On a general concept of multifractality: Multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity

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We introduce the mathematical concept of multifractality and describe various multifractal spectra for dynamical systems, including spectra for dimensions and spectra for entropies. We support the study by providing some physical motivation and describing several nontrivial examples. Among them are subshifts of finite type and one-dimensional Markov maps. An essential part of the article is devoted to the concept of multifractal rigidity. In particular, we use the multifractal spectra to obtain a ''physical'' classification of dynamical systems. For a class of Markov maps, we show that, if the multifractal spectra for dimensions of two maps coincide, then the maps are differentiably equivalent. © *1997 American Institute of Physics*. [S1054-1500(97)01201-9]

In the study of chaos one often encounters invariant sets with a very complicated geometry. In general, these sets are not self-similar, but can often be decomposed into subsets each possessing some scaling symmetry. This decomposition is called a *multifractal decomposition* and is an essential part of the multifractal analysis of dynamical systems. The physical data obtained in the numerical study of dynamical systems contain ''hidden'' information about multifractal decompositions. In order to reveal this information in a way which is convenient for the numerical analysis one can use the so-called *multifractal spectra*. Since the available data comes often only through ''physical'' observables, it is an important and challenging problem to recover information from the ''raw'' data about the dynamical system. We believe that for dynamical systems of hyperbolic type one can use a finite number of ''independent'' multifractal spectra to fully ''restore'' the dynamics. In this article, we present several results towards the solution of this problem.

I. INTRODUCTION

The multifractal analysis, i.e., the analysis of invariant sets and measures with multifractal structure, has been recently developed as a powerful tool for numerical study of dynamical systems. Its main constituent component is dimension spectra which include Rényi spectrum for dimensions, Hentschel-Procaccia spectrum for dimensions, and $f(\alpha)$-spectrum for (pointwise) dimensions. These spectra capture information about various dimensions associated with the dynamics. Among them are the well-known Hausdorff dimension, correlation dimension, and information dimension of invariant measures.

There is another dimension spectrum that is used to describe the distribution of Lyapunov exponents. It is called the dimension spectrum for Lyapunov exponents and it yields integrated information on the instability of trajectories.

Dimension spectra are examples of more general multifractal spectra that we introduce in this article. Another example of multifractal spectra is entropy spectra. They provide integrated information on the distribution of topological entropy associated with pointwise dimensions, Lyapunov exponents, etc.

One of the main points of the article is to demonstrate that multifractal spectra can be used in a sense to ''restore'' the dynamics—the phenomenon that we call the *multifractal rigidity*. There are two main problems related to multifractal rigidity. First, given a dynamical system of hyperbolic type there exist *finitely* many *independent* multifractal spectra that uniquely identify the main *macro-characteristics* of the system (such as its invariant measure, geometric structure of its invariant sets, their dimensions, etc.). These spectra can be viewed as a special type of *degrees of freedom*, called *multifractal degrees of freedom*, and can be effectively used in the numerical study of dynamical systems.

We demonstrate, in particular, that, for subshifts of finite type and some conformal expanding maps, the dimension spectrum alone is sufficient to determine all other multifractal spectra and, thus, these systems have one multifractal degree of freedom.

Another problem is inspired by an attempt to produce a ''physically meaningful'' classification of dynamical systems that takes care of various aspects of the dynamics (chaoticity, instability, geometry, etc.) simultaneously.

In the theory of dynamical systems there are various types of classifications. The most prominent ones seem to be

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topological classification (up to homeomorphisms) and *measure-theoretic classification* (up to measure preserving automorphisms). From a physical point of view, these classifications trace separate ''independent'' characteristics of the dynamics. We suggest a new type of classification that is based upon multifractal spectra and combines features of each of the above classifications, in what we call the *multifractal classification*. The new classification has a strong physical content and identifies two systems up to a change of variables. From a mathematical point of view, we establish the smooth equivalence of two dynamical systems that are *a priori* only topologically equivalent and have the same multifractal degrees of freedom. The multifractal classification is much more rigid than the topological and measure-theoretic classifications. Besides the smooth equivalence of the two dynamical systems, it establishes the coincidence of their dimension characteristics as well as the correspondence between their invariant measures.

II. A GENERAL CONCEPT OF MULTIFRACTALITY

We begin with the general concept of multifractal spectrum.

Let X be a set and let g: $X\to[-\infty,+\infty]$ be a function. The level sets of g,

$$K_\alpha^g=\{x\in X: g(x)=\alpha\}, \quad -\infty\leqslant\alpha\leqslant+\infty$$

are disjoint and produce a *multifractal decomposition* of X, that is,

$$X=\bigcup_{-\infty\leqslant\alpha\leqslant+\infty} K_\alpha^g. \tag{1}$$

Let G now be a set function, i.e., a real function that is defined on subsets of X. Assume that $G(Z_1)\leqslant G(Z_2)$ if $Z_1\subset Z_2$. We define the function $\mathscr{F}$:$[-\infty,+\infty]\to\mathbb{R}$ by

$$\mathscr{F}(\alpha)=G(K_\alpha^g).$$

We call $\mathscr{F}$ the *multifractal spectrum* specified by the pair of functions (g,G), or the (g,G)-multifractal spectrum. The function $\mathscr{F}$ captures important information about the structure of the set X generated by the function g.

It often happens that the function g is defined only on a subset $Y\subset X$. In this case the decomposition (1) should be replaced by

$$X=(X\backslash Y)\cup\bigcup_{-\infty\leqslant\alpha\leqslant+\infty} K_\alpha^g.$$

We still call this decomposition of X a multifractal decomposition.

Given $-\infty\leqslant\alpha\leqslant+\infty$, let ν_α be a probability measure on X such that $\nu_\alpha(K_\alpha^g)=1$. If

$$\mathscr{F}(\alpha)=\inf\{G(Z): \nu_\alpha(Z)=1\},$$

we call ν_α a (g,G)-*full measure*. Constructing a one-parameter family of (g,G)-full probability measures ν_α seems the most effective way of studying multifractal decompositions.

When X is a smooth compact manifold and g is a smooth function, each level set K_α^g is a hypersurface for all α but at most the critical values of g (see Figure 1); moreover, the set of numbers α for which the set K_α^g is nonempty is an interval. For smooth functions, typically the Lebesgue measure on K_α^g is a full measure, and the spectrum $\mathscr{F}$ is a delta function.

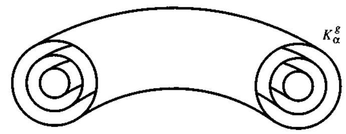

FIG. 1. A typical multifractal decomposition for a smooth function g.

We are mostly interested in the case where g is not even continuous and, thus, the sets K_α^g can have a very complicated structure (see Figure 2). In this case, we are going to establish, in some situations, that

(1) the set of numbers α for which the set K_α^g is nonempty is an interval;
(2) given α (with nonempty K_α^g), there is a measure ν_α supported on K_α^g;
(3) the set $X\backslash Y$ is negligible (in some sense);
(4) the function $\mathscr{F}$ is analytic and strictly convex.

We furthermore use families of full measures to classify multifractal decompositions and the corresponding multifractal spectra. Namely, let $\mathscr{F}$ and $\mathscr{F}'$ be two multifractal spectra specified by the pairs of functions (g,G) and (g',G'), respectively. Assume that there exist families $\{\nu_\alpha\}_{\alpha\in\mathbb{R}}$ and $\{\nu'_\alpha\}_{\alpha\in\mathbb{R}}$ of (g,G)-full measures and (g',G')-full measures, respectively. We say that the spectra $\mathscr{F}$ and $\mathscr{F}'$ are *equivalent* (with respect to the families $\{\nu_\alpha\}_{\alpha\in\mathbb{R}}$ and $\{\nu'_\alpha\}_{\alpha\in\mathbb{R}}$) and we write $\mathscr{F}\sim\mathscr{F}'$ if there exists a bijective map π:$[-\infty,+\infty]\to[-\infty,+\infty]$ such that $\nu_\alpha = \nu'_{\pi(\alpha)}$. The function π is called a $(\mathscr{F},\mathscr{F}')$-*parametrization*.

In this article we will be interested in multifractal decompositions associated with dynamical systems acting on X. There may exist many such decompositions generated by different ''naturally chosen'' functions g and G (see Section III). We believe that, in studying dynamical systems with chaotic behavior, the equivalence class of every such spectrum contains crucial information about the dynamics of f on the invariant set X.

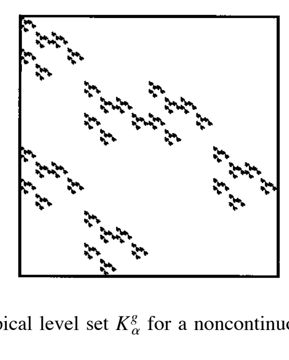

FIG. 2. A typical level set K_α^g for a noncontinuous function g.

III. EXAMPLES OF MULTIFRACTAL SPECTRA

We illustrate the general concept of multifractal spectra by studying several explicit spectra.

A. Dimension and entropy spectra

Let X be a complete separable metric space and let $f: X \to X$ be a continuous map. We begin with the choice of the set function G. There are two ''natural'' set functions on X. The first one is generated by the metric structure on X. Namely, given a subset $Z \subset X$, we set

$$G_D(Z) = \dim_H Z, \tag{2}$$

where $\dim_H Z$ is the Hausdorff dimension of Z (see the Appendix).

