k$ has a solution in the set of integers for any $k\in\mathbb{Z}$.

In the case of one-dimensional media ($t=1$) instead of the condition 2 we shall assume the following weaker condition:

(2′) the numbers $l,\ m$ are relatively prime; in particular, the equation $lx_1+mx_2=k$ has a solution in the set of integers for any $k\in\mathbb{Z}$.

The traveling wave is a solution of Eq. (2.3) in the form

$$u(\bar{j},n)=\psi((\bar{l},\bar{j})+mn)=\psi(k-m),$$

where $(\bar{l},\bar{j})=\Sigma_{i=1}^{t} l_ij_i,\ k=(\bar{l},\bar{j})+mn+m$ and $\psi:\mathbb{Z}\to\mathbb{R}^d$ is a function satisfying the "traveling wave equation"

$$\psi(k)=f(\psi(k-m))+\varkappa g(\{\psi(k-m+(\bar{l},\bar{i}))\}_{|\bar{i}|\leqslant s}). \qquad (2.4)$$

It is worthwhile to emphasize that the above assumption on the numbers $m_1,\ l_1,...,l_t$ represents the fact that the "velocity" of the traveling wave $m/\Sigma_{i=1}^{t} l_i$ is sufficiently large.

First we describe hyperbolic properties of Eq. (2.4). The function $\psi$ is uniquely defined by Eq. (2.4) if we know the values $x_p=\psi(-m-ls+p-1),\ p=1,...,\ ls+m$, where $l=\Sigma_{i=1}^{t} l_i$. Let us introduce the "traveling wave" map

$$F_{\varkappa}:(\mathbb{R}^d)^{ls+m}\to(\mathbb{R}^d)^{ls+m},$$

$$F_{\varkappa}(x_1,...,x_{ls+m})=(\bar{x}_1,...,\bar{x}_{ls+m}),$$

where

$$\bar{x}_1=x_2,\quad \bar{x}_2=x_3,...,\bar{x}_{ls+m-1}=x_{ls+m},$$

$$\bar{x}_{ls+m}=f(x_{ls+1})+\varkappa g(\{x_{p(\bar{i})}\}_{|\bar{i}|\leqslant s}),$$

$$p(\bar{i})=ls+(\bar{l},\bar{i})+1. \qquad (2.5)$$

Below we shall show that although the evolution operator $\Phi_{\varkappa}$ is infinite-dimensional the problem of study of traveling wave solutions of Eq. (2.3) can be reduced to the investigation of the finite-dimensional dynamical system (2.5).

We assume that for any $x=(x_i)\in(\mathbb{R}^d)^{(2s+1)^t}$ the function $g$ satisfies the following conditions

$$\det\frac{\partial g(x)}{\partial x_1}\neq 0,\quad \left|\left(\frac{\partial g(x)}{\partial x_1}\right)^{-1}\right|\leqslant \text{const}<\infty. \qquad (2.6)$$

These assumptions hold in many "practically interesting" cases, for example when $g$ is a "diffusionlike" potential.

Our main assumption is that $f$ possesses a hyperbolic closed invariant set $\Lambda$ (definition, cf. Ref. 11). We describe now the hyperbolic properties of the map $F_{\varkappa}$ given by Eq. (2.5) for small enough $\varkappa$.

*Proposition 2: (Ref. 9)* There exists $\varkappa_0>0$ such that for any $\varkappa,\ |\varkappa|\leqslant\varkappa_0,\ \varkappa\neq 0$ the map $F_{\varkappa}$ possesses a hyperbolic closed invariant set

$$\Lambda_{\varkappa}\subset\tilde{U}\overset{\text{def}}{=}\bigoplus_{i=1}^{ls+m}U_i,U_i\equiv U.$$

Denote by $V_{\varkappa}^{s}(x),\ V_{\varkappa}^{u}(x)$ the local stable and unstable smooth manifolds passing through a point $x\in\Lambda_{\varkappa}$.

Denote also by $\Psi_{\varkappa}$ the set of all solutions of Eq. (2.4) and by $\Psi_{\varkappa,q_1,q_2}\ q_1>1,\ q_2>1$ the set of those of them for which

$$\|\psi\|_{q_1,q_2}^2=\sum_{j\geqslant 0}\frac{\|\psi(j)\|^2}{q_1^j}+\sum_{j<0}\frac{\|\psi(j)\|^2}{q_2^{-j}}<\infty,\psi\in\Psi_{\varkappa}.$$

Let us emphasize that all bounded solutions of Eq. (2.4), of course, belong to $\Psi_{\varkappa,q_1,q_2}$. Define the mapping $\chi_{\varkappa}:(\mathbb{R}^d)^{ls+m}\to\Psi_{\varkappa}$ in the following way:

$$\chi_{\varkappa}(x)(k)=\begin{cases}(x)_k, & \text{if}\quad -ls-m\leqslant k\leqslant -1\\ (F_{\varkappa}^{k+1}(x))_{ls+m}, & \text{if}\quad k\geqslant 0\\ (F_{\varkappa}^{k+ls+m}(x))_1, & \text{if}\quad k\leqslant -ls-m-1\end{cases}.$$

We also consider the shift $S_{\varkappa}:\Psi_{\varkappa}\to\Psi_{\varkappa},\ (S_{\varkappa}\psi)(k)=\psi(k+1),\ \psi\in\Psi_{\varkappa}$. This map describes dynamics of the system along traveling coordinate.

