JOURNAL OF DYNAMICAL AND CONTROL SYSTEMS, Vol. 3, No. 1, 1997, 33-49

MULTIFRACTAL SPECTRA AND MULTIFRACTAL RIGIDITY FOR HORSESHOES

L. BARREIRA, YA. PESIN, J. SCHMELING

ABSTRACT. We discuss a general concept of multifractality, and give a complete description of the multifractal spectra for Gibbs measures on two-dimensional horseshoes. We discuss a multifractal characterization of surface diffeomorphisms.

INTRODUCTION

The multifractal analysis has been recently developed as a new powerful tool to study dynamical systems. Its main constituent component — dimension spectra — capture information about various dimensions associated with the dynamics. Among them are the Hausdorff dimension, correlation dimension, and information dimension of invariant measures.

Dimension spectra are examples of more general multifractal spectra that we have introduced in [4]. They provide information on the distribution of pointwise dimensions, local entropies, Lyapunov exponents, etc. In [4], we considered Gibbs measures invariant under conformal expanding maps and demonstrated that multifractal spectra can be used in a sense to "restore" the dynamics — the phenomenon that we call the *multifractal rigidity*. In this paper, we extend these results to Gibbs measures invariant under Axiom A surface diffeomorphisms.

1. A GENERAL CONCEPT OF MULTIFRACTALITY

1.1. Definitions. Let X be a set, and let $g\colon Y \to [-\infty, +\infty]$ be a function defined on a subset $Y \subset X$. The *level sets* of g,

$$K_\alpha^g = \{x \in X : g(x) = \alpha\} \quad \text{for } -\infty \le \alpha \le +\infty,$$

1991 *Mathematics Subject Classification.* 58F11, 58F12, 28D99, 28C99.

Key words and phrases. Gibbs measure, hyperbolic set, local entropy, Lyapunov exponents, multifractal analysis, multifractal rigidity, pointwise dimension.

This paper was written while L. B. and J. S. were visiting Penn State. J. S. was supported by the Leopoldina-Forderpreis. L. B. and Ya. P. were partially supported by the National Science Foundation, Grant No. DMS9403723.

are disjoint and produce a *multifractal decomposition* of X, that is,

$$X = \bigcup_{-\infty \leq \alpha \leq +\infty} K_\alpha^g \cup (X \setminus Y).$$

Let now G be a set function, i.e., a real function that is defined on subsets of X. Assume that $G(Z_1) \leq G(Z_2)$ if $Z_1 \subset Z_2$. We define the function $\mathcal{F}\colon [-\infty, +\infty] \to \mathbb{R}$ by

$$\mathcal{F}(\alpha) = G(K_\alpha^g).$$

We call $\mathcal{F}$ the *multifractal spectrum* specified by the pair of functions (g, G), or the (g, G)-multifractal spectrum. The function $\mathcal{F}$ captures important information about the structure of the set X generated by the function g.

Given $-\infty \leq \alpha \leq +\infty$, let ν_α be a probability measure on X such that $\nu_\alpha(K_\alpha^g) = 1$. If

$$\mathcal{F}(\alpha) = \inf\{G(Z) : \nu_\alpha(Z) = 1\},$$

we call ν_α a *(g, G)-full measure.* Constructing a one-parameter family of (g, G)-full probability measures ν_α seems to be the most effective way of studying multifractal decompositions.

We illustrate the general concept of multifractal spectra by studying several explicit spectra. We refer to [9] for references and more details.

1.2. Dimension and entropy spectra. Let X be a complete separable metric space and let $f\colon X \to X$ be a continuous map. We begin with the choice of the set function G. There are two "natural" set functions on X. The first one is defined by

$$G_D(Z) = \dim_H Z, \tag{1}$$

where $\dim_H Z$ is the Hausdorff dimension of the set $Z \subset X$. The second function is given by

$$G_E(Z) = h(f|Z), \tag{2}$$

where $h(f|Z)$ is the topological entropy of f on Z (see [9] for definitions). We call the multifractal spectra generated by the function G_D *dimension spectra*, and the multifractal spectra generated by the function G_E *entropy spectra*.

We now describe some "natural" choices for the function g.

1.3. Multifractal spectra for pointwise dimensions. Let μ be a Borel finite measure on X. Consider the subset $Y \subset X$ consisting of all points $x \in X$ for which the limit

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists, where $B(x,r)$ denotes the ball of radius r centered at x. The number $d_\mu(x)$ is called the pointwise dimension of μ at x. Whenever $x \in Y$, we say that the pointwise dimension of μ exists at the point x. We define the function g_D on Y by

$$g_D(x) = d_\mu(x).$$

We obtain two multifractal spectra $\mathcal{D}_D = \mathcal{D}_D^{(\mu)}$ and $\mathcal{D}_E = \mathcal{D}_E^{(\mu)}$ specified by the pairs of functions (g_D, G_D) and (g_D, G_E) respectively, where the set functions G_D and G_E are given by (1) and (2). These spectra are called multifractal spectra for (pointwise) dimensions. The spectrum $\mathcal{D}_D$ is known in the literature as the *dimension spectrum* or *$f_\mu(\alpha)$-spectrum for dimensions.*

In [3] (see also [2]), we prove the following statement.