The second function is generated by the dynamical system f acting on X and the metric on X. Namely,

$$G_E(Z) = h(f|_Z), \tag{3}$$

where $h(f|_Z)$ is the topological entropy of f on Z (see the Appendix; notice that Z need not be compact nor f-invariant). We call the multifractal spectra generated by the function G_D *dimension spectra*, and the multifractal spectra generated by the function G_E *entropy spectra*.

We now describe some ''natural'' choices for the function g.

B. Multifractal spectra for pointwise dimensions

Let μ be a Borel finite measure on X. Consider the subset $Y \subset X$ consisting of all points $x \in X$ for which the limit

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists, where $B(x,r)$ denotes the ball of radius r centered at x. The number $d_\mu(x)$ is called the *pointwise dimension* of μ at x. Whenever $x \in Y$, we say that the pointwise dimension of μ exists at the point x. We define the function g_D on Y by

$$g_D(x) = d_\mu(x).$$

We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_D} = \{x : d_\mu(x) = \alpha\}.$$

We obtain two multifractal spectra $\mathscr{D}_D = \mathscr{D}_D^{(\mu)}$ and $\mathscr{D}_E = \mathscr{D}_E^{(\mu)}$ specified by the pairs of functions (g_D, G_D) and (g_D, G_E), respectively, where the set functions G_D and G_E are given by (2) and (3). We call them *multifractal spectra for (pointwise) dimensions*.

Let us note that the spectrum $\mathscr{D}_D$ is known in the literature as the *dimension spectrum* or $f_\mu(\alpha)$-*spectrum for dimensions*. The concept of a multifractal analysis was suggested by a group of physicists in Ref. 1 (see Ref. 2 for more references and details).

In Ref. 3, Eckmann and Ruelle discussed the pointwise dimension of hyperbolic measures (that is, measures with nonzero Lyapunov exponents almost everywhere), invariant under diffeomorphisms. They conjectured that the pointwise dimension exists almost everywhere, that is, $\mu(X \setminus Y) = 0$. This claim has been known as the *Eckmann-Ruelle conjecture* and has become a celebrated problem in the dimension theory of dynamical systems. In Ref. 4, we establish the affirmative solution of this conjecture for $C^{1+\epsilon}$ diffeomorphisms (an announcement appeared in Ref. 5).

C. Multifractal spectra for local entropies

Let X be a complete separable metric space and let $f: X \to X$ be a continuous map preserving a Borel probability measure μ. Consider a finite measurable partition ξ of X. For every $n > 0$, we write $\xi_n = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n}\xi$, and denote by $\xi_n(x)$ the element of the partition ξ_n that contains point x. Consider the set $Y = Y_\xi \subset X$ consisting of all points $x \in X$ for which the limit

$$h_\mu(f,\xi,x) = \lim_{n \to \infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists. We call $h_\mu(f,\xi,x)$ the μ-*local entropy* of f at the point x (with respect to ξ). Clearly, Y is f-invariant and $h_\mu(f,\xi,fx) = h_\mu(f,\xi,x)$ for every $x \in Y$. By the Shannon-McMillan-Breiman theorem, $\mu(X \setminus Y) = 0$. In addition, if ξ is a *generating partition* and μ is ergodic, then

$$h_\mu(f) = h_\mu(f,\xi,x)$$

for μ-almost all $x \in X$, where $h_\mu(f)$ is the measure-theoretic entropy of f (with respect to μ). We define the function g_E on Y by

$$g_E(x) = h_\mu(f,\xi,x).$$

Let us stress that g_E may depend on ξ. We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_E} = \{x : h_\mu(f,\xi,x) = \alpha\}.$$

We obtain two multifractal spectra $\mathscr{E}_D = \mathscr{E}_D^{(\mu)}$ and $\mathscr{E}_E = \mathscr{E}_E^{(\mu)}$ specified by the pairs of functions (g_E, G_D) and (g_E, G_E), respectively, where the set functions are given by (2) and (3). We call them *multifractal spectra for (local) entropies*. In Sections IV and V we will observe that in some situations these spectra, in fact, do not depend on ξ for a broad class of partitions.

We note that, in the study of the multifractal spectra for local entropies, the Shannon-McMillan-Breiman theorem plays the same role as the Eckmann-Ruelle conjecture in the study of the multifractal spectra for pointwise dimensions.

D. Multifractal spectra for Lyapunov exponents

Let X be a differentiable manifold and let $f: X \to X$ be a C^1 map. Consider the subset $Y \subset X$ of all points $x \in X$ for which the limit

$$\lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \log \| d_x f^n \|$$

exists. By Kingman's subadditive ergodic theorem, if μ is an f-invariant Borel probability measure, then $\mu(X \setminus Y) = 0$. We define the function g_L on Y by

$$g_L(x) = \lambda(x).$$

We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_L} = \{x : \lambda(x) = \alpha\}.$$

We obtain two multifractal spectra $\mathscr{L}_D$ and $\mathscr{L}_E$ specified, respectively, by the pairs of functions (g_L, G_D) and (g_L, G_E), where the set functions G_D and G_E are given by (2) and (3). We call them *multifractal spectra for Lyapunov exponents*. The spectrum $\mathscr{L}_D$ was studied in Ref. 6 (see also the references in that paper). The spectrum $\mathscr{L}_E$ was introduced in Ref. 7.

In the following sections we show how to compute the above multifractal spectra in some particular cases.

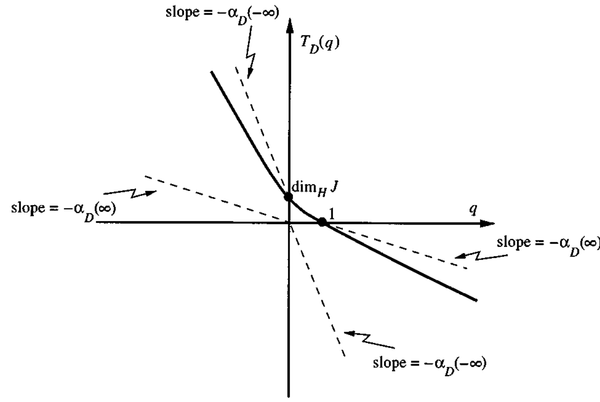

FIG. 3. A typical graph of the function $T_D(q)$.

IV. MULTIFRACTAL SPECTRA OF GIBBS MEASURES FOR SUBSHIFTS OF FINITE TYPE

Let A be a $p \times p$ matrix whose entries are either 0 or 1. The topological Markov chain Σ_A^+ consists of the sequences $\omega = (i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbf{N}}$ such that $a_{i_k i_{k+1}} = 1$ for every $k \geq 1$. Let $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$ be the shift map on Σ_A^+. We assume that A is *transitive*, i.e., there exists a positive integer k such that all entries of A^k are positive (this holds if and only if $\sigma|_{\Sigma_A^+}$ is topologically mixing).

Fix $a > 1$ and define a metric on Σ_A^+ by

$$d(\omega, \omega') = \sum_{k=1}^{\infty} a^{-k} |i_k - i_k'|.$$

Notice that $d(\sigma\omega, \sigma\omega') = a \cdot d(\omega, \omega')$ for all $\omega, \omega' \in \Sigma_A^+$ with $d(\omega, \omega') < a^{-1}$.

Given a continuous function φ on Σ_A^+, a measure μ on Σ_A^+ is said to be a *Gibbs measure* for φ if there exist constants $C_1, C_2 > 0$, such that

$$C_1 \leq \frac{\mu(C_{i_1 \ldots i_n})}{\exp(-nP(\varphi) + \sum_{k=0}^{n-1} \varphi(f^k \omega))} \leq C_2$$

for every $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbf{N}$, where

$$C_{i_1 \ldots i_n} = \{(j_1 j_2 \cdots) \in \Sigma_A^+ : j_k = i_k \quad \text{for} \quad k = 1, \ldots, n\}$$

is the cylinder set of length n containing ω, and P is the topological pressure with respect to σ (see the Appendix).

Let φ be a Hölder continuous function on Σ_A^+ and let μ be the corresponding Gibbs measure; the measure μ exists and is unique (because $\sigma|_{\Sigma_A^+}$ is topologically mixing). It is more convenient to work with the "normalized" function $\log \psi$ on Σ_A^+ defined by $\log \psi = \varphi - P(\varphi)$. Note that μ is also the Gibbs measure for $\log \psi$.

For each $q \in \mathbb{R}$ let us consider the function

$$\varphi_q = -T(q) \log a + q \log \psi,$$

where the number $T(q)$ is chosen in such a way that $P(\varphi_q) = 0$. See Figure 3 for a typical graph of the function $T(q)$. Clearly,

$$T(q) \log a = P(q \log \psi). \tag{4}$$

Let h be the spectral radius of A (which is also the topological entropy of $\sigma|_{\Sigma_A^+}$).

