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PARTIALLY HYPERBOLIC DYNAMICAL SYSTEMS

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Abstract. Smooth dynamical systems having contracting and expanding invariant foliations (of not necessarily complementary dimensions) are investigated. Ergodicity and the K-property are established for such dynamical systems under additional assumptions.

§1. Introduction

1. In the present paper we study the metric and topological properties of a certain class of smooth dynamical systems on compact Riemannian manifolds.

Consider the Banach space $\Gamma^0(TM^n)$ of continuous vector fields on a compact smooth n-dimensional Riemannian manifold M^n. It is a module over the ring of continuous functions $C(M^n)$. We define the dynamical systems of interest to us. Let f be an arbitrary diffeomorphism of M^n and let f_* denote the continuous and invertible linear operator in $\Gamma^0(TM^n)$ defined by the formula

$$(f_*v)(x) = dfv(f^{-1}(x)), \quad v \in \Gamma^0(TM^n).$$

By the *spectrum* of a diffeomorphism f is meant the spectrum of the complexification of the operator f_*.

Definition 1.1 (see [9]). A diffeomorphism f of a compact Riemannian manifold M^n is said to be *partially hyperbolic* if the spectrum of the operator f_* is contained in three annuli with radii λ_i and μ_i such that $0 < \lambda_1 \le \mu_1 < \lambda_2 \le 1 \le \mu_2 < \lambda_3 \le \mu_3$ and at least two of these annuli contain nonempty components of the spectrum.

Definition 1.2 (see [9]). A flow f^t on a compact Riemannian manifold is said to be *partially hyperbolic* if the diffeomorphism f^1 is partially hyperbolic.

By a *partially hyperbolic dynamical system* we will understand a partially hyperbolic diffeomorphism or flow.

A special case of partially hyperbolic dynamical systems is afforded by the Anosov systems (for a definition of which and proofs of the facts presented below, see

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[1–3, 5, 11–13]). The spectrum of an Anosov diffeomorphism f consists of two components S_1 and S_2 respectively lying strictly inside and strictly outside the unit circle. To these components there corresponds a splitting of $\Gamma^0(TM^n)$ into a direct sum of two submodules Γ_1 and Γ_2: $\Gamma^0(TM^n) = \Gamma_1 \oplus \Gamma_2$, such that $\mathrm{sp}\, f_*|_{\Gamma_1} = S_1$ and $\mathrm{sp}\, f_*|_{\Gamma_2} = S_2$. To the submodules Γ_1 and Γ_2 there correspond distributions E_1 and E_2 such that $T_xM^n = E_1(x) \oplus E_2(x)$ for each $x \in M^n$. The distributions E_1 and E_2 satisfy a Hölder condition and are integrable, their integral manifolds respectively forming contracting and expanding foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$. Each of these foliations is absolutely continuous and metrically transitive. An Anosov diffeomorphism f preserving a smooth measure μ is a K-automorphism of the measure space (M^n, μ). Analogous facts are valid for Anosov flows. We note that a diffeomorphism of an Anosov flow is not necessarily an Anosov diffeomorphism, whereas any diffeomorphism of a partially hyperbolic flow is partially hyperbolic.

2. Partially hyperbolic dynamical systems have a number of metric and topological properties, many of which generalize the corresponding properties of Anosov systems. However, a similar generalization of the characteristic properties of Anosov systems to partially hyperbolic dynamical systems encounters significant difficulties and apparently cannot be fully realized.

In §2 we show that to the components, of the spectrum of a partially hyperbolic dynamical system f, lying in annuli with radii λ_i and μ_i, $i = 1, 2, 3$, there correspond certain distributions E_i. These distributions satisfy a Hölder condition and, under certain conditions on the numbers λ_i and μ_i, even turn out to be smooth. Moreover, the distributions E_1 and E_3 are integrable and their integral manifolds respectively form contracting and expanding foliations of the dynamical systems.

In §3 we prove that these foliations are absolutely continuous. We note that the notion itself of absolute continuity of foliations, as it was introduced, for example, in [2], does not carry over verbatim to our case. However, this notion admits a natural modification, which is formulated in §3.1. The general direction of our proof of absolute continuity of the foliations follows the main ideas of the corresponding proof presented in [2] for the foliations of Anosov systems but differs from it in certain essential respects. It is also proved in this section that a translation along the trajectories of a vector field subordinate to one of these foliations (see Definition 3.2) is an absolutely continuous homeomorphism of the manifold.