Theorem 1.1. *If μ is a hyperbolic measure invariant under a $C^{1+\varepsilon}$ diffeomorphism, then the pointwise dimension of μ exists almost everywhere.*

The theorem implies that if μ is a hyperbolic measure invariant under a $C^{1+\varepsilon}$ diffeomorphism, then $\mu(X \setminus Y) = 0$. This claim is known as the *Eckmann–Ruelle conjecture.*

1.4. Multifractal spectra for local entropies. Let X be a complete separable metric space and let $f \colon X \to X$ be a continuous map preserving a Borel probability measure μ. Consider a finite measurable partition ξ of X. For every $n > 0$, we write $\xi_n = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n}\xi$, and denote by $\xi_n(x)$ the element of the partition ξ_n that contains the point x. Consider the set $Y = Y_\xi \subset X$ consisting of all points $x \in X$ for which the limit

$$h_\mu(f,\xi,x) = \lim_{n \to \infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists. We call $h_\mu(f,\xi,x)$ the *μ-local entropy* of f at the point x (with respect to ξ). Clearly, Y is f-invariant and $h_\mu(f,\xi,fx) = h_\mu(f,\xi,x)$ for every $x \in Y$. By the Shannon–McMillan–Breiman theorem, $\mu(X \setminus Y) = 0$. In addition, if ξ is a *generating partition* and μ is ergodic, then

$$h_\mu(f) = h_\mu(f,\xi,x)$$

for μ-almost all $x \in X$, where $h_\mu(f)$ is the measure-theoretic entropy of f (with respect to μ). We define the function g_E on Y by

$$g_E(x) = h_\mu(f,\xi,x).$$

Let us stress that g_E may depend on ξ. We obtain two multifractal spectra $\mathcal{E}_D = \mathcal{E}_D^{(\mu)}$ and $\mathcal{E}_E = \mathcal{E}_E^{(\mu)}$ specified by the pairs of functions (g_E, G_D) and (g_E, G_E), respectively, where the set functions are given by (1) and (2). These spectra are called *multifractal spectra for (local) entropies*. In Sec. 2 we will observe that in some situations these spectra, in fact, do not depend on ξ for a broad class of partitions.

We remark that in the study of the multifractal spectra for local entropies, the Shannon–McMillan–Breiman theorem plays the same role as Theorem 1.1 in the study of the multifractal spectra for pointwise dimensions. Namely, the first can be considered as the primary reason for the coincidence of the different definitions of entropy, due to Kolmogorov and Sinai, Katok, Brin and Katok, and Pesin (see [9] for references). Similarly, Theorem 1.1 implies the coincidence of all characteristics of dimension type.

1.5. Multifractal spectra for Lyapunov exponents. Let X be a differentiable manifold of dimension n, and let $f\colon X \to X$ be a C^1 map. Consider the subset $Y \subset X$ of all points $x \in X$ for which the limit

$$\lambda(x, v) = \lim_{n\to+\infty} \frac{1}{n} \log \|d_x f^n v\|$$

exists for every $v \in T_x X$. The number $\lambda(x, v)$ is called the *Lyapunov exponent* of v (specified by f) at the point x. By Oseledets' multiplicative ergodic theorem, if μ is an f- invariant Borel probability measure and $\log^+ \|df\| \in L^1(X, \mu)$, then $\mu(X \setminus Y) = 0$. Let $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ be the values of the Lyapunov exponents at x, counted with multiplicities. For each $i = 1, \cdots, n$, we define the function g_L^i on Y by

$$g_L^i(x) = \lambda_i(x).$$

For each $i = 1, \cdots, n$, we obtain two multifractal spectra $\mathcal{L}_D^i$ and $\mathcal{L}_E^i$ specified, respectively, by the pairs of functions (g_L^i, G_D) and (g_L^i, G_E), where the set functions G_D and G_E are given by (1) and (2). These spectra are called *multifractal spectra for Lyapunov exponents*.

2. Multifractal spectra of Gibbs measures for horseshoes

We consider Gibbs measures invariant under Axiom A surface diffeomorphisms and describe the associated multifractal spectra for dimensions, entropies, and Lyapunov exponents. Our approach is an extension and a modification of the approach in [4] for conformal expanding maps.

2.1. Preliminaries. Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism of a smooth manifold M, and let Λ be a compact locally maximal hyperbolic set for f. From now on we assume that $f|\Lambda$ is topologically mixing. The general case can be reduced to this one by using the Spectral Decomposition Theorem. See [6] for definitions.

For each $x \in M$, we write

$$a^u(x) = \|df|E^u(x)\| \quad \text{and} \quad a^s(x) = \|df|E^s(x)\|.$$

The functions a^s and a^u are Hölder continuous and satisfy $a^u(x) > 1$ and $a^s(x) < 1$ for every $x \in \Lambda$.

Let κ be a Hölder continuous function on Λ, and let ν be the unique Gibbs measure for κ with respect to $f|\Lambda$. We define the function η on Λ by $\log \eta = \kappa - P_\Lambda(\kappa)$.

2.2. Multifractal spectra for pointwise dimensions. It is an experimental fact that the orbit distribution in attractors of dynamical systems is often not uniform. Instead one can observe places of high and low density, sometimes called *hot* and *cold* spots. The same phenomenon has been observed for the more general class of hyperbolic attractors with singularities which includes the Lorenz attractor, the Lozi attractor, the Belikh attractor, etc. In the multifractal analysis of dynamical systems one encodes all the (experimental) data in multifractal spectra. It is a challenging problem to travel in the opposite direction and obtain information about the dynamical system from its multifractal spectra.

Let now Λ be a compact locally maximal hyperbolic set of a $C^{1+\varepsilon}$ diffeomorphism on a smooth surface. For each $q \in \mathbb{R}$, we define the function $\varphi^u_{D,q}$ on Λ by

$$\varphi^u_{D,q} = -T^u_D(q) \log a^u + q \log \eta,$$

where $T^u_D(q)$ is chosen such that

$$P_\Lambda(\varphi^u_{D,q}) = 0.$$

Similarly, for each $q \in \mathbb{R}$, we define the function $\varphi^s_{D,q}$ on Λ by

$$\varphi^s_{D,q} = -T^s_D(q) \log a^s + q \log \eta,$$

where $T^s_D(q)$ is chosen such that

$$P_\Lambda(\varphi^s_{D,q}) = 0.$$

It is obvious that the functions $\varphi^u_{D,q}$ and $\varphi^s_{D,q}$ are Hölder continuous. We set

$$T_D(q) = T^u_D(q) + T^s_D(q). \tag{3}$$

The Legendre transforms of T^u_D and T^s_D are denoted by $\mathcal{D}^u_D = \mathcal{D}^{u,(\mu)}_D$ and $\mathcal{D}^s_D = \mathcal{D}^{s,(\mu)}_D$ respectively.