Proposition 4.1. *The function T is a real analytic on $\mathbb{R}$, and satisfies $T'(q) \leq 0$ and $T''(q) \geq 0$ for every $q \in \mathbb{R}$. Moreover, $T(0) = h/\log a$ and $T(1) = 0$.*

Proof. Since $q \mapsto P(q \log \psi)$ is analytic (see Ref. 8), the function T is analytic. All the remaining statements are consequences of well-known properties of the topological pressure (see, for example, Ref. 9). If ν_q is the Gibbs measure for φ_q (and hence, for $q \log \psi$), we obtain

$$\frac{d}{dq} P(q \log \psi) = \int_{\Sigma_A^+} \log \psi \, d\nu_q = \int_{\Sigma_A^+} \varphi \, d\nu_q - P(\varphi) \leq -h_{\nu_q}(\sigma|_{\Sigma_A^+}) \leq 0.$$

Therefore, $T'(q) \leq 0$ for every $q \in \mathbb{R}$. Since the topological pressure is convex, the function T is convex and hence $T''(q) \geq 0$ for every $q \in \mathbb{R}$. The identities in the proposition follow immediately from (4). □

Only the four spectra $\mathscr{D}_D$, $\mathscr{D}_E$, $\mathscr{E}_D$, and $\mathscr{E}_E$ make sense for subshifts of finite type. We provide a complete description of all these spectra.

Denote by $\mathfrak{B}$ the class of finite partitions of Σ_A^+ into disjoint cylinder sets (not necessarily of the same length). Clearly, each $\xi \in \mathfrak{B}$ is a generating partition. We use it to define the spectra for entropies $\mathscr{E}_D$ and $\mathscr{E}_E$.

The following theorem establishes the relations between the multifractal spectra for dimensions and entropies.

Theorem 4.2. *For every $\alpha \in \mathbb{R}$, we have*

$$\mathscr{E}_E(\alpha) = \mathscr{E}_D(\alpha) \log a = \mathscr{D}_E(\alpha/\log a) = \mathscr{D}_D(\alpha/\log a) \log a, \tag{5}$$

and the common value is independent of the partition $\xi \in \mathfrak{B}$. Moreover, the multifractal decompositions of the spectra $\mathscr{E}_E$, $\mathscr{E}_D$, $\mathscr{D}_E$, and $\mathscr{D}_D$ coincide, that is, the families of level sets of these four spectra are equal up to the parametrizations given by (5).

Proof. Notice that there exists a constant $C>0$ such that $\operatorname{diam}\xi_n(x)=Ca^{-n}$ for every $x\in\Sigma_A^+$ and $n\geqslant 1$. If the pointwise dimension $d_\mu(x)$ or the local entropy $h_\mu(f,\xi,x)$ exists for some $x\in\Sigma_A^+$, then

$$h_\mu(f,\xi,x)=\lim_{n\to\infty}-\frac{1}{n}\log\mu(\xi_n(x))$$
$$=\log a\lim_{n\to\infty}\frac{\log\mu(\xi_n(x))}{\operatorname{diam}\xi_n(x)}=\log a\cdot d_\mu(x). \tag{6}$$

This shows that $d_\mu(x)$ exists at a point x if and only if $h_\mu(f,\xi,x)$ exists. The previous identity and the lemma in the Appendix immediately imply the relations (5).

Since the spectra for dimensions $\mathscr{D}_D$ and $\mathscr{D}_E$ are independent of the partition $\xi\in\mathfrak{B}$, each of the values in (5) is independent of $\xi\in\mathfrak{B}$. □

We begin with the description of the dimension spectrum for pointwise dimensions $\mathscr{D}_D$. The following theorem shows (with a minor exception) that $\mathscr{D}_D$ is defined on an interval, is analytic, and is strictly convex. It also establishes a relationship between the functions $\mathscr{D}_D(\alpha)$ and $T_D(q)$; namely, they form a Legendre pair. By virtue of Theorem 4.2 this also provides a description of the spectra $\mathscr{D}_E$, $\mathscr{E}_D$, and $\mathscr{E}_E$.

Let m_E be the measure of maximal entropy. The following statement is a consequence of a general result proved by Pesin and Weiss[10] (see also their paper in this volume[11]). We set $\alpha(q)=-T'(q)$. The range of the function $\alpha(q)$ is the interval $[\alpha_1,\alpha_2]$, where $\alpha_1=\alpha(+\infty)$ and $\alpha_2=\alpha(-\infty)$.

Theorem 4.3.

(1) *For μ-almost every $x\in\Sigma_A^+$, the pointwise dimension of μ at x exists and*

$$g_D(x)=d_\mu(x)=-\frac{1}{\log a}\int_{\Sigma_A^+}\log\psi\,d\mu.$$

(2) *The domain of the function $\alpha\mapsto\mathscr{D}_D(\alpha)$ is a closed interval in $[0,+\infty)$ and coincides with the range of the function $\alpha(q)$. For every $q\in\mathbb{R}$, we have*

$$\mathscr{D}_D(\alpha(q))=T(q)+q\alpha(q).$$

(3) *If $\mu\neq m_E$, then $\mathscr{D}_D$ and T are analytic strictly convex functions, and hence, $(\mathscr{D}_D,T)$ is a Legendre pair with respect to the variables α, q (see the Appendix).*

Using Theorems 4.2 and 4.3 we describe the spectra $\mathscr{D}_E$, $\mathscr{E}_D$, and $\mathscr{E}_E$.

Theorem 4.4.

(1) *There exists a set $S\subset\Sigma_A^+$ with $\mu(S)=1$ such that for every partition $\xi\in\mathfrak{B}$ and every $x\in S$, the local entropy of μ at x exists, has the same value for every ξ, and*

$$g_E(x)=h_\mu(f,\xi,x)=-\int_{\Sigma_A^+}\log\psi\,d\mu.$$

(2) *The domain of each of the functions $\alpha\mapsto\mathscr{D}_E(\alpha)$, $\alpha\mapsto\mathscr{E}_D(\alpha\log a)$, and $\alpha\mapsto\mathscr{E}_E(\alpha\log a)$ is the range of the function $\alpha(q)$. For every $q\in\mathbb{R}$, we have*

$$\mathscr{D}_E(\alpha(q))=T(q)\log a+q\alpha(q)\log a,$$
$$\mathscr{E}_D(\alpha(q)\log a)=T(q)+q\alpha(q),$$
$$\mathscr{E}_E(\alpha(q)\log a)=T(q)\log a+q\alpha(q)\log a.$$

(3) *If $\mu\neq m_E$, then $\mathscr{D}_E$, $\mathscr{E}_D$, and $\mathscr{E}_E$ are analytic strictly convex functions, and hence,*
$(\mathscr{D}_E/\log a,T)$, $(\mathscr{E}_D(\cdot\log a),T)$, $(\mathscr{E}_E(\cdot\log a)/\log a,T)$
are Legendre pairs with respect to the variables α, q.

Proof. The existence of the set S follows from the identity (6), which is valid for a set of full μ-measure (notice that $d_\mu(x)$ does not depend on the partitions $\xi\in\mathfrak{B}$).

All the remaining statements follow easily from Theorem 4.3 and Equation (6). □

Remarks.

(1) Let HP_μ and R_μ be respectively the Hentschel-Procaccia and Rényi spectra for dimensions (see Ref. 2). In Ref. 10, Pesin and Weiss proved that for every $q\in\mathbb{R}$,

$$T(q)=(1-q)HP_\mu(q)=(1-q)R_\mu(q)$$
$$=\lim_{n\to\infty}\frac{1}{n\log a}\log\sum_C\mu(C)^q,$$

where the sum is taken over all cylinder sets of length n.

(2) It follows from results of Schmeling (see Ref. 12) that the level sets $K_\alpha^{g_D}$ are empty for every $\alpha\notin[\alpha_1,\alpha_2]$, the function $q\mapsto\alpha(q)$ is invertible, and the numbers $\mathscr{D}_D(\alpha_1)$ and $\mathscr{D}_D(\alpha_2)$ may not be zero. More precisely, for any numbers τ_1, $\tau_2\in[0,\dim_H\Sigma_A^+)$ there exists a Gibbs measure corresponding to a Hölder continuous function such that $\mathscr{D}_D(\alpha_1)=\tau_1$ and $\mathscr{D}_D(\alpha_2)=\tau_2$. On the other hand, ''generically'' $\mathscr{D}_D(\alpha_1)=\mathscr{D}_D(\alpha_2)=0$. Similar conclusions hold for the spectra $\mathscr{D}_E$, $\mathscr{E}_D$, and $\mathscr{E}_E$.

(3) One can show that the unique Gibbs measure ν_q corresponding to the Hölder continuous function φ_q is a (g_D,G_D)-full measure for $\alpha(q)$. More precisely, for every $q\in\mathbb{R}$ we have $\nu_q(K_{\alpha(q)}^{g_D})=1$ and $d\nu_q(x)=T(q)+q\alpha(q)$ for ν_q-almost all $x\in K_{\alpha(q)}^{g_D}$. See Section V for more general results.

V. MULTIFRACTAL SPECTRA OF GIBBS MEASURES FOR CONFORMAL REPELLERS

We consider Gibbs measures invariant under conformal expanding maps, and describe the associated multifractal spectra for dimensions, entropies, and Lyapunov exponents.

A. Preliminaries

Let M be a smooth Riemannian manifold and let $f:M\to M$ be a C^1 map. Consider a compact subset J of M. We say that f is *expanding* and J is a *repeller* of f if

(1) there exist constants $C>0$ and $\beta>1$ such that $\|d_xf^nu\|\geqslant C\beta^n\|u\|$ for all $x\in J$, $u\in T_xM$, and $n\geqslant 1$;
(2) $J=\bigcap_{n\geqslant 0}f^{-n}V$ for some open neighborhood V of J.

One can easily show that $fJ=J$.

We recall that a finite cover $\{R_1,\ldots,R_p\}$ of J by closed sets is called a *Markov partition* of J (with respect to f) if

(1) int $R_i=R_i$ for each $i=1,\ldots,p$;
(2) int $R_i\cap$int $R_j=\varnothing$ if $i\neq j$;
(3) each fR_i is a union of sets R_j.