In §4 we describe a series of topological properties that the contracting and expanding foliations of a partially hyperbolic dynamical system can possess; in particular, the absolute nonintegrability and transitivity of a pair of foliations. The first of these properties pertains to the case of sufficiently smooth foliations and corresponds to the absolute nonintegrability of a pair of distributions (see [14]). The property of transitivity of a pair of foliations is a generalization of the property of absolute nonintegrability for nonsmooth foliations and consists, roughly, in the fact that any two points of the manifold can be joined by a finite chain of leaves of the contracting and expanding foliations. Clearly, the foliations of an Anosov diffeomorphism possess a similar property.

An important result is contained in Theorem 4.1, which asserts that the property of transitivity of a pair of foliations is preserved under a small perturbation of the dynamical system in the class C^2 if the original system has a pair of foliations of a sufficiently high smoothness class.

If a partially hyperbolic dynamical system has the indicated topological properties, then, under certain additional assumptions, it is ergodic, mixing, and even a K-system. For example, if the contracting and expanding foliations are absolutely nonintegrable, then, as was shown in [14], a dynamical system is mixing and, in addition (see Theorem 5.2 of the present paper), has the K-property. Apparently, additional assumptions of this kind are essential, and the conjecture that any partially hyperbolic dynamical system with a transitive pair of foliations is a K-system is not true.

In §5 we consider the group G of transformations of a manifold M^n generated by translations along the trajectories of the vector fields subordinate to the contracting and expanding foliations of a partially hyperbolic dynamical system f. If the pair of foliations is transitive, G acts transitively on M^n. If G acts metrically transitively on M^n, the dynamical system f has the K-property. In §5 we cite a series of conditions that are sufficient for a group G to act metrically transitively on a manifold (see Theorems 5.1–5.4).

3. As was mentioned above, Anosov systems are examples of partially hyperbolic dynamical systems. It should be noted that Anosov systems can be regarded in some cases as nontrivial examples of partially hyperbolic dynamical systems. Suppose, for example, that the component S_1 of the spectrum of an Anosov diffeomorphism splits into two nonempty sets S_1' and S_1'' lying respectively in annuli with radii λ_1', μ_1' and λ_1'', μ_1'' such that $0 < \lambda_1' \le \mu_1' < \lambda_1'' \le \mu_1'' < 1$. Such an Anosov diffeomorphism can be regarded as partially hyperbolic with the three components of its spectrum respectively contained in the annuli with radii λ_1', μ_1'; λ_1'', 1 and λ_2, μ_2 (here λ_2 and μ_2 are the radii of the annulus containing the second component of the spectrum of the Anosov diffeomorphism).

4. Another important example of partially hyperbolic dynamical systems is afforded by fiber bundles over Anosov systems for which the hyperbolic properties of an Anosov system acting in the base space of a fiber bundle are "stronger" than the hyperbolic properties in the direction of the fiber (for the definitions and corresponding results, see §§2.6 and 5.6). If the pair of foliations of such a fiber bundle is transitive, then, under certain restrictions on the spectrum, this dynamical system has the K-property (see Corollary 5.3). Moreover, in §5 we show that the class of fiber bundles over Anosov systems contains an open set of dynamical systems having the K-property.

We consider a concrete example of a fiber bundle over an Anosov flow, viz. a k-frame flow on a manifold of negative curvature (see [4, 10]). Let M^n be a smooth Riemannian manifold. By a *k-frame* on M^n is meant a pair $w = (x, \xi^k)$, where $x \in M^n$ and ξ^k is an ordered set $(\xi_1, \dots, \xi_k)$ of k orthonormal vectors $\xi_i \in T_xM^n$. The set of all k-frames on M^n is a locally trivial bundle $\pi\colon \Omega_k \to W^{2n-1}$ whose base space is the manifold W^{2n-1} of unit linear elements $\omega = (x, \xi_1)$ and whose fiber is the Stiefel manifold $V_{n-1,k-1}$ (viz. the manifold of ordered sets of $k-1$ orthonormal vectors in T_xM^n that are orthogonal to ξ_1). We denote by $P^t(\omega, \eta^l)$ the parallel translation in time t of an ordered set $\eta^l = (\eta_1, \ldots, \eta_l)$ of l orthonormal vectors along the trajectory passing through the linear element ω of the geodesic flow S^t.