The following properties are proved in [10].

Proposition 2.1. *The function T_D is real analytic and satisfies $T'_D(q) \leq 0$ and $T''_D(q) \geq 0$ for every $q \in \mathbb{R}$. We have $T_D(0) = \dim_H \Lambda$ and $T_D(1) = 0$.*

Set

$$\alpha_D(q) = -T'_D(q).$$

Let m^u_D be the equilibrium measure on Λ corresponding to the Hölder continuous function $t^u \log a^u$, where t^u is the unique root of Bowen's equation

$$P_\Lambda(t \log a^u) = 0.$$

Similarly, let m^s_D be the equilibrium measure on Λ corresponding to the Hölder continuous function $-t^s \log a^s$, where t^s is the unique root of Bowen's equation

$$P_\Lambda(-t \log a^s) = 0.$$

A measure μ is called a *measure of maximal dimension* for Λ if $\dim_H \Lambda = \dim_H \mu$.

Let $h_\nu(f)$ be the measure-theoretic entropy of f, and let λ^u_ν and λ^s_ν be the positive and negative Lyapunov exponents of ν. Simpelaere [11] and Pesin and Weiss [10] effected independently a multifractal analysis of the spectrum $\mathcal{D}_D$.

Theorem 2.2.

(1) *For ν-almost every $x \in \Lambda$, the pointwise dimension of ν at x exists and*

$$g_D(x) = d_\nu(x) = -\frac{\int_\Lambda \log \eta \, d\nu}{\int_\Lambda \log a^u \, d\nu} + \frac{\int_\Lambda \log \eta \, d\nu}{\int_\Lambda \log a^s \, d\nu} = \\ = h_\nu(f) \left(\frac{1}{\lambda^u_\nu} - \frac{1}{\lambda^s_\nu} \right).$$

(2) *The domain of the function $\alpha \mapsto \mathcal{D}_D(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_D(q)$. For every $q \in \mathbb{R}$, we have*

$$\mathcal{D}_D(\alpha_D(q)) = T_D(q) + q\alpha_D(q). \tag{4}$$

(3) *If ν is not a measure of maximal dimension, then $\mathcal{D}_D$ and T_D are strictly convex, and hence, $(\mathcal{D}_D, T_D)$ is a Legendre pair with respect to the variables α, q.*

(4) *If ν is a measure of maximal dimension, then $\mathcal{D}_D$ is the delta function*

$$\mathcal{D}_D(\alpha) = \begin{cases} \dim_H \Lambda & \text{if } \alpha = \dim_H \Lambda, \\ 0 & \text{if } \alpha \neq \dim_H \Lambda. \end{cases}$$

We note that if ν is an equilibrium measure of maximal dimension, then it is also an equilibrium measure both for $t^u \log a^u$ and $-t^s \log a^s$, and hence, $\nu = m_D^u = m_D^s$.

Statement 1 in Theorem 2.2 was first obtained by Young [14]. Setting $q = 0$ in (4), we obtain $\dim_H \Lambda = t^u + t^s$. This formula was first established by McCluskey and Manning [7] for C^1 diffeomorphisms. Other proofs were given independently by Takens [12], Barreira [1], and Pesin [9].

2.3. Multifractal spectra for local entropies. For each $p \in \mathbb{R}$, we set

$$T_E(p) = P_\Lambda(p \log \eta).$$

The following property is immediate from the definitions.

Proposition 2.3. *The function T_E is real analytic and satisfies $T_E'(p) \leq 0$ and $T_E''(p) \geq 0$ for every $p \in \mathbb{R}$. We have $T_E(0) = h$ and $T_E(1) = 0$.*

Set

$$\alpha_E(p) = -T_E'(p).$$

Any Markov partition is a generating partition. The same is true for any partition of Λ by rectangles obtained from a Markov partition (not necessarily all at the same level) and corresponding to disjoint cylinder sets in the underlying symbolic dynamics (see Appendix). We denote the class of such partitions by $\mathfrak{P}_f$.

We now give a full description of the spectrum $\mathcal{E}_E$ for Gibbs measures supported on locally maximal hyperbolic sets. Let m_E be the measure of maximal entropy, i.e., the unique equilibrium measure for $f|\Lambda$.

Theorem 2.4.

(1) *There exists a set $S \subset M$ with $\nu(S) = 1$ such that for every partition $\xi \in \mathfrak{P}_f$ and every $x \in \Lambda$, the local entropy of ν at x exists, does not depend on x and ξ, and*

$$g_E(x) = h_\nu(f, \xi, x) = -\int_\Lambda \log \eta \, d\nu.$$

(2) *The domain of the function $\alpha \mapsto \mathcal{E}_E(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_E(p)$. For every $p \in \mathbb{R}$, we have*

$$\mathcal{E}_E(\alpha_E(p)) = T_E(p) + p\alpha_E(p).$$

(3) *If $\nu \neq m_E$, then $\mathcal{E}_E$ and T_E are analytic strictly convex functions, and hence, $(\mathcal{E}_E, T_E)$ is a Legendre pair with respect to the variables α, q.*

(4) *If $\nu = m_E$, then $\mathcal{E}_E$ is the delta function*

$$\mathcal{E}_E(\alpha) = \begin{cases} h & \text{if } \alpha = h, \\ 0 & \text{if } \alpha \neq h. \end{cases}$$

According to (3), the dimension spectrum for pointwise dimensions admits a decomposition into "stable" and "unstable" dimension spectra for pointwise dimensions. In the appendix we will show that one can also define "stable" and "unstable" entropy spectra for local entropies, which coincide with $\mathcal{E}_E$.

2.4. Multifractal spectra for Lyapunov exponents. Note that in the two-dimensional case $g_L^1(x) = \lambda_1(x)$ is the positive value of the Lyapunov exponent, and $g_L^2(x) = \lambda_2(x)$ is the negative value.

We consider the spectra $\mathcal{L}_D^u$ and $\mathcal{L}_D^s$ specified by the pairs of functions (g_L^1, G_D) and $(-g_L^2, G_D)$ respectively. Weiss [13] effected a multifractal analysis of the spectrum $\mathcal{L}_D^u$. The two spectra $\mathcal{L}_D^u$ and $\mathcal{L}_D^s$ can be described in the following way.