*Proposition 3: (Ref. 9)* Assume that $f$ and $g$ satisfy conditions (2.1) and (2.6). There exist $\varkappa_1>0$ with the following property: if $|\varkappa|\leqslant\varkappa_1,\ \varkappa\neq 0$ then one can find $q_1^{(0)}>1$, $q_2^{(0)}(\varkappa)>1$ such that for any $q_1\geqslant q_1^{(0)},\ q_2\geqslant q_2^{(0)}(\varkappa)$

(1) $\chi_{\varkappa}$ is a smooth imbedding of $(\mathbb{R}^d)^{ls+m}$ into $\mathscr{M}_{q_1,q_2}$;

(2) $\Psi_{\varkappa,q_1,q_2}=\chi_{\varkappa}(\mathbb{R}^d)^{ls+m}$ is $d(ls+m)$-dimensional smooth submanifold in $\mathscr{M}_{q_1,q_2}$;

(3) the diagram

$$\begin{array}{ccc}\mathbb{R}^{d(ls+m)} & \xrightarrow{\chi_{\varkappa}} & \Psi_{\varkappa,q_1,q_2}\subset\mathscr{M}_{q_1,q_2}\\ F_{\varkappa}\downarrow & & \downarrow S_{\varkappa}\\ \mathbb{R}^{d(ls+m)} & \xrightarrow{\chi_{\varkappa}} & \Psi_{\varkappa,q_1,q_2}\subset\mathscr{M}_{q_1,q_2}\end{array}$$

is commutative;

(4) the set $\Lambda_{\varkappa,q_1,q_2}=\chi_{\varkappa}(\Lambda_{\varkappa})\subset\Psi_{\varkappa,q_1,q_2}$ is a hyperbolic closed invariant set for $S_{\varkappa}$.

(5) the sets $V_{\varkappa,q_1,q_2}^{s}(\psi)=\chi_{\varkappa}(V_{\varkappa}^{s}(x))$ and $V_{\varkappa,q_1,q_2}^{u}(\psi)=\chi_{\varkappa}(V_{\varkappa}^{u}(x)),\ \psi=\chi_{\varkappa}(x)$ are local stable and unstable smooth manifolds passing through $\psi$.

(6) $\chi_{\varkappa}\in C^1,\ \chi_{\varkappa}^{-1}|\Psi_{\varkappa,q_1,q_2}\in C^1$.

The above statements allow us to transfer the entire hyperbolic picture for the generalized Henon-type map (2.3) to the space $\mathscr{M}_{q_1,q_2}$ containing traveling wave solutions moving with the same wave vector. The next step is to display this picture in the phase space of the original system.

We now return to Eq. (2.3) and describe hyperbolic properties of its solutions in the form of traveling waves. From now on we assume that $|\varkappa|\leqslant\varkappa_1$ and $\varkappa\neq 0$. Given $\psi\in\Psi_\varkappa$ define

$$\alpha_\varkappa(\psi)=(u(\psi)(\bar{j}))=(\psi(\bar{l},\bar{j})).$$

*Proposition 4: Ref. 9* (1) If $\psi\in\mathscr{M}_{q_1,q_2}$ for some $q_1>1$, $q_2>1$ then $\alpha_\varkappa(\psi)\in\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ where $\bar{q}_1(\bar{l}) = (q_1^{l_i})$, $\bar{q}_2(\bar{l}) = (q_2^{l_i})$, $i=1,...,t$.

(2) Assume that $f$, $g$ satisfy conditions (2.1) and (2.6) and $q_1\geqslant q_1^{(0)}$, $q_2\geqslant q_2^{(0)}(\varkappa)$ (cf. Proposition 3). Then

(a) $\alpha_\varkappa|\Psi_{\varkappa,q_1,q_2}$ is a smooth imbedding into $\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$; the set $\mathscr{A}_{\varkappa,q_1q_2} = \alpha_\varkappa(\Psi_{\varkappa,q_1,q_2})$ is a smooth $d(ls+m)$ dimensional submanifold in $\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$;

(b) $\mathscr{L}_{\varkappa,q_1,q_2} \overset{\text{def}}{=} \alpha_\varkappa(\Lambda_{\varkappa,q_1,q_2}) \subset \mathscr{A}_{\varkappa,q_1,q_2}$ is a hyperbolic closed invariant set for $\Phi_\varkappa$ on $\mathscr{A}_{\varkappa,q_1q_2}$;

(c) the sets $\mathscr{V}^s_{\varkappa,q_1,q_2}(u) = \alpha_\varkappa(V^s_{\varkappa,q_1,q_2}(\psi))$ and $\mathscr{V}^u_{\varkappa,q_1,q_2}(u) = \alpha_\varkappa(V^u_{\varkappa,q_1,q_2}(\psi))$, $u=\alpha_\varkappa(\psi)$ are local stable and unstable smooth manifolds passing through $u$;

(3) $\alpha_\varkappa\circ S^m_\varkappa=\Phi_\varkappa\circ\alpha_\varkappa$.

(4) $\alpha_\varkappa\in C^1$, $\alpha_\varkappa^{-1}|\mathscr{A}_{\varkappa,q_1,q_2}\in C^1$.

The above statements show that in the infinite-dimensional phase space $\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$ of the original system there exists a smooth $d(ls+m)$-dimensional submanifold which inherits all the hyperbolic properties of the Henon-type map $F_\varkappa$. In Ref. 9 we also proved that this submanifold is stable in a weak sense with respect to perturbations lying in an infinite-dimensional subspace "transversal" to it. This subspace is everywhere dense in the phase space. This picture and Proposition 4 exhibit the hyperbolic structure of traveling wave solutions of Eq. (2.3) in $\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$.

The lattice $\mathbb{Z}^t$ acts on $\mathscr{M}_{\bar{q}_1,\bar{q}_2}$ by translations $S^{\bar{i}}$, $\bar{i}\in\mathbb{Z}^t$ such that $(S^{\bar{i}}u)(\bar{j})=u(\bar{i}+\bar{j})$. This action commutes with the evolution operator $\Phi_\varkappa$, i.e., $S^{\bar{i}}\circ\Phi_\varkappa=\Phi_\varkappa\circ S^{\bar{i}}$.

**THEOREM 1**: The sets $\mathscr{A}_{\varkappa,q_1,q_2}$ and $\mathscr{L}_{\varkappa,q_1,q_2}$ as well as local stable and unstable manifolds at points $u\in\mathscr{L}_{\varkappa,q_1,q_2}$ are invariant under the $\mathbb{Z}^t$ action.