It is well known that repellers admit Markov partitions of an arbitrarily small diameter. Markov partitions are used to build symbolic models of repellers by subshifts of finite type (see Section IV).

Let J be a repeller of an expanding map f, and let $\xi=\{R_1,\ldots,R_p\}$ be a Markov partition of J with respect to f. We define a $p\times p$ *transfer matrix* $A=(a_{ij})$ by setting $a_{ij}=1$ if $R_i\cap f^{-1}R_j\neq\varnothing$, and $a_{ij}=0$ otherwise. Consider the associated subshift of finite type (Σ_A^+,σ). For each $\omega=(i_1 i_2\ldots)\in\Sigma_A^+$, we set

$$\chi(\omega)=\{x\in X: f^{k-1}x\in R_{i_k}\quad\text{for}\quad\text{every}\quad k\geqslant 1\}.$$

The set $\chi(\omega)$ consists of a single point in J, and we obtain a *coding map* $\chi:\Sigma_A^+\to J$ for the repeller. The map χ is continuous, onto, and the following diagram is commutative:

$$\begin{array}{ccc}\Sigma_A^+ & \xrightarrow{\sigma} & \Sigma_A^+\\ \chi\downarrow & & \downarrow\chi\\ J & \xrightarrow{f} & J.\end{array}$$

We assume that the matrix A is transitive (and thus, f is topologically mixing).

It is clear that any Markov partition ξ is a generating partition. The same is true for any partition of J by rectangles obtained from a Markov partition (not necessarily all of the same level) and corresponding to disjoint cylinder sets in Σ_A^+. We denote the class of such partitions by $\mathfrak{B}_f$. It is easy to see that, for every partition $\xi\in\mathfrak{B}_f$, there is a partition $\eta\in\mathfrak{B}$ such that $\chi\eta=\xi$.

A smooth map $f:M\to M$ is called conformal if $d_x f$ is a multiple of an isometry at every point $x\in M$. Well-known examples of conformal expanding maps include one-dimensional Markov maps (see Section VII) and holomorphic maps. We write $a(x)=\|d_x f\|$ for each $x\in M$.

B. Multifractal spectra

Let J be a repeller of a conformal $C^{1+\epsilon}$ expanding map f for some $\epsilon>0$. Also let m_D be the unique Gibbs measure corresponding to the function $x\mapsto -\dim_H J\cdot\log a(x)$ on J. It is known that m_D is a measure of maximal dimension, i.e., $\dim_H J=\dim_H m_D$ (see Ref. 8). We denote by m_E the measure of maximal entropy for $f:J\to J$, and by h the topological entropy of f on J.

Let φ be a Hölder continuous function on J and let μ be the corresponding Gibbs measure with respect to f. Write $\log\psi=\varphi-P(\varphi)$.

For each q, $p\in\mathbb{R}$, consider the functions

$$\varphi_{D,q}=-T_D(q)\log a+q\log\psi$$

and

$$\varphi_{E,p}=-T_E(p)+p\log\psi,$$

where the numbers $T_D(q)$ and $T_E(p)$ are chosen such that

$$P(\varphi_{D,q})=P(\varphi_{E,p})=0.$$

Clearly, $T_E(p)=P(p\log\psi)$. See Figure 3 for a typical graph of the function $T_D(q)$.

Proposition 5.1. *The following properties hold:*

(1) *The function T_D is real analytic and satisfies $T_D'(q)\leqslant 0$ and $T_D''(q)\geqslant 0$ for every $q\in\mathbb{R}$. We have $T_D(0)=\dim_H J$ and $T_D(1)=0$.*
(2) *The function T_E is real analytic, and satisfies $T_E'(p)\leqslant 0$ and $T_E''(p)\geqslant 0$ for every $p\in\mathbb{R}$. We have $T_E(0)=h$ and $T_E(1)=0$.*

The first property is proved in Ref. 10 [see also the paper of Pesin and Weiss in this volume[11]; note that the equality $T_D(0)=\dim_H J$ follows the formula for the dimension of a conformal repeller established by Ruelle in Ref. 8]. The second property is a rewriting of Proposition 4.1. Set

$$\alpha_D(q)=-T_D'(q)\quad\text{and}\quad\alpha_E(p)=-T_E'(p).$$

We now give a full description of the four multifractal spectra $\mathscr{D}_D$, $\mathscr{E}_E$, $\mathscr{L}_D$, and $\mathscr{L}_E$ for Gibbs measures supported on repellers of conformal smooth expanding maps.

We begin with the dimension spectrum for pointwise dimensions $\mathscr{D}_D$. The following theorem shows (with minor exceptions) that $\mathscr{D}_D$ is defined on an interval, is analytic, and is strictly convex. It also establishes a relationship between the functions $\mathscr{D}_D(\alpha)$ and $T_D(q)$; namely, they form a Legendre pair. In Ref. 10, Pesin and Weiss effected a description of the spectrum $\mathscr{D}_D$ (see also their paper in this volume[11]).

Theorem 5.2.

(1) *For μ-almost every $x\in J$, the pointwise dimension of μ at x exists and*
$$g_D(x)=d_\mu(x)=-\frac{\int_J\log\psi\,d\mu}{\int_J\log a\,d\mu}.$$
(2) *The domain of the function $\alpha\mapsto\mathscr{D}_D(\alpha)$ is a closed interval in $[0,+\infty)$ and coincides with the range of the function $\alpha_D(q)$. For every $q\in\mathbb{R}$, we have*
$$\mathscr{D}_D(\alpha_D(q))=T_D(q)+q\alpha_D(q).\qquad(7)$$
(3) *If $\mu\neq m_D$, then $\mathscr{D}_D$ and T_D are analytic strictly convex functions and, hence, $(\mathscr{D}_D,T_D)$ is a Legendre pair with respect to the variables α, q.*
(4) *If $\mu=m_D$, then $\mathscr{D}_D$ is the delta function*
$$\mathscr{D}_D(\alpha)=\begin{cases}\dim_H J, & \text{if}\quad\alpha=\dim_H J,\\ 0, & \text{if}\quad\alpha\neq\dim_H J.\end{cases}$$

The identity (7) is a consequence of property 1 in Theorem 5.4. See Figure 4 for a typical graph of the function $\mathscr{D}_D(\alpha)$.

We now provide a description of the spectrum $\mathscr{E}_E$. The following result is an immediate consequence of Theorem 4.4.

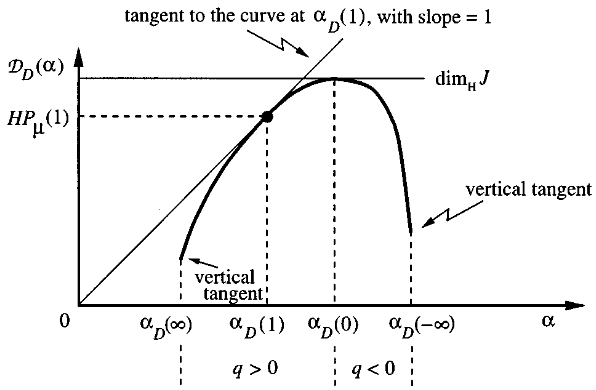

FIG. 4. A typical graph of the function $\mathscr{D}_D(\alpha)$.

Theorem 5.3.

(1) *There exists a set* $S\subset X$ *with* $\mu(S)=1$ *such that for every partition* $\xi\in\mathfrak{B}_f$ *and every* $x\in S$*, the local entropy of* μ *at* x *exists, does not depend on* x *and* ξ*, and*

$$g_E(x)=h_\mu(f,\xi,x)=-\int_J \log \psi\, d\mu.$$

(2) *The domain of the function* $\alpha\mapsto\mathscr{E}_E(\alpha)$ *is a closed interval in* $[0,+\infty)$ *and coincides with the range of the function* $\alpha_E(p)$*. For every* $p\in\mathbb{R}$*, we have*

$$\mathscr{E}_E(\alpha_E(p))=T_E(p)+p\,\alpha_E(p). \tag{8}$$

(3) *If* $\mu\neq m_E$*, then* $\mathscr{E}_E$ and T_E *are analytic strictly convex functions and, hence,* $(\mathscr{E}_E,T_E)$ *is a Legendre pair with respect to the variables* α*,* q*.*

(4) *If* $\mu=m_E$*, then* $\mathscr{E}_E$ *is the delta function*

$$\mathscr{E}_E(\alpha)=\begin{cases} h, & \text{if } \alpha=h,\\ 0, & \text{if } \alpha\neq h.\end{cases}$$

The identity (8) is a consequence of property 2 in Theorem 5.4.

We now describe the full measures for the spectra $\mathscr{D}_D$ and $\mathscr{E}_E$. It turns out that these are the unique Gibbs measures $\nu_q^{\mathscr{D}_D}$ and $\nu_p^{\mathscr{E}_E}$ for the (Hölder continuous) functions $\varphi_{D,q}$ and $\varphi_{E,p}$, respectively.