Theorem 2.5. *For every $\alpha \in \mathbb{R}$, we have*

$$\mathcal{L}_D^u(\alpha) = \mathcal{D}_D^{u,(m_E)}(h/\alpha) \quad \text{and} \quad \mathcal{L}_D^s(\alpha) = \mathcal{D}_D^{s,(m_E)}(h/\alpha).$$

Moreover:

(1) *if m_E is not equivalent to the measure m_D^u (respectively, m_D^s), then $\mathcal{L}_D^u$ (respectively, $\mathcal{L}_D^s$) is an analytic strictly convex function defined on a closed interval containing h/t^u (respectively, h/t^s);*

(2) *if m_E is equivalent to the measure m_D^u (respectively, m_D^s), then the Lyapunov spectrum $\mathcal{L}_D^u$ (respectively, $\mathcal{L}_D^s$) is the delta function*

$$\mathcal{L}_D^u(\alpha) = \begin{cases} t^u & \text{if } \alpha = h/t^u \\ 0 & \text{if } \alpha \neq h/t^u \end{cases} \left(\text{respectively, } \mathcal{L}_D^s(\alpha) = \begin{cases} t^s & \text{if } \alpha = h/t^s \\ 0 & \text{if } \alpha \neq h/t^s \end{cases} \right).$$

We now give a complete description of the multifractal spectra $\mathcal{L}_E^u$ and $\mathcal{L}_E^s$ specified by the pairs of functions (g_L^1, G_E) and $(-g_L^2, G_E)$ respectively.

Theorem 2.6. *For every $\alpha \in \mathbb{R}$ and $\xi \in \mathfrak{P}_f$, we have*

$$\mathcal{L}_E^u(\alpha) = \mathcal{E}_E^{(m_D^u)}(\alpha t^u) \quad \text{and} \quad \mathcal{L}_E^s(\alpha) = \mathcal{E}_E^{(m_D^s)}(\alpha t^s).$$

Moreover:

(1) *if m_E is not equivalent to the measure m_D^u (respectively, m_D^s), then $\mathcal{L}_E^u$ (respectively, $\mathcal{L}_E^s$) is an analytic strictly convex function defined on a closed interval containing h/t^u (respectively, h/t^s);*

(2) *if m_E is equivalent to the measure m_D^u (respectively, m_D^s), then the Lyapunov spectrum $\mathcal{L}_E^u$ (respectively, $\mathcal{L}_E^s$) is the delta function*

$$\mathcal{L}_E^u(\alpha) = \begin{cases} h & \text{if } \alpha = h/t^u \\ 0 & \text{if } \alpha \neq h/t^u \end{cases} \left(\textit{respectively, } \mathcal{L}_D^s(\alpha) = \begin{cases} h & \text{if } \alpha = h/t^s \\ 0 & \text{if } \alpha \neq h/t^s \end{cases} \right).$$

As immediate consequences of Theorems 2.5 and 2.6, we obtain the following

Corollary 2.7.

(1) *If the measure* m_E *is not equivalent to the measure* m_D^u*, then the range of the function* g_L^1 *contains an open interval, and hence,* g_L^1 *attains uncountably many distinct values.*

(2) *If* g_L^1 *attains only countably many values, then* m_E *is equivalent to* m_D^u*.*

One can formulate similar statements for the function g_L^2.

It is an open problem in dimension theory to obtain a description of the spectra $\mathcal{D}_E$ and $\mathcal{E}_D$ for Gibbs measures on locally maximal hyperbolic sets.

3. Multifractal rigidity for horseshoes

In this section we consider another interesting phenomenon in dimension theory of dynamical systems which we regard as a multifractal rigidity phenomenon. Roughly speaking, it states that if two dynamical systems are topologically equivalent and some of their multifractal spectra coincide, then they are smoothly equivalent. In particular, not only topological, but also measure-theoretical and dimensional properties of the two systems coincide. Thus, given a dynamical system one can use multifractal spectra to "identify" the metric structure of the phase space as well as the corresponding invariant measure. We illustrate the multifractal rigidity phenomenon by the following example.

Let f be a "linear horseshoe map," i.e., a C^∞ map $f\colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ such that there exist two disjoint horizontal strips H_1 and H_2, and two disjoint vertical strips V_1 and V_2 such that $f\colon H_i \to V_i$ is a linear onto map for $i = 1, 2$. We denote the constant values of the partial derivatives on H_i by

$$c_i = \|\partial_2 f | H_i\| \quad \text{and} \quad d_i = \|\partial_1 f | H_i\|^{-1}$$

for $i = 1, 2$. We consider the horseshoe defined by f, i.e., the locally maximal hyperbolic set for f given by

$$\Lambda = \bigcap_{k=-\infty}^{\infty} f^k(H_1 \cup H_2).$$

The restriction of f to Λ is topologically conjugate to the full shift on 2 symbols $\sigma|\Sigma_2$. Let $\chi\colon \Sigma_2 \to \Lambda$ be the corresponding coding map (see Appendix).

We also consider the Bernoulli measure μ on Σ_2 with probabilities β_1 and $\beta_2 = 1 - \beta_1$.

We define the functions a^u, a^s, and η for $i = 1, 2$ by

$$a^u(x) = c_i \quad \text{if } x \in H_i,$$

$$a^s(x) = d_i \quad \text{if } x \in V_i,$$

$$\eta(x) = \log \beta_i \quad \text{if } x \in H_i.$$

For every $q \in \mathbb{R}$, the functions $T^u_D(q)$ and $T^s_D(q)$ satisfy the identities

$$c_1{}^{-T^u_D(q)} \beta_1{}^q + c_2{}^{-T^u_D(q)} \beta_2{}^q = 1 \tag{5}$$

and

$$d_1{}^{-T^s_D(q)} \beta_1{}^q + d_2{}^{-T^s_D(q)} \beta_2{}^q = 1. \tag{6}$$

Let $\widehat{f}$ be another "linear horseshoe map" with its horseshoe given by

$$\widehat{\Lambda} = \bigcap_{k=-\infty}^{\infty} \widehat{f}^k(\widehat{H}_1 \cup \widehat{H}_2).$$

Let $\widehat{\chi}\colon \Sigma_2 \to \widehat{\Lambda}$ be the corresponding coding map. We consider the Bernoulli measure $\widehat{\mu}$ on Σ_2 with probabilities $\widehat{\beta}_1$ and $\widehat{\beta}_2 = 1 - \widehat{\beta}_1$.