*Proof:* First of all we shall check that

$$\alpha_\varkappa\circ S_\varkappa^{(\bar{l},\bar{i})}=S^{\bar{i}}\circ\alpha_\varkappa. \tag{2.7}$$

In fact, let $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ and $\psi = \alpha_\varkappa^{-1}(u)\in\Psi_{\varkappa,q_1,q_2}$. We have that $u(\bar{j})=\psi((\bar{l},\bar{j}))$ for any $\bar{j}\in\mathbb{Z}^t$. This gives us that

$$(\alpha_\varkappa\circ S_\varkappa^{(\bar{l},\bar{i})}\psi)(\bar{j})=S_\varkappa^{(\bar{l},\bar{i})}\psi((\bar{l},\bar{j}))=\psi((\bar{l},\bar{j})+(\bar{l},\bar{i})).$$

On the other hand

$$S^{\bar{i}}(\alpha_\varkappa\psi)(\bar{j})=\alpha_\varkappa\psi(\bar{j}+\bar{i})=\psi((\bar{l},(\bar{j}+\bar{i})))$$
$$=\psi((\bar{l},\bar{j})+(\bar{l},\bar{i})).$$

This implies Eq. (2.7). Now the desired results follows immediately from Eq. (2.7). □

## III. ERGODIC PROPERTIES

From now on we assume that $q_1\geqslant q_1^{(0)}$, $q_2\geqslant q_2^{(0)}(\varkappa)$.

We start from a finite measure $\mu$ in $\mathbb{R}^{d(ls+m)}$ invariant under $F_\varkappa$. Consider the measure $\tilde{\mu}_{q_1,q_2} = (\chi_\varkappa)_*\mu$. By virtue of Proposition 3 it is concentrated on $\Psi_{\varkappa,q_1,q_2}$ and invariant under $S_\varkappa$. Consider now the measure $\mu_{q_1,q_2} =(\alpha_\varkappa)_*\tilde{\mu}_{q_1,q_2}$.

**THEOREM 2**: (1) $\mu_{q_1,q_2}$ is concentrated on $\mathscr{A}_{\varkappa,q_1,q_2}$ and invariant under $\Phi_\varkappa$.

(2) If the measure $\mu$ is weakly mixing then $\mu_{q_1,q_2}$ is ergodic; if $\mu$ is mixing then $\mu_{q_1,q_2}$ is the same.

*Proof:* The first statement follows directly from Proposition 4. If $\mu$ is weakly mixing or mixing then $\tilde{\mu}_{q_1,q_2}$ is the same. That is why the second statement is also the consequence of Proposition 4 (let us recall that if $\tilde{\mu}_{q_1,q_2}$ is weakly mixing then the map $S^m_\varkappa$ is ergodic with respect to $\tilde{\mu}_{q_1,q_2}$). □

We now describe ergodic properties of $\mu_{q_1,q_2}$ with respect to the $\mathbb{Z}^t$ action.

**THEOREM 3**: (1) $\mu_{q_1,q_2}$ is invariant under the $\mathbb{Z}^t$ action.

(2) In the case of multidimensional media ($t\geqslant 2$) if the measure $\mu$ is ergodic or weakly mixing or mixing then $\mu_{q_1,q_2}$ is the same; in the case of one-dimensional media ($t=1$) if $\mu$ is weakly mixing then $\mu_{q_1,q_2}$ is ergodic; if $\mu$ is mixing then $\mu_{q_1,q_2}$ is also mixing.

*Proof:* Since the measure $\tilde{\mu}_{q_1,q_2}$ is invariant under $S^k_\varkappa$ for any $k$ the first statement follows from Eq. (2.7). Consider now the case $t\geqslant 2$ and assume that $\mu$ is ergodic and, hence, $\tilde{\mu}_{q_1,q_2}$ is the same. Let $A \subset \mathscr{A}_{\varkappa,q_1,q_2}$ be invariant under the $\mathbb{Z}^t$ action. There exists $\bar{i}\in\mathbb{Z}^t$ such that $(\bar{l},\bar{i})=1$. By virtue of Eq. (2.7) this means that $(\alpha_\varkappa^{-1})(A)$ is invariant under $S_\varkappa$ and, consequently, its $\tilde{\mu}_{q_1,q_2}$ measure is 1 or 0. Therefore $\mu_{q_1,q_2}(A) = 1$ or 0 and $\mu_{q_1,q_2}$ is ergodic. The other statements follow from Eq. (2.7). Let us consider the case $t=1$. If $\mu$ is weakly mixing then $\tilde{\mu}_{q_1,q_2}$ is also weakly mixing and, consequently, $S^k_\varkappa$ is ergodic for any $k$ with respect to $\tilde{\mu}_{q_1,q_2}$. By virtue of Eq. (2.7) this implies that $\mu_{q_1,q_2}$ is ergodic. The last statement is obvious. □

*Remark:* Theorems 2 and 3 express the fact that according to Refs. 5 and 6 if the Henon-type map $F_\varkappa$ admits a measure $\mu$ which is mixing then the measure $\mu_{q_1,q_2}$ displays a space–time chaos on the set of traveling wave solutions $\mathscr{A}_{\varkappa,q_1,q_2}$.

We now describe some special class of measures in $\mathscr{A}_{\varkappa,q_1,q_2}$ concentrated on $\mathscr{L}_{\varkappa,q_1,q_2}$. Given Hölder continuous function $\varphi$ on $\Lambda_\varkappa$ there exists the uniquely defined measure $\mu_{\varkappa,\varphi}$ invariant under $F_\varkappa$ whose ergodic properties are described as follows.

*Proposition 6: (cf. Ref. 11)* (1) There exists the splitting $\Lambda_\varkappa=\cup_{i=1}^{p}\Lambda_{\varkappa,i}$ into disjoint invariant sets $\Lambda_{\varkappa,i}$, $i=1,...,p$ such that $F_\varkappa|\Lambda_{\varkappa,i}$ is ergodic with respect to $\mu_{\varkappa,\varphi}$.

(2) For any $i=1,...,p$ there exists the splitting $\Lambda_{\varkappa,i} =\cup_{j=1}^{k}\Lambda^j_{\varkappa,i}$ into disjoint invariant sets $\Lambda^j_{\varkappa,i}$, $j=1,...,k_i$ such that

(a) $F_\varkappa(\Lambda_{\varkappa,i}^j)=\Lambda_{\varkappa,i}^{j+1}$, for $j=1,\dots,k_i-1$ and $F_\varkappa(\Lambda_{\varkappa,i}^{k_i-1}) = \Lambda_{\varkappa,i}^1$,

(b) $F_\varkappa^{k_i}|\Lambda_{\varkappa,i}^1$ is isomorphic to a Bernoulli automorphism with respect to $\mu_{\varkappa,\varphi}|\Lambda_{\varkappa,i}^1$.