Definition 1.3. By a *k-frame flow* on a smooth Riemannian manifold is meant a one parameter group Φ^t of transformations of Ω_k acting as follows:

$$\Phi^t w = (S^t\omega, P^t(\omega, \xi^{k-1})), \text{ где } w = (x, \xi^k) = (\omega, \xi^{k-1}).$$

Clearly, a k-frame flow is a fiber bundle over a geodesic flow.

In §6 we will show that if a manifold M^n has negative curvature, then a k-frame flow Φ^t on M^n is a partially hyperbolic system, and if the Riemannian metric on M^n is sufficiently close to a metric of constant negative curvature, then Φ^t is a K-flow.

5. Another example of partially hyperbolic dynamical systems is afforded by flows on homogeneous spaces of simple and semisimple Lie groups. Let $X = G/\Gamma$ be a homogeneous space of a simple Lie group G (Γ is discrete) and let x be an element of the Lie algebra of G such that the spectrum of its adjoint representation ad x contains at least one eigenvalue with a nonzero real part. It can be shown that the flow induced by the element x in the space X is partially hyperbolic. Inasmuch as the contracting and expanding foliations of this flow (corresponding to its horospherical subgroups) are transitive, the flow $\exp tx$ is a K-flow.

A concrete example of such a flow, which was proposed by L. Auslander, is given in [14].

Analogous results are valid for flows on homogeneous spaces of semisimple Lie groups. Namely, if in some simple ideal the operator ad x has at least one not purely imaginary eigenvalue, the corresponding flow is partially hyperbolic. If this is true for every simple ideal, the flow $\exp tx$ is a K-flow. Similar dynamical systems are studied in more detail in [20, 21].

6. The authors are grateful to D. V. Anosov for the attention he has given to the work and for many valuable discussions. The general direction of the investigations was suggested by A. B. Katok, under whose guidance the work was carried out. To him the authors express their most heartfelt thanks.

The results of §3, as well as Theorems 4.1 and 4.2, are due to M. I. Brin. The results of §2 are due to Ja. B. Pesin. The remaining results were obtained jointly.

§2. The existence of invariant distributions and foliations

1. Let f be a dynamical system (i.e. a diffeomorphism or flow) of class C^r, $r \geq 1$, on a compact n-dimensional smooth Riemannian manifold M^n. We assume that the spectrum of f consists of p components $S_i^f = S_i$ and that each set S_i is contained in an annulus with radii λ_i, μ_i such that

$$0<\lambda_1\leqslant\mu_1<\ldots<\lambda_p\leqslant\mu_p.$$

The results of [8] and [12] imply the following propositions.

Proposition 2.1. *Suppose that the nonperiodic points of f are everywhere dense in M^n, that $\lambda \in \operatorname{sp} f_*$ and that $|\xi| = 1$. Then $\lambda\xi \in \operatorname{sp} f_*$.*

In other words, the spectrum of f consists of circles $|\lambda| = \text{const}$.

Proposition 2.2. *The following decomposition holds:*

$$\Gamma^0(TM^n) = \Gamma_1 \oplus \dots \oplus \Gamma_p,$$

where the Γ_i, $i = 1, \dots, p$, are the invariant (relative to f_*) submodules of $\Gamma^0(TM^n)$ for which $\operatorname{sp} f_*|_{\Gamma_i} = S_i$.

To each submodule Γ_i there corresponds a distribution, which we denote by $E_i^f = E_i$.

From Proposition 2.2 it follows that $p \leq n$ and $TM^n = \bigoplus_1^p E_i$.

Proposition 2.3. *If $\mu_1 < 1$, then the distribution E_1 is integrable and its integral manifolds form a continuous invariant foliation $\mathfrak{S}$ with smooth leaves (see [11] as well as [1, 2]) and $T_x\mathfrak{S}(x) = E_1(x)$. Moreover, there exist $C_1 > 0$ and $C_2 > 0$ such that for any $x_1, x_2 \in \mathfrak{S}(x)$*

$$C_1\lambda_1^n \rho_{\mathfrak{S}(x)}(x_1, x_2) \leqslant \rho_{\mathfrak{S}(x)}(f^n(x_1), f^n(x_2)) \leqslant C_2\mu_1^n \rho_{\mathfrak{S}(x)}(x_1, x_2). \tag{2.1}$$

Here and below $\mathfrak{S}(x)$ is the leaf of $\mathfrak{S}$ through x and $\rho_{\mathfrak{S}(x)}$ is the internal metric induced on $\mathfrak{S}(x)$ by the Riemannian metric ρ on the manifold M^n.