Set $\widehat{V}_i = \widehat{f}(\widehat{H}_i)$ for $i = 1, 2$. We define the functions $\widehat{a}^u$, $\widehat{a}^s$, and $\widehat{\eta}$ for $i = 1, 2$ by

$$\widehat{a}^u(x) = \widehat{c}_i \quad \text{if } x \in \widehat{H}_i,$$

$$\widehat{a}^s(x) = \widehat{d}_i \quad \text{if } x \in \widehat{V}_i,$$

$$\widehat{\eta}(x) = \log \widehat{\beta}_i \quad \text{if } x \in \widehat{H}_i.$$

We note that $f|\Lambda$ and $\widehat{f}|\widehat{\Lambda}$ are topologically equivalent under the homeomorphism $\widehat{\chi}\circ\chi^{-1}$. If ρ is an automorphism of Σ_2, then the homeomorphism $\widehat{\chi}\circ\rho\circ\chi^{-1}$ is also a topological conjugacy between $f|\Lambda$ and $\widehat{f}|\widehat{\Lambda}$, and all topological conjugacies are of this form. An important question is whether the class of all conjugacies contains a homeomorphism ζ preserving the differentiable structure, i.e., $a^u = \widehat{a}^u \circ \zeta$ and $a^s = \widehat{a}^s \circ \zeta$. One can ask if, in addition, ζ is measure preserving, i.e., $\nu = \widehat{\nu}\circ\zeta$. We give a complete answer to these questions below.

We consider the spectra $\mathcal{D}^u_D$ and $\mathcal{D}^s_D$ specified by f (as defined above). Similarly, we consider the spectra $\widehat{\mathcal{D}}^u_D$ and $\widehat{\mathcal{D}}^s_D$ specified by $\widehat{f}$.

Theorem 3.1. *If $\mathcal{D}^u_D(\alpha) = \widehat{\mathcal{D}}^u_D(\alpha)$ and $\mathcal{D}^s_D(\alpha) = \widehat{\mathcal{D}}^s_D(\alpha)$ for every α and these spectra are not delta functions, then there is a homeomorphism $\zeta\colon \Lambda \to \widehat{\Lambda}$ such that:*

(1) $\zeta \circ f = \widehat{f} \circ \zeta$ *on* Λ;

(2) *the automorphism* ρ *of* Σ_2 *satisfying* $\zeta \circ \chi = \widehat{\chi} \circ \rho$ *is either the identity or the involution automorphism*;
(3) $a^u = \widehat{a}^u \circ \zeta$, $a^s = \widehat{a}^s \circ \zeta$, $\eta = \widehat{\eta} \circ \zeta$, *and* $\nu = \widehat{\nu} \circ \zeta$.

Proof. The proof of the theorem is a modification of the proof of Theorem 7.1 in [4].

It is suffices to prove that the spectrum $\mathcal{D}_D^u$ uniquely determines the numbers β_1, β_2, c_1, and c_2 up to a permutation of the indices 1 and 2.

By the uniqueness of the Legendre transform, the spectrum $\mathcal{D}_D^u$ uniquely determines $T_D^u(q)$ for every $q \in \mathbb{R}$, and hence, it is enough to prove that Eq. (5) uniquely determines the numbers β_1, β_2, c_1, and c_2 up to a permutation of the indices 1 and 2.

One can verify that the numbers $\alpha_\pm = \alpha_D^u(\pm\infty)$ can be computed by

$$\alpha_\pm = -\lim_{q\to\pm\infty}(T_D^u(q)/q).$$

We observe that since the spectrum $\mathcal{D}_D^u$ is not a delta function, and hence, $T_D^u(q)$ is not linear and is strictly convex, then $\alpha_+ < t^u < \alpha_-$. Therefore, raising both sides of (5) to the power $1/q$ and letting $q \to \pm\infty$, we obtain

$$\max\{\beta_1 {c_1}^{\alpha_+}, \beta_2 {c_2}^{\alpha_+}\} = \min\{\beta_1 {c_1}^{\alpha_-}, \beta_2 {c_2}^{\alpha_-}\} = 1.$$

We assume that $\beta_1 {c_1}^{\alpha_+} = 1$ (the case when $\beta_2 {c_2}^{\alpha_+} = 1$ can be treated in a similar way; in this case, ρ is the involution automorphism). Since $\alpha_+ < \alpha_-$, we must have $\beta_2 {c_2}^{\alpha_-} = 1$.

Setting $q = 0$ and $q = 1$ in Eq. (5), we obtain, respectively,

$${c_1}^{-t^u} + {c_2}^{-t^u} = 1 \quad \text{and} \quad \beta_1 + \beta_2 = 1.$$

Set $x = {c_1}^{-t^u}$, $a = \alpha_+/t^u < 1$, and $b = \alpha_-/t^u > 1$. Then, one can easily derive the equation

$$x^a + (1-x)^b = 1.$$

One can verify with standard calculus arguments that this equation has a unique solution $x \in (0,1)$, which uniquely determines the numbers c_1 and c_2, and hence, also the numbers β_1 and β_2.