(3) If $F_\varkappa|\Lambda_\varkappa$ is topologically transitive then $\mu_{\varkappa,\varphi}$ is ergodic; if $F_\varkappa|\Lambda_\varkappa$ is topologically mixing then $\mu_{\varkappa,\varphi}$ is mixing (and, in fact, has the Bernoulli property; cf. definitions of topological transitivity and mixing in Ref. 11).

(4) $\mu_{\varkappa,\varphi}$ satisfies the variational principle, i.e., it is the only one measure for which

$$\sup_\mu\left(h_\mu(F_\varkappa)+\int\varphi\,d\mu\right)=h_{\mu_{\varkappa,\varphi}}(F_\varkappa)+\int\varphi\,d\mu_{\varkappa,\varphi},$$

where $h_\mu(F_\varkappa)$ is the Kolmogorov–Sinai metric entropy of the map $F_\varkappa$ and sup is taken over all invariant measures.

The measure $\mu_{\varkappa,\varphi}$ is called the equilibrium state (or the Gibbs measure) corresponding to the function $\varphi$.

Let us consider the measure $\mu_{q_1,q_2,\varphi} = (\chi_\varkappa)_*\mu_{\varkappa,\varphi}$ on $\Lambda_{\varkappa,q_1,q_2}$ and $\mu_{q_1,q_2,\varphi} = (\alpha_\varkappa)_*\tilde{\mu}_{q_1,q_2,\varphi}$ on $\mathscr{L}_{\varkappa,q_1,q_2}$. The following result can be derived from Propositions 3, 4, and 6 and Theorems 2 and 3.

**THEOREM 4**: The measure $\tilde{\mu}_{q_1,q_2,\varphi}$ is the equilibrium state corresponding to the Hölder continuous function $(\chi_\varkappa)_*\varphi$ on $\mathscr{L}_{\varkappa,q_1,q_2}$ (with respect to $S_\varkappa$). The measure $\mu_{q_1,q_2,\varphi}$ is the equilibrium state corresponding to the Hölder continuous function $(\alpha_\varkappa)_*(\chi_\varkappa)_*\varphi$ (with respect to $\Phi_\varkappa$). These measures possess properties listed in Proposition 6 (with obvious replacement of $F_\varkappa$, $\Lambda_\varkappa$ by $S_\varkappa$, $\Lambda_{\varkappa,q_1,q_2}$ and $\Phi_\varkappa$, $\mathscr{L}_{\varkappa,q_1,q_2}$, respectively). The measure $\mu_{q_1,q_2,\varphi}$ also invariant under the space translations $S^{\bar{i}}$, $\bar{i}\in\mathbb{Z}^l$ with ergodic properties stated in Theorem 3.

Let us consider the special case when $\Lambda_\varkappa$ is a hyperbolic attractor for $F_\varkappa$. This means that there exist a bounded open domain $\tilde{U}$ in $\mathbb{R}^{d(ls+m)}$ such that

$$\overline{F_\varkappa(\tilde{U})}\subset\tilde{U},\quad \Lambda_\varkappa=\bigcap_{n>0}F_\varkappa^n(\tilde{U}).$$

We have this situation if $\Lambda$ is a hyperbolic attractor for $f$. It is easy to see that $\Lambda_{\varkappa,q_1,q_2}$ and $\mathscr{L}_{\varkappa,q_1,q_2}$ are hyperbolic attractors, respectively, for $S_\varkappa$ (in $\Psi_{\varkappa,q_1,q_2}$) and $\Phi_\varkappa$ (in $\mathscr{A}_{\varkappa,q_1,q_2}$). Denote by "mes" the Lebesque measure in $(\mathbb{R}^d)^{ls+m}$ and set $\underset{\sim}{\nu}=[1/\text{mes}(\tilde{U})]\text{mes}$. It is a normalized Borel measure on $\tilde{U}$. Consider its evolution under $F_\varkappa$, i.e., the sequence of measures

$$\nu_n=\frac{1}{n}\sum_{k=0}^{n-1}(F_\varkappa)_*^k\nu.$$

It is known that it converges to the Bowen–Ruelle–Sinai measure $\mu_\varkappa$ concentrated on $\Lambda_\varkappa$ and invariant under $F_\varkappa$. This measure is also the equilibrium state corresponding to the function

$$\varphi(x)=\log\text{Jac}(dF_\varkappa|T_xV^u(x)),\quad x\in\Lambda_\varkappa.$$

Thus its ergodic properties are described in Proposition 6.

Let us define the measure $\tilde{\nu}_{q_1,q_2} = (\chi_\varkappa)_*\nu$. It is a smooth measure on $\Psi_{\varkappa,q_1,q_2}\subset\mathscr{M}_{q_1,q_2}$ whose evolution under $S_\varkappa$ is given by the sequence of measures

$$\tilde{\nu}_{q_1,q_2,n}=\frac{1}{n}\sum_{k=0}^{n-1}(S_\varkappa)_*^k\tilde{\nu}_{q_1,q_2}=(\chi_\varkappa)_*\nu_n.$$

It converges to the Bowen–Ruelle–Sinai measure $\tilde{\mu}_{q_1,q_2} =(\chi_\varkappa)_*\mu_\varkappa$.

The last step is to consider the measure $\nu_{q_1,q_2} = (\alpha_\varkappa)_*\tilde{\nu}_{q_1,q_2}$. It is a smooth measure on $\mathscr{A}_{\varkappa,q_1,q_2}\subset\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$. Its evolution under $\Phi_\varkappa$ is the sequence of measures

$$\nu_{q_1,q_2,n}=\frac{1}{n}\sum_{k=0}^{n-1}(\Phi_\varkappa)_*^k\nu_{q_1,q_2}.$$

One can prove that it converges to the Bowen–Ruelle–Sinai measure $\mu_{q_1,q_2} = (\alpha_\varkappa)_*\tilde{\mu}_{q_1,q_2}$. The ergodic properties of the measures $\tilde{\mu}_{q_1,q_2}$ and $\mu_{q_1,q_2}$ are described in Theorem 3.