Theorem 5.4. *The following properties hold:*

(1) *For every* $q\in\mathbb{R}$*, we have* $\nu_q^{\mathscr{D}_D}(K^{g_D}_{\alpha_D(q)})=1$ *and*

$$d_{\nu_q^{\mathscr{D}_D}}(x)=T_D(q)+q\,\alpha_D(q)$$

for $\nu_q^{\mathscr{D}_D}$*-almost all* $x\in K^{g_D}_{\alpha_D(q)}$*. Moreover,* $\nu_q^{\mathscr{D}_D}$ *is a* (g_D,G_D)*-full measure for* $\alpha_D(q)$*.*

(2) *For every* $p\in\mathbb{R}$*, we have* $\nu_p^{\mathscr{E}_E}(K^{g_E}_{\alpha_E(p)})=1$ *and*

$$h_{\nu_p^{\mathscr{E}_E}}(f,\xi,x)=T_E(p)+p\,\alpha_E(p)$$

for $\nu_p^{\mathscr{E}_E}$*-almost all* $x\in K^{g_E}_{\alpha_E(p)}$ *and every* $\xi\in\mathfrak{B}_f$*. Moreover,* $\nu_p^{\mathscr{E}_E}$ *is a* (g_E,G_E)*-full measure for* $\alpha_E(p)$*.*

The first statement is proved in Ref. 10. The proof of the second statement is similar.

We now give a description of the spectra for Lyapunov exponents. In Ref. 6, Weiss effected a complete analysis of the spectrum $\mathscr{L}_D$.

Theorem 5.5. *For every* $\alpha\in\mathbb{R}$*, we have*

$$\mathscr{L}_D(\alpha)=\mathscr{D}_D^{(m_E)}(h/\alpha).$$

Moreover,

(1) *if* $m_E\neq m_D$*, then* $\mathscr{L}_D$ *is an analytic strictly convex function, defined on a closed interval containing* $h/\dim_H J$*;*

(2) *if* $m_E=m_D$*, then* $\mathscr{L}_D$ *is the delta function*

$$\mathscr{L}_D(\alpha)=\begin{cases} \dim_H J, & \text{if } \alpha=h/\dim_H J,\\ 0, & \text{if } \alpha\neq h/\dim_H J.\end{cases}$$

We now give a complete description of the spectrum $\mathscr{L}_E$.

Theorem 5.6. *For every* $\alpha\in\mathbb{R}$ *and* $\xi\in\mathfrak{B}_f$*, we have*

$$\mathscr{L}_E(\alpha)=\mathscr{E}_E^{(m_D)}(\alpha\,\dim_H J).$$

Moreover,

(1) *if* $m_D\neq m_E$*, then* $\mathscr{L}_E$ *is an analytic strictly convex function, defined on a closed interval containing* $h/\dim_H J$*;*

(2) *if* $m_D=m_E$*, then* $\mathscr{L}_E$ *is the delta function*

$$\mathscr{L}_E(\alpha)=\begin{cases} h, & \text{if } \alpha=h/\dim_H J,\\ 0, & \text{if } \alpha\neq h/\dim_H J.\end{cases}$$

Proof. Since f is conformal and of class $C^{1+\epsilon}$, there exist positive constants C_1 and C_2 such that for each $x=\chi(i_1 i_2\cdots)\in J$ and integer $n\geq 1$,

$$C_1\leq \frac{\operatorname{diam} R_{i_1\cdots i_n}}{\prod_{k=0}^{n-1} a(f^k x)^{-1}}\leq C_2.$$

Since m_D is the Gibbs measure for $-\dim_H J\cdot\log a$, for every $x\in J$ we have

$$d_{m_D}(x)=\lim_{n\to\infty}\frac{\log m_D(R_{i_1\cdots i_n})}{\log \operatorname{diam} R_{i_1\cdots i_n}}$$

$$=\lim_{n\to\infty}\frac{\log m_D(R_{i_1\cdots i_n})}{\log \prod_{k=0}^{n-1} a(f^k x)^{-1}}=\dim_H J.$$

Therefore, for every $x\in J\cap K_\alpha^{g_L}$ and $\xi\in\mathfrak{B}_f$,

$$h_{m_D}(f,\xi,x)=\lim_{n\to\infty}-\frac{1}{n}\log m_D(R_{i_1\cdots i_n})$$

$$=\dim_H J\ \lim_{n\to\infty}-\frac{1}{n}\log \operatorname{diam} R_{i_1\cdots i_n}$$

$$=\alpha\,\dim_H J$$

and, hence, $x\in K^{g_E}_{\alpha\,\dim_H J}$. This implies that

$$\mathscr{L}_E(\alpha)=\mathscr{E}_E^{(m_D)}(\alpha\,\dim_H J).$$

Property 1 follows from Theorem 5.3. To obtain property 2 observe that for every $x\in J$,

$$h_{m_E}(f,\xi,x)=\lim_{n\to\infty}-\frac{1}{n}\log m_E(R_{i_1\cdots i_n})=h.$$

This completes the proof of the theorem. □

We now describe the full measures for the spectra $\mathscr{L}_D$ and $\mathscr{L}_E$. In the case of $\mathscr{L}_D$ these are the Gibbs measures $\nu_r^{\mathscr{L}_D}$ corresponding to the functions

$$-T_{\mathscr{L}_D}(r)\log a - rh, \tag{9}$$

where $T_{\mathscr{L}_D}(r)$ is chosen to satisfy $P(-T_{\mathscr{L}_D}(r)\log a) = rh$. Similarly, the full measures for $\mathscr{L}_E$ are the Gibbs measures $\nu_s^{\mathscr{L}_E}$ corresponding to the functions

$$-T_{\mathscr{L}_E}(s) + s \dim_H J \cdot \log a, \tag{10}$$

where $T_{\mathscr{L}_E}(s) = P(s \dim_H J \cdot \log a)$. One can check that $T_{\mathscr{L}_D}(r)$ and $T_{\mathscr{L}_E}(s)$ are real analytic functions. We set

$$\alpha_{\mathscr{L}_D}(r) = -T'_{\mathscr{L}_D}(r) \quad \text{and} \quad \alpha_{\mathscr{L}_E}(s) = -T'_{\mathscr{L}_E}(s).$$

Theorem 5.7. *The following properties hold:*

(1) *For every $r \in \mathbb{R}$, we have $\nu_r^{\mathscr{L}_D}(K^{g_L}_{\alpha_{\mathscr{L}_D}(r)}) = 1$ and*
$$d_{\nu_r^{\mathscr{L}_D}}(x) = T_{\mathscr{L}_D}(r) + r\alpha_{\mathscr{L}_D}(r)$$
for $\nu_r^{\mathscr{L}_D}$-almost all $x \in K^{g_L}_{\alpha_{\mathscr{L}_D}(r)}$. Moreover, $\nu_r^{\mathscr{L}_D}$ is a (g_L, G_D)-full measure for $\alpha_{\mathscr{L}_D}(r)$.

(2) *For every $s \in \mathbb{R}$, we have $\nu_s^{\mathscr{L}_E}(K^{g_L}_{\alpha_{\mathscr{L}_E}(s)}) = 1$ and*
$$h_{\nu_s^{\mathscr{L}_E}}(f,\xi,x) = T_{\mathscr{L}_E}(s) + s\alpha_{\mathscr{L}_E}(s)$$
for $\nu_s^{\mathscr{L}_E}$-almost all $x \in K^{g_L}_{\alpha_{\mathscr{L}_E}(s)}$ and every $\xi \in \mathfrak{B}_f$. Moreover, $\nu_s^{\mathscr{L}_E}$ is a (g_L, G_E)-full measure for $\alpha_{\mathscr{L}_E}(s)$.

(3) *$\mathscr{L}_D \sim \mathscr{L}_E$, i.e., $\nu_r^{\mathscr{L}_D} = \nu_s^{\mathscr{L}_E}$, where the parametrizations $r = r(s) = T_{\mathscr{L}_E}(s)/h$ and $s = s(r) = -T_{\mathscr{L}_D}(r)/\dim_H J$ are strictly monotonic and analytic.*

Proof. We note that

$$\begin{aligned} d_{\nu_r^{\mathscr{L}_D}}(x) &= \lim_{n\to\infty} \frac{\log \nu_r^{\mathscr{L}_D}(R_{i_1\cdots i_n})}{\log \operatorname{diam} R_{i_1\cdots i_n}} \\ &= \lim_{n\to\infty} \frac{T_{\mathscr{L}_D}(r)\log(\prod_{k=0}^{n-1} a(f^k x)^{-1}) - rhn}{\log \operatorname{diam} R_{i_1\cdots i_n}} \\ &= T_{\mathscr{L}_D}(r) - rh/\lambda(x) \\ &= T_{\mathscr{L}_D}(r) + rh \Big/ \int_J \log a \, d\nu_r^{\mathscr{L}_D}. \end{aligned}$$

Since $P(-T_{\mathscr{L}_D}(r)\log a) = rh$, taking derivatives with respect to r we obtain

$$\alpha_{\mathscr{L}_D}(r) \int_J \log a \, d\nu_r^{\mathscr{L}_D} = h,$$

and, hence,

$$d_{\nu_r^{\mathscr{L}_D}}(x) = T_{\mathscr{L}_D}(r) + r\alpha_{\mathscr{L}_D}(r)$$

for $\nu_r^{\mathscr{L}_D}$-almost all $x \in K^{g_L}_{\alpha_{\mathscr{L}_D}(r)}$. The remaining properties in statement 1 can be easily checked.

In a similar way, we obtain

$$\begin{aligned} h_{\nu_s^{\mathscr{L}_E}}(f,\xi,x) &= T_{\mathscr{L}_E}(s) - s \dim_H J \cdot \lambda(x) \\ &= T_{\mathscr{L}_E}(s) - s \dim_H J \int_J \log a \, d\nu_s^{\mathscr{L}_E}. \end{aligned}$$

Since $\alpha_{\mathscr{L}_E}(s) = -\dim_H J \int_J \log a \, d\nu_s^{\mathscr{L}_E}$, we conclude statement 2.