Similar arguments show that Eq. (6) uniquely determines the numbers d_1 and d_2. $\square$

We believe that the following conjecture holds. Let Λ be a locally maximal hyperbolic set, and let a and η be Hölder continuous functions on Λ such that $a(x) > 1$ for each $x \in \Lambda$, and $P_\Lambda(\log\eta) = 0$. For any $q \in \mathbb{R}$, we define the number $T(a,\eta)(q)$ as the unique root of the equation

$$P_\Lambda(-T(a,\eta)(q)\log a + q\log\eta) = 0.$$

Let now Λ and $\widehat{\Lambda}$ be locally maximal hyperbolic sets of topologically equivalent diffeomorphisms f and $\widehat{f}$ respectively. We consider the functions a^u

and a^s defined by f on Λ, as well as the functions $\widehat{a}^u$ and $\widehat{a}^s$ defined by $\widehat{f}$ on $\widehat{\Lambda}$. We also consider the functions η and $\widehat{\eta}$ as above, corresponding to f and $\widehat{f}$ respectively. We define $T_D = T_D^u + T_D^s$ as in (3). Likewise, we set $\widehat{T}_D^u = T(\widehat{a}^u, \widehat{\eta})$, $\widehat{T}_D^s = T(\widehat{a}^s, \widehat{\eta})$, and $\widehat{T}_D = \widehat{T}_D^u + \widehat{T}_D^s$.

Conjecture. One of the following alternatives holds:

(1) there exist functions a and η as above, such that $T(a, \eta) = T_D$; in this case, there is a constant $\gamma \in (0,1)$ such that $T_D^u = \gamma T_D$ and $T_D^s = (1-\gamma)T_D$;

(2) the function T_D can be decomposed into a sum $T_D^s + T_D^u$ in a unique way, i.e., if
$$T_D^s + T_D^u = \widehat{T}_D^s + \widehat{T}_D^u = T_D,$$
then $T_D^s = \widehat{T}_D^s$ or $T_D^s = \widehat{T}_D^u$.

If the first alternative holds, one can verify that $\gamma \log a^u = (1-\gamma) \log a^s = \log a$, up to an automorphism of the shift space. If the second alternative holds, one can substitute the hypotheses of Theorem 3.1 by the requirement that $\mathcal{D}_D^{(\mu)}(\alpha) = \widehat{\mathcal{D}}_D^{(\widehat{\mu})}(\alpha)$ for every α.

Appendix A.

We describe the "coordinate-wise" approach to the multifractal spectrum for local entropies by decomposing the potential function κ onto "stable" and "unstable" parts.

Let $\{R_1, \dots, R_k\}$ be a Markov partition of Λ. It generates a symbolic model of Λ by a subshift of finite type (Σ_A, σ). Here Σ_A is the set of two-sided infinite sequences on k symbols, which are admissible with respect to the transfer matrix $A = (a_{ij})$ (i.e., $a_{ij} = 1$ if $\operatorname{int} R_i \cap f^{-1}(\operatorname{int} R_j) \neq \varnothing$, and $a_{ij} = 0$ otherwise), and σ is the shift map. We define the *coding map* $\chi \colon \Sigma_A \to \Lambda$ by
$$\chi(\cdots i_{-1} i_0 i_1 \cdots) = \bigcap_{n=-\infty}^{+\infty} f^{-n} R_{i_n}.$$
We note that the map χ is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$. Moreover, χ is injective on the set of points whose trajectories never hit the boundary of any element of the Markov partition.

The pullback of κ by the coding map χ is a Hölder continuous function $\varphi = \kappa \circ \chi$ on Σ_A. Let μ be the unique Gibbs measure for φ with respect to the shift σ. Its push forward is the measure ν defined above.

We recall that two functions φ_1 and φ_2 on X are called *cohomologous* (with respect to f) if there exist a Hölder continuous function $b \colon X \to \mathbb{R}$ and a constant c such that
$$\varphi_1 - \varphi_2 = b - b \circ f + c.$$

In this case we write $\varphi_1 \sim \varphi_2$. We recall that for two Hölder continuous functions φ_1 and φ_2 on X, we have $\varphi_1 \sim \varphi_2$ if and only if the equilibrium measures corresponding to φ_1 and φ_2 on X coincide.

Given a point $x \in \Lambda$, we set

$$A^u(x) = V^u(x) \cap R(x) \quad \text{and} \quad A^s(x) = V^s(x) \cap R(x),$$

where $V^u(x)$ and $V^s(x)$ are local unstable and stable manifolds at x respectively, and $R(x)$ is a fixed element of the Markov partition containing x. Let

$$[\cdot,\cdot]\colon A^u(x) \times A^s(x) \to [A^u(x), A^s(x)]$$

be the Hölder homeomorphism defined by $[y,z] = V^s(y) \cap V^u(z)$. For surface diffeomorphisms, $[\cdot,\cdot]$ is, in fact, Lipschitz and has Lispchitz inverse.

We denote by Σ_A^+ and Σ_A^- the sets of right-sided and left-sided infinite sequences on k symbols respectively. Note that Σ_A^- is canonically identified with $\Sigma_{A^t}^+$, where A^t denotes the transpose of A, by the bijective map

$$\Sigma_A^- \ni (\cdots i_{-1} i_0) \mapsto (i_0 i_{-1} \cdots) \in \Sigma_{A^t}^+.$$

We consider the subshifts of finite type $\sigma_+\colon \Sigma_A^+ \to \Sigma_A^+$ and $\sigma_-\colon \Sigma_A^- \to \Sigma_A^-$ defined by $\sigma_+(i_0 i_1 \cdots) = (i_1 i_2 \cdots)$ and $\sigma_-(\cdots i_{-1} i_0) = (\cdots i_{-2} i_{-1})$ respectively.

Let $x \in \Lambda$ and choose $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A$ such that $\chi(\omega) = x$. For any point $\omega' = (\cdots i'_{-1} i'_0 i'_1 \cdots) \in \Sigma_A$ with the same past as ω (i.e., $i'_j = i_j$ for any $j \leq 0$), we have $\chi(\omega') \in A^u(x)$. Similarly, for any point $\omega' = (\cdots i'_{-1} i'_0 i'_1 \cdots) \in \Sigma_A$ with the same future as ω (i.e., $i'_j = i_j$ for any $j \geq 0$), we have $\chi(\omega') \in A^s(x)$. Thus, the manifold $A^u(x)$ can be identified via the coding map χ with the cylinder C_{i_0} in Σ_A^+ and the manifold $A^s(x)$ can be identified via the coding map χ with the cylinder C_{i_0} in Σ_A^-.