We denote by ρ_{C^1} the metric in the space $\operatorname{Diff}^1(M^n)$ of C^1 diffeomorphisms of M^n induced by the Riemannian metric, and by d the metric in the space of continuous distributions on M^n:

$$d(E_1, E_2) = \max_{x \in M^n} \tilde{\rho}(E_1(x), E_2(x))$$

$$= \max_{x \in M^n} \max_{v_1 \in E_1} \min_{v_2 \in E_2} \left\| \frac{v_1}{\|v_1\|} - \frac{v_2}{\|v_2\|} \right\|.$$

Proposition 2.4. *Let $f \in \operatorname{Diff}^1(M^n)$. Then for every sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that for any diffeomorphism g with $\rho_{C^1}(f, g) < \delta$*

$$\operatorname{sp} g_* = \bigcup_{i=1}^p S_i^g,$$

where S_i^g is contained in the annulus with radii $\lambda_i - \epsilon$ and $\mu_i + \epsilon$, and $d(E_i^f, E_i^g) < \epsilon$.

Proposition 2.4 shows that the properties of dynamical systems formulated in Propositions 2.2 and 2.3 are preserved under a small perturbation in the class C^1 (in regard to this assertion, see also [13]).

2. Let f be a partially hyperbolic dynamical system. We denote by E^s, E^0 and E^u the invariant distributions corresponding to the submodules Γ_1, Γ_2 and Γ_3 for which $\operatorname{sp} f_*|_{\Gamma_i}$ lies in the annulus with radii λ_i and μ_i, $i = 1, 2, 3$, and by $\mathfrak{S}^s$ and $\mathfrak{S}^u$ the invariant foliations corresponding to the distributions E^s and E^u. We have

$$TM^n = E^s \oplus E^0 \oplus E^u, \quad T_x\mathfrak{S}^s(x) = E^s(x), \quad T_x\mathfrak{S}^u(x) = E^u(x).$$

3. **Definition 2.1** (see [3]). A distribution E satisfies a *Hölder δ-condition* with constant C and exponent α, $0 < \alpha < 1$, if for any $x, y \in M^n$ such that $\rho(x, y) < \delta$

$$\tilde{\rho}(\Pi_{xy}E(x), E(y)) < C\rho(x, y)^\alpha.$$

Here Π_{xy} denotes parallel translation along the geodesic joining x and y. If δ is sufficiently small, this geodesic is unique and the definition makes sense. In the sequel, when it will not cause confusion, we will drop the symbol Π_{xy}.

Theorem 2.1. *Let f be a dynamical system of class C^2 on a manifold M^n whose spectrum consists of two nonintersecting components S_1 and S_2 respectively lying in annuli with radii λ_1, μ_1 and λ_2, μ_2 such that $0 < \lambda_1 \le \mu_1 < \lambda_2 \le \mu_2$. Then for certain δ, C and α the distributions E_i satisfy a Hölder δ-condition with constant C and exponent α, i.e.*

$$\tilde{\rho}(\Pi_{xy}E_i(x), E_i(y)) \leqslant C\rho(x, y)^\alpha \tag{2.2}$$

when $\rho(x, y) < \delta$.

Proof. It suffices to establish the assertion of the theorem for diffeomorphisms. We will also assume that the diameter of the manifold is less than 1.

Lemma 1. *Let E be an arbitrary continuous distribution such that* $\dim E(x) = \dim E_1(x)$ *and $E(x) \cap E_2(x) = 0$ for all $x \in M^n$ and $d(E_1, E) \le \frac{1}{2}$. Then the sequence of distributions $f_*^{-n}E$ converges to the distribution E_1 in the metric d.*

Proof. Let $x \in M^n$. The invariance of the distributions E_1 and E_2 relative to f_* implies

$$\tilde{\rho}(f_*^{-n}E(x), E_1(x)) = \tilde{\rho}(f_*^{-n}E(x), f_*^{-n}E_1(x))$$

$$= \max_{v\in E(f^n(x))} \min_{w\in E_1(f^n(x))} \left\| \frac{df^{-n}v}{\|df^{-n}v\|} - \frac{df^{-n}w}{\|df^{-n}w\|} \right\|$$

$$\leqslant \max_{v\in E(f^n(x))} \min_{w\in E_1(f^n(x))} \left\| \frac{df^{-n}v}{\|df^{-n}v\|} - \frac{df^{-n}v_1}{\|df^{-n}v_1\|} \right\|$$

$$+ \max_{v\in E(f^n(x))} \min_{w\in E_1(f^n(x))} \left\| \frac{df^{-n}v_1}{\|df^{-n}v_1\|} - \frac{df^{-n}w}{\|df^{-n}w\|} \right\|$$