*Remark:* For arbitrary $q_1>1$, $q_2>1$ (which not necessarily satisfy our assumption to be large enough) one can still construct the mappings $\chi_\varkappa$ and $\alpha_\varkappa$. In general, $\chi_\varkappa$ is only continuous such that $\Lambda_{\varkappa,q_1,q_2}$ and $\mathscr{L}_{\varkappa,q_1,q_2}$ are topologically hyperbolic closed invariant sets (cf. details in Ref. 9). However, an invariant measure $\mu$ on $\Lambda_\varkappa$ can be still transformed to $\Lambda_{\varkappa,q_1,q_2}$ and $\mathscr{L}_{\varkappa,q_1,q_2}$ to produce the measures $\tilde{\mu}_{q_1,q_2}$ and $\mu_{q_1,q_2}$, respectively. If $\mu$ is mixing then $\tilde{\mu}_{q_1,q_2}$ and $\mu_{q_1,q_2}$ are the same. Moreover $\mu_{q_1,q_2}$ is also invariant under the $\mathbb{Z}^l$ action and mixing. So it can still display the space–time chaos. However, the sets $\Psi_{\varkappa,q_1,q_2}$ and $\mathscr{A}_{\varkappa,q_1,q_2}$ are no longer manifolds (either smooth or topological). If $\Lambda_\varkappa$ is a hyperbolic attractor then $\Lambda_{\varkappa,q_1,q_2}$ and $\mathscr{L}_{\varkappa,q_1,q_2}$ are attracting sets and the above measures are limit distributions for the evolution of initial measures concentrated on the sets $\Psi_{\varkappa,q_1,q_2}$ and $\mathscr{A}_{\varkappa,q_1,q_2}$, respectively.

When $q_1$ becomes bigger than $q_1^{(0)}$ (but $1<q_2<q_2^{(0)}(\varkappa)$) the first transition occurs: one can show that the mapping $\chi_\varkappa$ is differentiable along $V^u_{\varkappa,q_1,q_2}(\psi)$. Since $\alpha_\varkappa$ is also smooth this implies the existence of local unstable manifold $\mathscr{V}^u_{\varkappa,q_1,q_2}(u)$ on any point $u\in\mathscr{L}_{\varkappa,q_1,q_2}$. Moreover, the measure $\mu_{q_1,q_2}$ inherits the following property of Bowen–Ruelle–Sinai measures: its conditional measure on unstable local manifolds are equivalent to the Lebesque measure. When $q_2$ becomes bigger than $q_2^{(0)}(\varkappa)$ the second transition takes place: the mapping $\chi_\varkappa$ is differentiable at any point $x\in\mathbb{R}^{d(ls+m)}$. Therefore $\Psi_{\varkappa,q_1,q_2}$ and $\mathscr{A}_{\varkappa,q_1,q_2}$ become finite-dimensional smooth submanifolds in $\mathscr{M}_{\varkappa,q_1,q_2}$ and $\mathscr{M}_{\bar{q}_1(\bar{l}),\bar{q}_2(\bar{l})}$, respectively, so that the above results hold.

## IV. DIMENSION

Following Refs. 13 and 15 we define the correlation dimension of a measure $\mu$ on $\mathscr{A}_{\varkappa,q_1,q_2}$ invariant under the evolution operator $\Phi_\varkappa$. Given $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ let us set

$$C_t(u,n,r)=\frac{2}{n^2}\,\mathrm{card}\{(i,j):\rho(\Phi_\varkappa^i(u),\Phi_\varkappa^j(u))\leqslant r \quad \text{for } 0\leqslant i\leqslant j\leqslant n\}$$

(card denotes the number of elements in a set and $\rho$ is the metric in $\mathscr{A}_{\varkappa,q_1,q_2}$ induced by the norm $\|\cdot\|_{q_1,q_2}$).

The quantities

$$\bar{\beta}_t(u)=\overline{\lim_{r\to 0}}\lim_{n\to\infty}\frac{\log C_t(u,n,r)}{\log r},$$

$$\underline{\beta}_t(u)=\underline{\lim_{r\to 0}}\lim_{n\to\infty}\frac{\log C_t(u,n,r)}{\log r} \tag{4.1}$$

are called, respectively, the upper and lower correlation dimensions at point $u$ with respect to the evolution operator. (So far we assume that the limit as $n\to\infty$ exists; below we will see that this is true for $\mu$-almost every $u\in\mathscr{A}_{\varkappa,q_1,q_2}$.) It is easy to see that $\underline{\beta}_t(u)\leqslant\bar{\beta}_t(u)$. If for $\mu$-almost every $u\in\mathscr{A}_{\varkappa,q_1,q_2}$

$$\underline{\beta}_t(u)=\bar{\beta}_t(u)\overset{\text{def}}{=}\beta_t$$

($\beta$ does not depend on $u$) then it is called the correlation dimension (with respect to $\mu$).

The following result gives a condition for the correlation dimension to be correctly defined. It also gives us a formula to calculate it.

**THEOREM 3:** (1) For $\mu$-almost every $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ the limit exists for any $r>0$

$$\lim_{n\to\infty}C_t(u,n,r)=\int_{\mathscr{A}_{\varkappa,q_1,q_2}}\mu(B(u,r))d\mu(u)\overset{\text{def}}{=}\varphi(r)$$

and does not depend on $u$.

(2) For $\mu$-almost every $u\in\mathscr{A}_{\varkappa,q_1,q_2}$

$$\underline{\beta}_t(u)=\underline{\lim_{r\to 0}}\log\varphi(r)/\log r\overset{\text{def}}{=}\underline{\beta}_t,$$

$$\bar{\beta}_t(u)=\overline{\lim_{r\to 0}}\log\varphi(r)/\log r\overset{\text{def}}{=}\bar{\beta}_t$$

and the limits do not depend on $u$.