Choose k points $\omega_1, \ldots, \omega_k \in \Sigma_A$ such that $\omega_i \in \Sigma_A \cap C_i$ for each i, and set $\Omega = (\omega_1, \ldots, \omega_k)$. We define the function $r_\Omega\colon \Sigma_A \to \Sigma_A$ by

$$r_\Omega(\cdots i_{-1} i_0 i_1 \cdots) = (\cdots j_{-2} j_{-1} i_0 i_1 i_2 \cdots),$$

where $(\cdots j_{-1} j_0 j_1 \cdots) = \omega_{i_0}$. We now define the function $\theta^u = \theta^u_\Omega$ on Σ_A by

$$\theta^u(\omega) = \varphi(r_\Omega(\omega)) + \sum_{j=0}^{\infty} \left[\varphi(\sigma^{j+1} r_\Omega(\omega)) - \varphi(\sigma^j r_\Omega(\sigma\omega))\right].$$

The functions θ^u and φ are cohomologous and have the same pressure (see Lemma 1.6 in [5]). Hence μ is also the Gibbs measure of θ^u.

Let $\pi_+\colon \Sigma_A \to \Sigma_A^+$ and $\pi_-\colon \Sigma_A \to \Sigma_A^-$ be the projections defined, respectively, by

$$\pi_+(\cdots i_{-1} i_0 i_1 \cdots) = (i_0 i_1 \cdots) \quad \text{and} \quad \pi_-(\cdots i_{-1} i_0 i_1 \cdots) = (\cdots i_{-1} i_0).$$

One can easily check that $\theta^u(\cdots i_{-1}i_0i_1\cdots) = \theta^u(\cdots i'_{-1}i'_0i'_1\cdots)$ whenever $i_j = i'_j$ for every $j \geq 0$. This means that there is a function φ^u on Σ_A^+ such that $\theta^u = \varphi^u \circ \pi_+$ on Σ_A.

In a similar fashion, we define the function $\theta^s = \theta^s_\Omega$ on Σ_A; and there is a function φ^s on Σ_A^- such that $\theta^s = \varphi^s \circ \pi_-$.

The functions φ^u and φ^s are Hölder continuous (see [5]). Let μ^u be the Gibbs measure for φ^u on Σ_A^+, and μ^s the Gibbs measure for φ^s on Σ_A^-. We note that $\mu^u = \mu \circ {\pi_+}^{-1}$ since they are both Gibbs measures for the function φ^u, and that $\mu^s = \mu \circ {\pi_-}^{-1}$ since they are both Gibbs measures for the function φ^s. This implies that

$$\mu^u(C^+_{i_0\cdots i_n}) = \mu(C_{i_0\cdots i_n}) \quad \text{and} \quad \mu^s(C^-_{i_0\cdots i_n}) = \mu(C_{i_0\cdots i_n}), \tag{7}$$

where $C^+_{i_0\cdots i_n} = \pi_+ C_{i_0\cdots i_n}$ and $C^-_{i_{-n}\cdots i_0} = \pi_- C_{i_{-n}\cdots i_0}$. The measures μ^u and μ^s can be identified (μ-almost everywhere) with the conditional measures of μ on unstable and stable sets of Σ_A respectively.

Define the function ψ^u on Σ_A^+ by

$$\log\psi^u = \varphi^u - P_{\Sigma_A^+}(\varphi^u),$$

and the function ψ^s on Σ_A^- by

$$\log\psi^s = \varphi^s - P_{\Sigma_A^-}(\varphi^s).$$

Clearly, $P_{\Sigma_A^+}(\log\psi^u) = P_{\Sigma_A^-}(\log\psi^s) = 0$. By Proposition 3.2 in [8] and (7), it follows that

$$\log\psi^u(\omega_+) = \lim_{n\to\infty}\log\frac{\mu^u(C^+_{i_0\cdots i_n})}{\mu^u(C^+_{i_1\cdots i_n})} = \lim_{n\to\infty}\log\frac{\mu(C_{i_0\cdots i_n})}{\mu(C_{i_1\cdots i_n})}$$

for each $\omega_+ = (i_0i_1\cdots) \in \Sigma_A^+$ (the uniform convergence), and that

$$\log\psi^s(\omega_-) = \lim_{n\to\infty}\log\frac{\mu^s(C^-_{i_{-n}\cdots i_0})}{\mu^s(C^-_{i_{-n}\cdots i_1})} = \lim_{n\to\infty}\log\frac{\mu(C_{i_{-n}\cdots i_0})}{\mu(C_{i_{-n}\cdots i_1})}$$

for each $\omega_- = (\cdots i_{-1}i_0) \in \Sigma_A^-$ (the uniform convergence). This implies that the functions ψ^u and ψ^s *do not depend on the choice of the point Ω.*

The following statement shows that Gibbs measures on compact locally maximal hyperbolic sets have a product structure.

Proposition A.1. *The following properties hold:*

(1) $P_{\Sigma_A}(\varphi) = P_{\Sigma_A^+}(\varphi^u) = P_{\Sigma_A^-}(\varphi^s)$;

(2) *there exist positive constants* K_1 *and* K_2 *such that for all integers* n, $m \geq 0$ *and any* $(\cdots i_{-1}i_0i_1\cdots) \in \Sigma_A$,

$$K_1 \leq \frac{\mu(C_{i_{-m}\cdots i_n})}{\mu^u(C^+_{i_0\cdots i_n}) \times \mu^s(C^-_{i_{-m}\cdots i_0})} \leq K_2.$$

Proof. The first property follows immediately from the former discussion. The second property follows from the identities in (7). □

For any point $x \in \Lambda \cap \chi(C_{i_0})$ we define the measure $\nu^s_x = \chi_*(\mu^s|C_{i_0})$ on $A^s(x)$ and the measure $\nu^u_x = \chi_*(\mu^u|C_{i_0})$ on $A^u(x)$. The following statement follows immediately from Proposition A.1.