Statement 3 follows immediately from the definitions. □

Remarks

(1) In Ref. 10, Pesin and Weiss proved that the μ-measure of any ball centered at any point of J is positive, and for every $q \in \mathbb{R}$ we have
$$\begin{aligned} T_D(q) &= (1-q)HP_\mu(q) = (1-q)R_\mu(q) \\ &= -\lim_{r\to 0} \frac{1}{\log r} \log \inf \sum_{B \in \mathfrak{D}_r} \mu(B)^q, \end{aligned}$$
where the infimum is taken over all finite covers $\mathfrak{D}_r$ of J by balls of radius r.

(2) All the results in this section extend to continuous expanding maps (see Ref. 2 for the definition). Similar results can also be obtained for hyperbolic sets (see Ref. 13).

It is an open problem in dimension theory to obtain a description of the spectra $\mathscr{D}_E$ and $\mathscr{E}_D$ for Gibbs measures on repellers of conformal smooth expanding maps.

VI. MULTIFRACTAL RIGIDITY I

We will show in Theorem 6.2 that if, for instance, the spectra $\mathscr{D}_D$ and $\mathscr{E}_E$ are equivalent, with respect to the ''canonical'' families of measures $\{\nu_q^{\mathscr{D}_D}\}_{q\in\mathbb{R}}$ and $\{\nu_p^{\mathscr{E}_E}\}_{p\in\mathbb{R}}$, then ''all other'' spectra are equivalent. We call this phenomenon *multifractal rigidity*. It indicates that the spectra $\mathscr{D}_D$ and $\mathscr{E}_E$ are essentially independent.

We recall that two functions φ_1 and φ_2 on X are called *cohomologous* (with respect to f) if there exist a Hölder continuous function $\eta: X \to \mathbb{R}$ and a constant κ such that

$$\varphi_1 - \varphi_2 = \eta - \eta \circ f + \kappa.$$

In this case we write $\varphi_1 \sim \varphi_2$. We recall some well-known properties of cohomologous functions (see, for example, Ref. 9).

Proposition 6.1. *Let φ_1 and φ_2 be Hölder continuous functions on X. Then the following properties hold:*

(1) *if $\varphi_1 \sim \varphi_2$, then for every $x \in X$, we have*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} [\varphi_1(f^k x) - \varphi_2(f^k x)] = \kappa;$$

(2) *if $\varphi_1 \sim \varphi_2$, then $\varphi_1 - \varphi_2 \sim 0$, $c\varphi_1 \sim c\varphi_2$, and $\varphi_1 \sim \varphi_2 + c$ for any real number c;*

(3) *$\varphi_1 \sim \varphi_2$ if and only if the equilibrium measures corresponding to φ_1 and φ_2 on X coincide.*

Let $f: M \to M$ be a conformal $C^{1+\epsilon}$ expanding map and let φ be a Hölder continuous function on M. We consider the "canonical" families of full measures $\{\nu_q^{\mathscr{D}_D}\}_{q\in\mathbb{R}}$, $\{\nu_p^{\mathscr{E}_E}\}_{p\in\mathbb{R}}$, $\{\nu_r^{\mathscr{L}_D}\}_{r\in\mathbb{R}}$, and $\{\nu_s^{\mathscr{L}_E}\}_{s\in\mathbb{R}}$, for the spectra $\mathscr{D}_D$, $\mathscr{E}_E$, $\mathscr{L}_D$, and $\mathscr{L}_E$, respectively.

Theorem 6.2. *Assume that* $\log a \not\sim 0$ (*i.e.*, $m_D \neq m_E$). *Consider the following five pairs of multifractal spectra*

$$(\mathscr{E}_E,\mathscr{D}_D),\quad (\mathscr{E}_E,\mathscr{L}_D),\quad (\mathscr{E}_E,\mathscr{L}_E),$$

$$(\mathscr{D}_D,\mathscr{L}_D),\quad (\mathscr{D}_D,\mathscr{L}_E).$$

If in at least one of these pairs the two spectra are equivalent, then

(1) $\log\psi \sim \tau\log a$ *for some constant* $\tau\neq 0$;
(2) $\mathscr{D}_D\sim\mathscr{E}_E\sim\mathscr{L}_D\sim\mathscr{L}_E$, *i.e.*, $\nu_q^{\mathscr{D}_D}=\nu_p^{\mathscr{E}_E}=\nu_r^{\mathscr{L}_D}=\nu_s^{\mathscr{L}_E}$, *where the parametrizations*

$$p=-\frac{T_D(q)}{\tau}+q,\quad r=\frac{qP(\tau\log a)}{h},\quad s=\frac{-T_D(q)+q\tau}{\dim_H J}$$

are analytic coordinate transformations;
(3) $T_{\mathscr{L}_D}(r)=T_D(q)-q\tau=-s\ \dim_H J$,

and

$$T_{\mathscr{L}_E}(s)=qP(\tau\log a)=rh.$$

Proof. We assume that $\mathscr{D}_D\sim\mathscr{E}_E$. The proofs in the other cases are similar. There exists a parametrization $p=\pi(q)$ such that

$$-T_D(q)\log a+q\log\psi\sim -T_E(p)+p\log\psi.$$

It follows that

$$-T_D(q)\log a\sim(\pi(q)-q)\log\psi.$$

Since $\log a\not\sim 0$, we have that the function $-T_D(q)$ is strictly monotonic, analytic, and convex. Moreover, $T_D(q)\neq 0$ if $q\neq 1$.

Furthermore, there exists $q\neq 1$ for which $\pi(q)\neq q$ (otherwise, $\log a\sim 0$). It follows that

$$\frac{-T_D(q)}{\pi(q)-q}\log a\sim\log\psi.$$

Set

$$\tau=-T_D(q)/(\pi(q)-q)$$

and notice that $\tau\neq 0$ is a constant independent of $q\neq 1$. Otherwise, $\tau(q_1)\log a\sim\tau(q_2)\log a$ for some $q_1\neq q_2$ which is impossible because $\log a\not\sim 0$. We conclude that

$$\log\psi\sim\tau\log a.$$

Therefore, $\nu_q^{\mathscr{D}_D}$ is the Gibbs measure corresponding to the function

$$-T_D(q)\log a+q(\tau\log a-P(\tau\log a)),\tag{11}$$

where $T_D(q)$ satisfies

$$P((-T_D(q)+q\tau)\log a)=qP(\tau\log a).\tag{12}$$

Consider

$$r=r(q)=\frac{qP(\tau\log a)}{h}$$

and

$$s=s(q)=\frac{-T_D(q)+q\tau}{\dim_H J}.$$

These functions are strictly monotonic and analytic.

Substituting $r=r(q)$ into (12) we obtain that $T_{\mathscr{L}_D}(r)=T_D(q)-q\tau$. Comparing (9) and (11) we conclude that $\nu_r^{\mathscr{L}_D}=\nu_q^{\mathscr{D}_D}$. In a similar way, substituting $s=s(q)$ into (12) and comparing (10) and (11), we obtain that $T_{\mathscr{L}_E}(s)=qP(\tau\log a)$ and that $\nu_s^{\mathscr{L}_E}=\nu_q^{\mathscr{D}_D}$. The remaining statements in properties 2 and 3 follow from property 3 in Theorem 5.7. □

We now describe the case when $\log a\sim 0$, i.e., when $m_D=m_E$.

Theorem 6.3. *The following statements are equivalent:*

(1) $\log a\sim 0$;
(2) *the spectrum* $\mathscr{L}_D$ *is a delta function;*
(3) *the spectrum* $\mathscr{L}_E$ *is a delta function;*
(4) $\mathscr{E}_D\sim\mathscr{D}_D\sim\mathscr{E}_E\sim\mathscr{D}_E$ *with respect to the family of measures* $\{\nu_\alpha^{\mathscr{D}_D}\}_{\alpha\in\mathbb{R}}$, *and* $\nu_p^{\mathscr{D}_D}=\nu_p^{\mathscr{E}_E}$ *for every* p.

Proof. We first assume that $\log a\sim 0$. It follows from statements 2 and 3 in Proposition 6.1 that

$$-T_D(p)\log a+p\log\psi\sim p\log\psi\sim -T_E(p)+p\log\psi\tag{13}$$

and, hence, $\nu_p^{\mathscr{D}_D}=\nu_p^{\mathscr{E}_E}$. This implies that $\mathscr{D}_D\sim\mathscr{E}_E$. On the other hand, it follows from statement 1 in Proposition 6.1 that $\lambda(x)=\kappa$ for every $x\in J$ and some constant κ, because $\log a\sim 0$. This implies that the spectra $\mathscr{L}_D$ and $\mathscr{L}_E$ are delta functions. Moreover, $K_\alpha^{g_E}=K_{\alpha\kappa}^{g_D}$ for every number α. This implies that $\mathscr{E}_D\sim\mathscr{D}_D$ and that $\mathscr{E}_E\sim\mathscr{D}_E$, and the proof of statement 4 is complete.