*Remark:* This theorem was proved in Ref. 13 under the additional assumption that the function $\varphi(r)$ is continuous. In Ref. 16 it was proved that $\varphi(r)$ has this property if $\mu$ is a Borel measure on a finite-dimensional Riemannian manifold. It is our case if the norm in $\mathbb{R}^d$ is generated by a scalar product (thus $\|\cdot\|_{q_1,q_2}$ is also generated by a scalar product in $\mathscr{M}_{\varkappa,q_1,q_2}$ which becomes a Hilbert space).

We describe another equivalent approach to the definition of the correlation dimension.

Given $u$, $v\in\mathscr{A}_{\varkappa,q_1,q_2}$ define

$$C_t(u,v,n,r)=\frac{2}{n^2}\,\mathrm{card}\{(i,j):\rho(\Phi_\varkappa^i(u),\Phi_\varkappa^j(v))\leqslant r \quad \text{for } 0\leqslant i\leqslant j\leqslant n\}.$$

Consider the space $X=\mathscr{A}_{\varkappa,q_1,q_2}\times\mathscr{A}_{\varkappa,q_1,q_2}$ with the metric $\tilde{\rho}((\tilde{u},\tilde{v}),(u,v))=\rho(\tilde{u},v)+\rho(\tilde{v},v)$ and the measure $\Delta=\mu\times\mu$ and define $\mathbb{Z}^2$ action on $X$ as follows

$$\Phi_\varkappa^{(i,j)}(u,v)=(\Phi_\varkappa^i(u),\Phi_\varkappa^j(v)).$$

It is not difficult to verify that this action preserves the measure $\Delta$ and is ergodic. This implies that for $\Delta$-almost every pair $(u,v)$ the limit exists for any $r>0$

$$\lim_{n\to\infty}C_t(u,v,n,r)=\Delta(U(\Lambda,r)),$$

where $\Lambda=\{(u,u):u\in\mathscr{L}_{\varkappa,q_1,q_2}\}$ is the diagonal and $U(\Lambda,r)=\{(u,v):\tilde{\rho}(u,v)\leqslant r\}$ is the $r$ neighborhood of $\Lambda$ in the direct-product space. Since $\Delta$ is the direct-product measure the Fubini theorem immediately implies that $\Delta(U(\Lambda,r))=\varphi(r)$. Define now for $u$, $v\in\mathscr{A}_{\varkappa,q_1,q_2}$

$$\underline{\beta}_t(u,v)=\underline{\lim_{r\to 0}}\lim_{n\to\infty}\frac{\log C_t(u,v,n,r)}{\log r},$$

$$\bar{\beta}_t(u,v)=\overline{\lim_{r\to 0}}\lim_{n\to\infty}\frac{\log C_t(u,v,n,r)}{\log r}$$

(we assume that the limit when $n\to\infty$ exists).

*Proposition 8: (cf. Ref. 13)* For $\Delta$-almost every pair $(u,v)\in X$

$$\underline{\beta}_t(u,v)=\underline{\lim_{r\to 0}}\log\varphi(r)/\log r=\underline{\beta}_t,$$

$$\bar{\beta}_t(u,v)=\overline{\lim_{r\to 0}}\log\varphi(r)/\log r=\bar{\beta}_t.$$

We now change the definition of the correlation dimension and introduce the notion of lower and upper limit correlation dimensions (specified by the evolution operator) by setting

$$\underline{\alpha}_t=\lim_{\delta\to 0}\inf_{Z}\underline{\lim_{r\to 0}}(\log\textstyle\int_Z\mu(B(u,r))d\mu(u)/\log r),$$

$$\bar{\alpha}_t=\lim_{\delta\to 0}\inf_{Z}\overline{\lim_{r\to 0}}(\log\textstyle\int_Z\mu(B(u,r))d\mu(u)/\log r), \tag{4.2}$$

where inf is taken over all sets $Z\subset\mathscr{A}_{\varkappa,q_1,q_2}$ with $\mu(Z)\geqslant 1-\delta$. If $\underline{\alpha}_t=\bar{\alpha}_t$ then the common value $\alpha_t$ is called the limit correlation dimension (with respect to $\mu$).

Given $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ define

$$\underline{d}(u)=\underline{\lim_{r\to 0}}\log\mu(B(u,r))/\log r,$$

$$\bar{d}(u)=\overline{\lim_{r\to 0}}\log\mu(B(u,r))/\log r,$$

where $B(u,r)$ is the ball in $\mathscr{A}_{\varkappa,q_1,q_2}$ centered at $u$ of radius $r$. These quantities are called, respectively, lower and upper pointwise dimensions at $u$.

The notion of pointwise dimension plays an important role in studying various characteristics of dimension type. It follows from results of Young (cf. Ref. 17) that if

$$\underline{d}(u)=\bar{d}(u)=d \tag{4.3}$$

for almost all $u\in\mathscr{A}_{\varkappa,q_1,q_2}$ then dimensionlike characteristics of the measure such as Hausdorff dimension, lower and upper box dimension, information dimension and some others coincide to the common value $d$. In fact it is also the common value for the generalized spectrum for dimensions (cf. details in Ref. 12). The conjecture due to Eckmann and Ruelle[18] claims that Eq. (4.1) holds for any ergodic Borel measure with nonzero Lyapunov exponents invariant under diffeomorphism of class $C^2$. This is proved in Ref. 17 for the two-dimensional case and in Ref. 19 for the Bowen–Ruelle–Sinai measures on hyperbolic attractors and Gibbs measures on hyperbolic sets.

The next statement follows from the results in Refs. 13 and 19.

**THEOREM 4**: If $\mu=\mu_{x,\varphi}$ is a Gibbs measure on $\mathscr{L}_{x,q_1,q_2}$ given by a function $\varphi$ then for $\mu$-almost every $u\in\mathscr{L}_{x,q_1,q_2}$

$$\underline{d}(u)=\bar{d}(u)=d=\underline{\alpha}_t=\bar{\alpha}_t=\alpha_t.$$

In particular, if $\mathscr{L}_{x,q_1,q_2}$ is a hyperbolic attractor and $\mu=\mu_{q_1,q_2}$ is the Bowen–Ruelle–Sinai measure then the above relations hold.