Proposition A.2. *There are positive constants* K_1 *and* K_2 *such that for any point* $x \in \Lambda$, *and any Borel sets* $E \in A^s(x)$ *and* $F \in A^u(x)$,

$$K_1(\nu^u_x(E) \times \nu^s_x(F)) \leq \nu([E,F]) \leq K_2(\nu^u_x(E) \times \nu^s_x(F)).$$

For ν-almost every point $x \in \Lambda$ let $\widetilde{\nu}^u_x$ and $\widetilde{\nu}^s_x$ be the conditional measures of ν on $A^u(x)$ and $A^s(x)$ respectively. We note that $\widetilde{\nu}^u_x = \nu^u_x$ and $\widetilde{\nu}^s_x = \nu^s_x$ for ν-almost every $x \in \Lambda$. In the two-dimensional case, the stable and unstable foliations are Lipschitz, and hence, there exist positive constants c_1, c_2 such that

$$c_1\widetilde{\nu}^u_x([E,x]) \leq \widetilde{\nu}^u_y([E,y]) \leq c_2\widetilde{\nu}^u_x([E,x]),$$
$$c_1\widetilde{\nu}^s_x([x,E]) \leq \widetilde{\nu}^s_y([y,E]) \leq c_2\widetilde{\nu}^s_x([x,E]),$$

for any point $y \in R(x)$ and any set $E \subset M$ such that $E \cap R(x) \neq \varnothing$. One can use these inequalities to obtain another proof of Proposition A.2 in the case of surfaces.

For each $p \in \mathbb{R}$, we define the function $\varphi^u_{E,p}$ on Σ^+_A by

$$\varphi^u_{E,p} = -T^u_E(p) + p\log\psi^u,$$

where $T^u_E(p)$ is chosen such that

$$P_{\Sigma^+_A}(\varphi^u_{E,p}) = 0.$$

Similarly, for each $p \in \mathbb{R}$, we define the function $\varphi^s_{E,p}$ on Σ^-_A by

$$\varphi^s_{E,p} = -T^s_E(p) + p\log\psi^s,$$

where $T^s_E(p)$ is chosen such that

$$P_{\Sigma^-_A}(\varphi^s_{E,p}) = 0.$$

Clearly,

$$T^u_E(p) = P_{\Sigma^+_A}(p\log\psi^u) \quad \text{and} \quad T^s_E(p) = P_{\Sigma^-_A}(p\log\psi^s).$$

It is obvious that the functions $\varphi^u_{E,p}$ and $\varphi^s_{E,p}$ are Hölder continuous.

Theorem A.3. $T_E^u(p) = T_E^s(p) = T_E(p)$.

Proof. It follows easily from the definitions that $P_{\Sigma_A^+}(p\varphi^u) = P_{\Sigma_A^-}(p\varphi^s) = P_{\Sigma_A}(p\varphi)$ for every p. We have

$$T_E^u(p) = P_{\Sigma_A^+}(p\varphi^u) - pP_{\Sigma_A^+}(\varphi^u) = P_{\Sigma_A}(p\varphi) - pP_{\Sigma_A}(\varphi) = T_E(p),$$

and similarly, $T_E^s(p) = T_E(p)$. □

REFERENCES

1. L. Barreira, A nonadditive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergodic Theory and Dynam. Syst.* (to appear).

2. L. Barreira, Ya. Pesin, and J. Schmeling, On the pointwise dimension of hyperbolic measures: A proof of the Eckmann–Ruelle conjecture. *Electronic Research Ann.* **2** (1996), No. 1, 69–72.

3. ______, Dimension of hyperbolic measures – A proof of the Eckmann–Ruelle conjecture. *WIAS Preprint* 245, 1996.

4. ______, On a general concept of multifractality: Multifractal spectra for dimensions, entropies, and Lyapunov exponents, multifractal rigidity. *Chaos* (to appear).

5. R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms. *Lect. Notes Math.* **470** (1975).

6. A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems. In: Encyclopedia of Mathematics and its Applications, Vol. 54, *Cambridge Univ. Press*, 1995.

7. H. McCluskey and A. Manning, Hausdorff dimension for horseshoes. *Ergodic Theory and Dynam. Syst.* **3** (1983), No. 2, 251–260, (errata in **5** (1985), No. 2, 319).

8. W. Parry and M. Pollicott, Zeta functions and the periodic orbit structures of hyperbolic dynamics. *Astérisque* (1990), 187–188.

9. Ya. Pesin, Dimension theory in dynamical systems: Rigorous results and applications. *Chicago Lect. Math.*, *Univ. Chicago Press*, 1997.

10. Ya. Pesin and H. Weiss, The multifractal analysis of Gibbs measures: Motivation, mathematical foundation, and examples. *Chaos* (to appear).

11. D. Simpelaere, Dimension spectrum of axiom A diffeomorphisms. II. Gibbs measures. *J. Statist. Phys.* **76** (1994), No. 5–6, 1359–1375.

12. F. Takens, Limit capacity and Hausdorff dimension of dynamically defined Cantor sets. *Lect. Notes Math.* **1331** (1988), 196–212.

13. H. Weiss, The Lyapunov and dimension spectra of equilibrium measures of conformal expanding maps. *Preprint*, 1996.

14. L.-S. Young, Dimension, entropy and Lyapunov exponents. *Ergodic Theory and Dynam. Syst.* **2** (1982), No. 1, 109–124.

(Received 28.11.1994)

Authors' addresses:
L. Barreira
Departamento de Matemática,
Instituto Superior Técnico,
1096 Lisboa, Portugal
E-mail: barreira@math.ist.utl.pt

Ya. Pesin
Department of Mathematics,
The Pennsylvania State University,
University Park, PA 16802, USA
E-mail: pesin@math.psu.edu

J. Schmeling
Weierstrass Institute of Applied Analysis and Stochastics,
Mohrenstrasse 39, D–10117 Berlin, Germany
E-mail: schmeling@wias-berlin.de