To complete the proof of the theorem it is sufficient to prove that statement 4 implies that $\log a\sim 0$. But $\nu_p^{\mathscr{D}_D}=\nu_p^{\mathscr{E}_E}$ implies that (13) holds and hence that $\log a\sim 0$. □

VII. MULTIFRACTAL RIGIDITY II

We now consider another interesting phenomenon in dimension theory of dynamical systems that we regard as a multifractal rigidity phenomenon. Roughly speaking, it states that if two dynamical systems are topologically equivalent (via a homeomorphism) and some of their multifractal spectra coincide, then they are smoothly equivalent (via a diffeomorphism). This leads to a classification of maps and Gibbs measures using multifractal spectra and/or multifractal decompositions. We believe that this type of classification fits well with the "physical" interpretation of the equivalence of dynamical systems. We think that this is a nontrivial and challenging problem, and we support that belief by the following observations.

Let f be a one-dimensional linear Markov map of the unit interval, modeled by the full shift on two symbols. This means that there are linear maps f_1 and f_2 defined, respectively, on two disjoint closed intervals I_1, $I_2\subset[0,1]$ such that

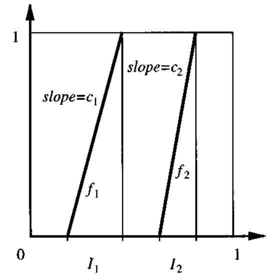

FIG. 5. A one-dimensional linear Markov map of the unit interval.

$f_1(I_1)=f_2(I_2)=[0,1]$, and the map $f\colon I_1\cup I_2\to\mathbb{R}$ is given by $f(x)=f_i(x)$ whenever $x\in I_i$, for $i=1, 2$ (see Figure 5).

We consider the f-invariant set

$$J=\bigcap_{k=1}^{\infty} f^{-k}(I_1\cup I_2).$$

Clearly, f extends to a C^∞ map on an open neighborhood of J. Moreover, $f|_J$ is conformal and J is a repeller of f.

The partition $\{J\cap I_1, J\cap I_2\}$ is a Markov partition of J (with respect to f) and $f|_J$ is topologically conjugate to the full shift $\sigma|_{\Sigma_2^+}$, where $\Sigma_2^+=\{1,2\}^{\mathbb{N}}$.

We consider the Bernoulli measure on Σ_2^+ with probabilities β_1 and $\beta_2=1-\beta_1$ (which is a Gibbs measure), and let $c_i=|f'|_{I_i}|=|f_i'|_{I_i}|$ for $i=1,2$.

We define the functions a and φ on J by

$$a(x)=c_i \quad\text{and}\quad \varphi(x)=\log\beta_i \quad \text{if } x\in I_i, \tag{14}$$

for $i=1, 2$. For every p, $q\in\mathbb{R}$, the functions $T_E(p)$ and $T_D(q)$ satisfy the identities

$$e^{-T_E(p)}(\beta_1{}^p+\beta_2{}^p)=1$$

and

$$c_1^{-T_D(q)}\beta_1{}^q+c_2^{-T_D(q)}\beta_2{}^q=1. \tag{15}$$

One can explicitly compute the measures $\nu_p^{\mathscr{E}_E}$ and $\nu_q^{\mathscr{D}_D}$: they are the Bernoulli measures with probabilities $e^{-T_E(p)}\beta_1{}^p$ and $e^{-T_E(p)}\beta_2{}^p$, and with probabilities $c_1^{-T_D(q)}\beta_1{}^q$ and $c_2^{-T_D(q)}\beta_2{}^q$, respectively.

Let f and $\hat{f}$ be two one-dimensional linear Markov maps of the unit interval, as above, with conformal repellers

$$J=\bigcap_{k=1}^{\infty} f^{-k}(I_1\cup I_2) \quad\text{and}\quad \hat{J}=\bigcap_{k=1}^{\infty}\hat{f}^{-k}(\hat{I}_1\cup\hat{I}_2),$$

respectively. The underlying symbolics dynamics of $f|_J$ and $\hat{f}|_{\hat{J}}$ coincide and are the full shift on two symbols $\sigma|_{\Sigma_2^+}$. Let $\chi\colon\Sigma_2^+\to J$ and $\hat{\chi}\colon\Sigma_2^+\to\hat{J}$ be the corresponding coding maps. We consider two Bernoulli measures μ and $\hat{\mu}$ on Σ_2^+ with probabilities β_1 and β_2, and with probabilities $\hat{\beta}_1$ and $\hat{\beta}_2$, respectively, where $\beta_1+\beta_2=\hat{\beta}_1+\hat{\beta}_2=1$. We also consider the numbers c_1, c_2 and the numbers $\hat{c}_1$, $\hat{c}_2$, which are the absolute values of the derivatives of the linear pieces of f and $\hat{f}$, respectively.

We define the functions a and φ on J by (14), as well as the functions $\hat{a}$ and $\hat{\varphi}$ on $\hat{J}$ by

$$\hat{a}(x)=\hat{c}_i \quad\text{and}\quad \hat{\varphi}(x)=\log\hat{\beta}_i \quad\text{if } x\in\hat{I}_i,$$

for $i=1, 2$. We note that the measures μ and $\hat{\mu}$ are the Gibbs measures for φ and $\hat{\varphi}$.

Recall that an *automorphism* ρ of Σ_2^+ is a homeomorphism $\rho\colon\Sigma_2^+\to\Sigma_2^+$ which commutes with the shift map σ. The *involution automorphism* is defined by $\rho(i_1i_2\cdots)=(i_1'i_2'\cdots)$, where $i_n'=2$ if $i_n=1$, and $i_n'=1$ if $i_n=2$, for each integer $n\geq 1$.

Since χ and $\hat{\chi}$ are invertible, one can define a homeomorphism $\theta\colon J\to\hat{J}$ by $\theta=\hat{\chi}\circ\chi^{-1}$. We note that $\theta\circ f=\hat{f}\circ\theta$ on J, and hence, θ is a *topological conjugacy* between $f|_J$ and $\hat{f}|_{\hat{J}}$, i.e., the two maps are *topologically equivalent*. If ρ is an automorphism of Σ_2^+, then the homeomorphism $\theta'=\hat{\chi}\circ\rho\circ\chi^{-1}$ is also a topological conjugacy between $f|_J$ and $\hat{f}|_{\hat{J}}$, and all topological conjugacies are of this form. An important question is whether the class of all conjugacies contains a homeomorphism ζ preserving the differentiable structure, i.e., $a=\hat{a}\circ\zeta$. One can ask if, in addition, ζ is measure preserving, i.e., $\mu=\hat{\mu}\circ\zeta$. We give below a complete answer to these questions.

We consider the spectra $\mathscr{D}_D=\mathscr{D}_D^{(\mu)}$ and $\mathscr{E}_E=\mathscr{E}_E^{(\mu)}$ specified by the measure μ, as well as the spectra $\hat{\mathscr{D}}_D=\mathscr{D}_D^{(\hat{\mu})}$ and $\hat{\mathscr{E}}_E=\mathscr{E}_E^{(\hat{\mu})}$ specified by the measure $\hat{\mu}$. Similarly, we consider the spectra $\mathscr{L}_D$ and $\mathscr{L}_E$, as well as the spectra $\hat{\mathscr{L}}_D$ and $\hat{\mathscr{L}}_E$.

As before, we use the functions a and φ to define the full measures $\nu_q^{\mathscr{D}_D}$, $\nu_p^{\mathscr{E}_E}$, $\nu_r^{\mathscr{L}_D}$, and $\nu_s^{\mathscr{L}_E}$. In a similar way, we use the functions $\hat{a}$ and $\hat{\varphi}$ to define the full measures $\nu_q^{\hat{\mathscr{D}}_D}$, $\nu_p^{\hat{\mathscr{E}}_E}$, $\nu_r^{\hat{\mathscr{L}}_D}$, and $\nu_s^{\hat{\mathscr{L}}_E}$.

Theorem 7.1. *If $\mathscr{D}_D(\alpha)=\hat{\mathscr{D}}_D(\alpha)$ for every α and these spectra are not delta functions, then there is a homeomorphism $\zeta\colon J\to\hat{J}$ such that*

1. *$\zeta\circ f=\hat{f}\circ\zeta$ on J, that is, ζ is a topological conjugacy between $f|_J$ and $\hat{f}|_{\hat{J}}$;*
2. *the automorphism ρ of Σ_2^+ satisfying $\zeta\circ\chi=\hat{\chi}\circ p$ is either the identity or the involution automorphism;*
3. *$a=\hat{a}\circ\zeta$, $\varphi=\hat{\varphi}\circ\zeta$, and $\mu=\hat{\mu}\circ\zeta$;*
4. *$\nu_q^{\mathscr{D}_D}=\nu_q^{\hat{\mathscr{D}}_D}\circ\zeta$, $\nu_p^{\mathscr{E}_E}=\nu_p^{\hat{\mathscr{E}}_E}\circ\zeta$, $\nu_r^{\mathscr{L}_D}=\nu_r^{\hat{\mathscr{L}}_D}\circ\zeta$, and $\nu_s^{\mathscr{L}_E}=\nu_s^{\hat{\mathscr{L}}_E}\circ\zeta$ for every q, p, r, and s.*

Proof. It is enough to prove that the spectrum $\mathscr{D}_D$ uniquely determines the numbers β_1, β_2, c_1, and c_2 up to a permutation of the indices 1 and 2.

By the uniqueness of the Legendre transform, the spectrum $\mathscr{D}_D$ uniquely determines $T_D(q)$ for every $q\in\mathbb{R}$ and, hence, it is enough to prove that equation (15) uniquely determines the numbers β_1, β_2, c_1, and c_2 up to a permutation of the indices 1 and 2.