*Remark:* (1) According to Refs. 13 and 17 one can conclude from here that the limit correlation dimension $\alpha_t$ coincides with many other dimensionlike characteristics of $\mu$ such as Hausdorff dimension, box dimension, and information dimension.

(2) One can easily derive from the definitions that $\underline{\alpha}_t\geqslant\underline{\beta}_t$, $\bar{\alpha}_t\geqslant\bar{\beta}_t$. However in general we can not expect these values to coincide (cf. Refs. 20 and 21).

We define now the correlation dimension of the measure $\mu$ with respect to translations $S^{\bar{i}}$, $\bar{i}\in\mathbb{Z}^t$. Our results generalize ones obtained in Ref. 14. Given $u\in\mathscr{A}_{x,q_1,q_2}$ let us put

$$C_s(u,n,r)=\frac{1}{n^{2t}}\operatorname{card}\{(\bar{i},\bar{j}):\rho(S^{\bar{i}}(u),S^{\bar{j}}(u))\leqslant r,\bar{i},\bar{j}\in B_t(n)\},$$

where $B_t(n)=\{\bar{i}\in\mathbb{Z}^t:\bar{i}=\{i_k\},1\leqslant i_k\leqslant n\}$ is the cube in $\mathbb{Z}^t$ of side $n$. Define

$$\bar{\beta}_s(u)=\overline{\lim_{r\to0}}\lim_{n\to\infty}\frac{\log C_s(u,n,r)}{\log r},$$
$$\underline{\beta}_s(u)=\underline{\lim_{r\to0}}\lim_{n\to\infty}\frac{\log C_s(u,n,r)}{\log r}.$$

(We assume again that the limit when $n\to\infty$ exists.) These quantities are called, respectively, upper and lower correlation dimensions at point $u$ with respect to the $\mathbb{Z}^t$ action. Obviously $\underline{\beta}_s(u)\leqslant\bar{\beta}_s(u)$. If they coincide almost everywhere (with respect to the measure $\mu$) the common value is called the correlation dimension (with respect to $\mu$). We shall state an analog of Theorem 3 for the $\mathbb{Z}^t$ action.

**THEOREM 5**: (1) For $\mu$-almost every $u\in\mathscr{A}_{x,q_1,q_2}$, the limit exists $\lim_{n\to\infty}C_s(u,n,r)=\varphi(r)$ for any $r>0$ and thus does not depend on $u$.

(2) For $\mu$-almost every $u\in\mathscr{A}_{x,q_1,q_2}$

$$\underline{\beta}_s(u)=\underline{\lim_{r\to0}}\frac{\log\varphi(r)}{\log r}\overset{\text{def}}{=}\underline{\beta}_s,$$

$$\bar{\beta}_s(u)=\lim_{r\to0}\frac{\log\varphi(r)}{\log r}\overset{\text{def}}{=}\bar{\beta}_s.$$

In particular, $\underline{\beta}_t=\underline{\beta}_s$, $\bar{\beta}_t=\bar{\beta}_s$.

Both Theorems 3 and 5 are special cases of the following general statement (cf. Ref. 16).

Let $(Y,\rho)$ be a separable metric space, $\mu$ a probability Borel measure on $Y$, and let $T^{\bar{k}}$, $\bar{k}\in Z^m(m\in\overline{1,\infty})$ be a dynamical system acting in the measure space $(Y,\mu)$, i.e., $T^{\bar{k}}$ are $\mu$-preserving transformations of $Y$ and $T^{\bar{k}}T^{\bar{l}}=T^{\bar{k}+\bar{l}}$, $\bar{k},\bar{l}\in Z^m$. We denote: $\Delta=\mu\times\mu$; $C(y,n,r)=(1/n^{2t})\operatorname{card}\{(\bar{k},\bar{l}):\bar{k},\bar{l}\in B_t(n),\ \rho(T^{\bar{k}}y,T^{\bar{l}}y)\leqslant r\}$, $y\in Y$; $\varphi(r)=\int_Y\mu(B(x,r))d\mu(x)$; $C_\varphi$ is the set of continuity points of $\varphi$. Since $\varphi(r)$ is a monotonic function over $r$ the set $C_\varphi$ can be only countable. [Let us emphasize again that in our case when the action is given by space or time translations in $\mathscr{A}_{x,q_1,q_2}$ the function $\varphi(r)$ is continuous.]

**THEOREM 6: (Ref. 16)** For $\mu$-almost every $y\in Y$ the limit exists for any $r\in C_\varphi$

$$\lim_{n\to\infty}C(y,n,r)=\varphi(r).$$

The proof of this theorem is given in Ref. 16. It consists of two steps. At first, we show that the set $U(\Lambda,r)$ can be approximated by "polygons" in $Y\times Y$. Then we use Wiener's multidimensional pointwise ergodic theorem (cf. Refs. 22 and 23) to prove the existence of the limit for any "polygon" $P$:

$$\lim_{n\to\infty}\frac{1}{n^{2m}}\sum_{\bar{k},\bar{l}\in B_m(n)}I_P(T^{\bar{k}}u,T^{\bar{l}}u)=\Delta(P)$$

for $\mu$-almost all $u$ ($I_P$ is the indicator of $P$).

Define now the lower and upper correlation dimensions specified by the dynamical system $T^{\bar{k}}$ as follows:

$$\underline{\beta}(y)=\underline{\lim_{r\to0}}(\log r)^{-1}\lim_{n\to\infty}\log C(y,n,r),$$

$$\bar{\beta}(y)=\underline{\lim_{r\to0}}(\log r)^{-1}\lim_{n\to\infty}\log C(y,n,r)$$

(as usual we assume that the limit as $n\to\infty$ exists). As Theorem 6 shows, for $\mu$-almost every $y\in Y$

$$\underline{\beta}(y)=\underline{\lim_{\substack{r\to0\\ r\in C_\varphi}}}\frac{\log\varphi(r)}{\log r},\quad\bar{\beta}(y)=\underline{\lim_{\substack{r\to0\\ r\in C_\varphi}}}\frac{\log\varphi(r)}{\log r}.$$

Thus the correlation dimension does not depend on the dynamical system $\{T^{\bar{k}},k\in\mathbb{Z}^m\}$ (in particular on the "time dimension" $m$) but only on the invariant measure $\mu$. That explains why the correlation dimensions, specified by the evolution operator and space translations, coincide.