Here $v = v_1 + v_2$, $v_1 \in E_1(f^n(x))$, $v_2 \in E_2(f^n(x))$, where, by virtue of the conditions of the lemma, $\|v_1\| \neq 0$ and, moreover, $\|v_2/\|v\|\,\| \leq ½$. Hence $\|v_1/\|v\|\,\| \geq ½$, and consequently $\|v_2\|/\|v_1\| \leq 1$. If v ranges over all of $E(f^n(x))$, then v_1 ranges over all of $E_1(f^n(x))$ and the second term of the sum is equal to zero:

$$\tilde{\rho}(E(f^n(x)),\ E_1(f^n(x))) = 0.$$

We also have

$$\left\| \frac{df^{-n}v}{\|df^{-n}v\|} - \frac{df^{-n}v_1}{\|df^{-n}v_1\|} \right\| = \left\| \frac{df^{-n}(v_1+v_2)}{\|df^{-n}v\|} - \frac{df^{-n}v_1}{\|df^{-n}v_1\|} \right\|$$

$$\leqslant \|df^{-n}v_1\|^{-1} \left[\left(1 - \frac{\|df^{-n}v_1\|}{\|df^{-n}v\|} \right) \right] \|df^{-n}v_1\| + \frac{\|df^{-n}v_1\|}{\|df^{-n}v\|} \|df^{-n}v_2\|$$

$$\leqslant 2\frac{\|df^{-n}v_2\|}{\|df^{-n}v\|} \leqslant 2\frac{q_2'\lambda_2^{-n}\|v_2\|}{q_1\mu_1^{-n}\|v_1\|} \leqslant 2\frac{q_2'}{q_1}\left(\frac{\mu_1}{\lambda_2}\right)^n.$$

Here we have made use of the following inequalities implied by Proposition 2.2:

$$\begin{aligned} q_1\mu_1^{-n}\|v_1\| &\leqslant \|df^{-n}v_1\| \leqslant q_1'\lambda_1^{-n}\|v_1\|, \\ q_2\mu_2^{-n}\|v_2\| &\leqslant \|df^{-n}v_2\| \leqslant q_2'\lambda_2^{-n}\|v_2\|, \end{aligned} \tag{2.3}$$

where n is a natural number and q_1, q_2, q_1', q_2' are certain positive numbers. Thus

$$d(f_*^{-n}E, E_1) = \max_{x \in M^n} \tilde{\rho}(f_*^{-n}E(x), E_1(x)) \leqslant 2\frac{q_2'}{q_1}\left(\frac{\mu_1}{\lambda_2}\right)^n \to 0$$

where $n \to \infty$. The lemma is proved.

Lemma 2. *There exist positive numbers ϵ and K such that the inequalities $d(E^i, E_1) < \epsilon$, $i = 1, 2$, are valid for any two distributions E^1 and E^2, and the inequality*

$$\tilde{\rho}(df^{-n}E^1(f^n(x)),\ df^{-n}E^2(f^n(x))) \leqslant K\left(\frac{\mu_1}{\lambda_2}\right)^n \tilde{\rho}(E^1(f^n(x)),\ E^2(f^n(x))) \tag{2.4}$$

is valid for all $x \in M^n$ and natural n.

Proof. Let $v \in E^1(f^n(x))$, $\|v\| = 1$, $v = v_1 + v_2$, $v_1 \in E_1(f^n(x))$, $v_2 \in E_2(f^n(x))$. By virtue of the conditions of the lemma, $\|v_1\| > 1 - \epsilon$ and $\|v_2\|/\|v_1\| < \epsilon/(1-\epsilon)$.