One can verify that the numbers $\alpha_\pm=\alpha_D(\pm\infty)$ can be computed by

$$\alpha_\pm=-\lim_{q\to\pm\infty}(T_D(q)/q).$$

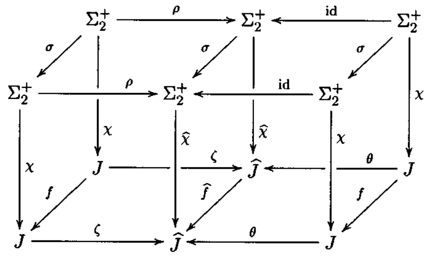

FIG. 6. The diagram gives a complete picture of the relation between the conjugacies θ and ζ.

We observe that, since the spectrum $\mathscr{D}_D$ is not a delta function and, hence, $T_D(q)$ is not linear and is strictly convex, then $\alpha_+ < \dim_H J < \alpha_-$. Therefore, raising both sides of (15) to the power $1/q$ and letting $q \to \pm\infty$, we obtain

$$\max\{\beta_1 c_1^{\alpha_+}, \beta_2 c_2^{\alpha_+}\} = \min\{\beta_1 c_1^{\alpha_-}, \beta_2 c_2^{\alpha_-}\} = 1.$$

We assume that $\beta_1 c_1^{\alpha_+} = 1$ (the case when $\beta_2 c_2^{\alpha_+} = 1$ can be treated in a similar way; in this case, ρ is the involution automorphism). Since $\alpha_+ < \alpha_-$, we must have $\beta_2 c_2^{\alpha_-} = 1$.

Setting $q=0$ and $q=1$ in equation (15), we obtain, respectively,

$$c_1^{-\dim_H J} + c_2^{-\dim_H J} = 1 \quad \text{and} \quad \beta_1 + \beta_2 = 1.$$

Set $x = c_1^{-\dim_H J}$, $a = \alpha_+/\dim_H J < 1$, and $b = \alpha_-/\dim_H J > 1$. Then, one can easily derive the equation

$$x^a + (1-x)^b = 1.$$

One can verify with standard calculus arguments that this equation has a unique solution $x \in (0,1)$, which uniquely determines the numbers c_1 and c_2 and, hence, also the numbers β_1 and β_2.

It follows from statement 2 in Theorem 7.1 that Figure 6 is commutative. A more general version of Theorem 7.1 can be found in Ref. 14.

Remarks

(1) Let f be a one-dimensional linear Markov map of the unit interval. It is straight-forward to verify that
 (a) $\mathscr{E}_E$ is a delta function if and only if $\beta_1 = \beta_2 = 1/2$;
 (b) $\mathscr{D}_D$ is a delta function if and only if $\log c_1/\log \beta_1 = \log c_2/\log \beta_2$;
 (c) $\mathscr{E}_E \sim \mathscr{D}_D$ if and only if $c_1 = c_2$.
 Notice that none of these three properties specifies all the numbers β_1, β_2, c_1, and c_2. On the other hand, the dynamical system is completely specified if and only if all these four numbers are known. In particular, we conclude that knowing any one of the three properties above is *not* sufficient to restore the system.

(2) We have shown that for a one-dimensional linear Markov map of the unit interval, one can determine the four numbers β_1, β_2, c_1, and c_1 using the spectrum $\mathscr{D}_D$. However, from the spectrum $\mathscr{E}_E$ one can only recover the two numbers β_1 and β_2: one can show that if $\beta_1 \geq \beta_2$, then

$$\beta_1 = \exp \lim_{p \to +\infty} (T_E(p)/p)$$

and

$$\beta_2 = \exp \lim_{p \to -\infty} (T_E(p)/p).$$

In a similar way, using one of the spectra $\mathscr{L}_D$ and $\mathscr{L}_E$, one can only recover the two numbers c_1 and c_2. For example, using the spectrum $\mathscr{L}_E$, one can prove that if $c_1 \geq c_2$, then

$$c_1 = \exp \lim_{s \to +\infty} (T_{\mathscr{L}_E}(s)/s)$$

and

$$c_2 = \exp \lim_{s \to -\infty} (T_{\mathscr{L}_E}(s)/s).$$

(3) Given two multifractal spectra $\mathscr{F}_1$ and $\mathscr{F}_2$ for the same map f, the condition $\mathscr{F}_1 \sim \mathscr{F}_2$ means that we can reparametrize the "physical" variables used to describe the dynamics and, hence, their multifractal decompositions are equal. On the other hand, if $\mathscr{F}$ and $\hat{\mathscr{F}}$ are two multifractal spectra for the maps f and $\hat{f}$, respectively, the condition $\mathscr{F} = \hat{\mathscr{F}}$ indicates the existence of a "symmetry" between the two dynamical systems (expressed by the existence of a homeomorphism ζ). Thus, it is a requirement of "physical" nature, and Theorem 7.1 has a strong physical content: if the spectra of two dynamical systems are equal, then the systems are the same up to a change of variables, and thus should be considered the same from the physical point of view.

Another fundamental open problem of multifractal rigidity is whether one can determine the main "macroscopic" characteristics of a given dynamical system using information "hidden" in its multifractal spectra. In particular, one can ask whether any subset of the six spectra is sufficient to "determine" the functions a and φ. For the class of one-dimensional linear Markov maps of the unit interval, the phenomena described in Theorem 7.1 and in Remark 2 give a complete affirmative answer to this question.

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APPENDIX: BASIC NOTIONS

Let (X,d) be a complete separable metric space. Consider a set $Z \subset X$ and a positive number δ. A cover of Z by sets of diameter at most δ is called a *δ-cover* of Z. For any $s>0$, we define the *s-dimensional Hausdorff measure* of Z by

$$m_H(Z,s) = \lim_{\delta \to 0} \inf_{\mathfrak{U}} \sum_{U \in \mathfrak{U}} (\operatorname{diam} U)^s,$$

where the infimum is taken over all finite or countable δ-covers $\mathfrak{U}$ of Z. There exists a unique value of s at which

$m_H(Z,s)$ jumps from $+\infty$ to 0. We call this value the *Hausdorff dimension* of Z and denote it by $\dim_H Z$. We have

$$\dim_H Z = \inf\{s : m_H(Z,s)=0\} = \sup\{s : m_H(Z,s)=+\infty\}.$$

Let $f: X \to X$ be a continuous map. If $\mathfrak{U}$ is a finite open cover of X, for each integer $n \geqslant 1$ we denote by $S_n(\mathfrak{U})$ the collection of strings $\mathbf{U}=U_1 \cdots U_n$, where $U_1,\ldots,U_n \in \mathfrak{U}$. For each $\mathbf{U} \in S_n(\mathfrak{U})$, we write $n(\mathbf{U})=n$ and define the open set

$$X(\mathbf{U}) = \{x \in X : f^{k-1}x \in U_k \ \text{ for } \ k=1,\ldots,n\}.$$

Consider a set $Z \subset X$. We say that a collection of strings Γ *covers* Z if the union $\cup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$. For every real number s, we define

$$M(Z,s,\mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-n(\mathbf{U})s),$$

where the infimum is taken over all collections $\Gamma \subset \cup_{k \geqslant n} S_k(\mathfrak{U})$ covering Z. There exists a unique value of s at which $M(Z,s,\mathfrak{U})$ jumps from $+\infty$ to 0, given by

$$h(Z,\mathfrak{U}) = \inf\{s : M(Z,s,\mathfrak{U})=0\} = \sup\{s : M(Z,s,\mathfrak{U})=+\infty\}.$$

We define the *topological entropy* of f on the set Z by

$$h(f|_Z) = \lim_{\operatorname{diam}\mathfrak{U}\to 0} h(Z,\mathfrak{U})$$

(one can show that the limit always exists). If Z is compact and f-invariant, then $h(f|_Z)$ coincides with the classical topological entropy (see, for example, Ref. 2). However, the set Z need not be compact nor f-invariant for our definition.

The following simple statement follows from the special type of metric on Σ_A^+ introduced in Section IV.

Lemma. *For any subset* $Z \subset \Sigma_A^+$ *we have* $h(f|_Z) = \dim_H Z \cdot \log a$.

For each $n \in \mathbf{N}$, we define the metric d_n on X by

$$d_n(x,y) = \max\{d(f^k x, f^k y) : 0 \leqslant k \leqslant n-1\}.$$

Given $\delta > 0$, we say that a finite set $E \subset X$ is a (n,δ)-*separated set* if $d_n(x,y) > \delta$ whenever $x, y \in E$ and $x \neq y$. We define the *topological pressure* of the continuous function $\varphi: X \to \mathbb{R}$ (with respect to f) by

$$P(\varphi) = \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k x),$$

where the supremum is taken over all (n,δ)-separated sets E.

The *Legendre transform* of the function T is the function $\mathscr{D}$ defined by $\mathscr{D}(\alpha) = \sup_q(\alpha q - T(q))$. We then say that the pair $(\mathscr{D},T)$ is a *Legendre pair* with respect to the variables α, q. We say that a C^2 function T is *strictly convex* if $T''>0$ everywhere on its domain. Given two strictly convex C^2 functions $\mathscr{D}$ and T, one can show that the pair $(\mathscr{D},T)$ is a Legendre pair with respect to the variables α, q if and only if $\mathscr{D}(\alpha) = T(q) + q\alpha$, where $\alpha = -T'(q)$ and $q = \mathscr{D}'(\alpha)$.

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