Let us consider the map $F_\varkappa$. Choose a Borel probability measure $\mu$ on $\mathbb{R}^{d(ls+m)}$ invariant under $F_\varkappa$ and denote by $\underline{\beta}^{(\varkappa)}(x)$, $\bar{\beta}^{(\varkappa)}(x)$ the lower and upper correlation dimensions at a point $x\in\mathbb{R}^{d(ls+m)}$ specified by $F_\varkappa$ [cf. Eq. (4.1)]. Let also $\underline{\alpha}^{(\varkappa)}(x)$ and $\bar{\alpha}^{(\varkappa)}(x)$ denote the lower and upper limit correlation dimensions at $x$ specified by $F_\varkappa$ [cf. Eq. (4.2)]. Consider the measure $\mu_{q_1,q_2}=(\alpha_\varkappa)_*(\chi_\varkappa)_*\mu$. We have seen that it is concentrated on $\mathscr{A}_{x,q_1,q_2}$ and is invariant under the evolution operator and space translations; moreover, its lower and upper correlation dimensions and limit correlation dimensions specified by the evolution operator [respectively, $\underline{\beta}_t(u)$, $\bar{\beta}_t(u)$ and $\underline{\alpha}_t(u)$, $\bar{\alpha}_t(u)$, $u\in\mathscr{A}_{\varkappa,q_1,q_2}$] as well as the same characteristics specified by the space translations [respectively, $\underline{\beta}_s(u)$, $\bar{\beta}_s(u)$ and $\underline{\alpha}_s(u)$, $\bar{\alpha}_s(u)$] coincide, i.e.,

$$\underline{\beta}_t(u)=\underline{\beta}_s(u), \quad \bar{\beta}_t(u)=\bar{\beta}_s(u),$$

$$\underline{\alpha}_t(u)=\underline{\alpha}_s(u), \quad \bar{\alpha}_t(u)=\bar{\alpha}_s(u).$$

If $\mu$ is weakly mixing and $q_1$, $q_2$ are big enough then

$$\underline{\beta}^{(\varkappa)}(x)=\underline{\beta}_t(u)=\underline{\beta}_t, \quad \bar{\beta}^{(\varkappa)}(x)=\bar{\beta}_t(u)=\bar{\beta}_t,$$

$$\underline{\alpha}^{(\varkappa)}(x)=\underline{\alpha}_t(u)=\underline{\alpha}_t, \quad \bar{\alpha}^{(\varkappa)}(x)=\bar{\alpha}_t(u)=\bar{\alpha}_t,$$

where $u=\alpha_\varkappa\circ\chi_\varkappa(x)$ (indeed, it is enough to notice that the maps $\alpha_\varkappa$ and $\chi_\varkappa$ are bi-Lipshitz maps, cf. statement 6 of Proposition 3 and statement 4 of Proposition 4). Thus, the correlation dimensions of $\mu_{q_1,q_2}$ specified by the evolution operator and space translations are completely determined by the corresponding correlation dimensions of $\mu$ specified by the map $F_\varkappa$. The same is true for the other dimension-like characteristics.

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[1] V. Afraimovich and V. Nekorkin, Chaos of traveling waves in a discrete chain of diffusively coupled maps, Preprint N330 of Institute of Applied Physics, Nizhny Novgorod (in Russian) (1991).
[2] J. A. Feroe, Math. Biosc. **55**, 3/4, 189 (1981).
[3] A. Maksimov and V. Nekorkin, Matematicheskoe Modelirovanie (in Russian) **2**, 129 (1990).
[4] V. Nekorkin, Izv. Vuzov—Radiofizika (in Russian) **31**, 41 (1988).
[5] L. Bunimovich and Ya. Sinai, Physica D **37**, 60 (1989).
[6] Ya. Pesin and Ya. Sinai, Adv. Sov. Math. **3** (1991).
[7] M. Rabinovich and M. Suschik, Russian Phys. Survey **160**, 1 (1990).
[8] V. Afraimovich and Ya. Pesin, Nonlinearity **3**, 1 (1990).
[9] V. Afraimovich and Ya. Pesin, Traveling waves in lattice models of multidimensional and multicomponent media: 1. General hyperbolic properties, Preprint PSU, 1992.
[10] M. Benedicks and L. S. Young, Sinai–Bowen–Ruelle measures for certain Henon maps, Preprint (1992).
[11] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes Math. 470 (Springer-Verlag, Berlin, 1975).
[12] Ya. Pesin, Generalized spectrum for the dimension: the approach based on Caratheodory's construction, in *Constantin Caratheodory, an Informational Tribute*, edited by Themisbocles Rassias, **2**, 1108 (1991).
[13] Ya. Pesin, On rigorous mathematical definitions of correlation dimension and generalized spectrum for dimensions, Preprint PSU (1992).
[14] V. Afraimovich, A. Ezersky, M. Rabinovich, M. Shereshevsky and A. Zhelernyak, Complexity and dimension of disordered states of nonequilibrium media, Preprint N291, Institute of Applied Physics, Nizhny Novgorod (in Russian) (1991) (to be published in Physica D 1992).
[15] I. Procaccia, P. Grassberger, and V. Hentschel, Lect. Notes Phys. **179**, 212 (1983).
[16] Ya. Pesin and A. Tempelmann, Correlation Dimension for Group Action, preprint, PSU (1992).
[17] L.-S. Young, Ergod. Theory and Dyn. Syst. **2**, 109 (1992).
[18] J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[19] Ya. Pesin and C. Yue, Hausdorff dimension of measures with nonzero Lyapunov exponents, preprint PSU (1993).
[20] E. Ott, T. Sauer, and I. Yorke, Phys. Rev. A **39**, 4212 (1989).
[21] C. Grebogi, E. Ott, and I. Yorke, Phys. Rev. A **38**, 3688 (1988).
[22] A. Tempelmann, *Ergodic Theorems for Group Actions* (Kluwer Academic, Dordrecht, 1992).
[23] N. Wiener, Duke Math J. **5**, 1 (1939).