It follows from (2.3) that if ϵ is sufficiently small,

$$\begin{aligned} \|df^{-n}v_1\| - \|df^{-n}v_2\| &\geqslant q_1\mu_1^{-n}\|v_1\| - q_2'\lambda_2^{-n}\|v_2\| \\ &\geqslant \mu_1^{-n}\left[q_1 - q_2'\frac{\varepsilon}{1-\varepsilon}\left(\frac{\mu_1}{\lambda_2}\right)^n\right](1-\varepsilon) \geqslant \frac{1}{2}\mu_1^{-n}. \end{aligned} \tag{2.5}$$

Consider a vector $w \in E^2(f^n(x))$, $\|w\| = 1$, $w = w_1 + w_2$, for which $v_1 = w_1$. It can be shown that if ϵ is sufficiently small, there exists a constant $C_1 > 0$ such that

$$\|v-w\|\leqslant C_1\tilde{\rho}(v, E^2(f^n(x))). \tag{2.6}$$

Inequalities (2.3), (2.5) and (2.6) imply

$$\tilde{\rho}\left(\frac{df^{-n}v}{\|df^{-n}v\|}, df^{-n}E^2(f^n(x))\right)\leqslant\left\|\frac{df^{-n}v}{\|df^{-n}v\|}-\frac{df^{-n}w}{\|df^{-n}w\|}\right\|$$

$$\leqslant\left\|\frac{df^{-n}v}{\|df^{-n}v\|}-\frac{df^{-n}w}{\|df^{-n}v\|}\right\|+\left\|\frac{df^{-n}w}{\|df^{-n}v\|}-\frac{df^{-n}w}{\|df^{-n}w\|}\right\|\leqslant 2\left\|\frac{df^{-n}(v-w)}{\|df^{-n}v\|}\right\|$$

$$\leqslant 2\frac{\|df^{-n}(v_2-w_2)\|}{\|df^{-n}v_1\|-\|df^{-n}v_2\|}\leqslant 4q_2^1\left(\frac{\mu_1}{\lambda_2}\right)^n\|v-w\|\leqslant K\left(\frac{\mu_1}{\lambda_2}\right)^n\tilde{\rho}(v, E^2(f^n(x))).$$

The assertion of the lemma follows.

Lemma 3. *There exist positive numbers δ, C and α and a natural number n_0 such that if a distribution E satisfies the Hölder δ-condition with constant C and exponent α and $d(E, E_1)<\epsilon$, then these same conditions are satisfied by the distribution $f_*^{-n_0}E$.*

Proof. Let E be such a distribution and choose n_0 so that $K(\mu_1/\lambda_2)^{n_0}=\eta<1$. Clearly $d(f_*^{-n_0}E, E_1)<\epsilon$.

Let $x, y\in M^n$ and let

$$\omega(f)=\tilde{\rho}(\Pi_{xy}f_*^{-n_0}E(x), f_*^{-n_0}E(y)),$$

$$\omega_1(f)=\tilde{\rho}(\Pi_{xy}df_{f^{n_0}(x)}^{-n_0}E(f^{n_0}(x)), df_{f^{n_0}(y)}^{-n_0}\Pi_{f^{n_0}(x)f^{n_0}(y)}E(f^{n_0}(x))),$$

$$\omega_2(f)=\tilde{\rho}(df_{f^{n_0}(y)}^{-n_0}\Pi_{f^{n_0}(x)f^{n_0}(y)}E(f^{n_0}(x)), df_{f^{n_0}(y)}^{-n_0}E(f^{n_0}(y))).$$

Clearly $\omega(f)\leq\omega_1(f)+\omega_2(f)$. Since $f\in\mathrm{Diff}^2(M^n)$, we have

$$\omega_1(f)\leqslant\|\overset{*}{\Pi}_{xy}df_{f^{n_0}(x)}^{-n_0}-df_{f^{n_0}(y)}^{-n_0}\|\leqslant K_1\rho(f^{n_0}(x), f^{n_0}(y))\leqslant K_1\mu_2^{n_0}\rho(x, y), \tag{2.7}$$

where the constant K_1 does not depend on x or y. For sufficiently small δ inequality (2.4) implies

$$\omega_2(f)\leqslant\eta\tilde{\rho}(\Pi_{f^{n_0}(x)f^{n_0}(y)}E(f^{n_0}(x)), E(f^{n_0}(y))). \tag{2.8}$$

We consider two cases:

a) $\rho(f^{n_0}(x), f^{n_0}(y))<\delta$,

b) $\rho(f^{n_0}(x), f^{n_0}(y))>\delta$.

In case a) the conditions of the lemma and inequality (2.8) imply

$$\omega_2(f)\leqslant\eta C\rho(f^{n_0}(x), f^{n_0}(y))^\alpha\leqslant\eta C(\mu_2^\alpha)^{n_0}\rho(x, y)^\alpha. \tag{2.9}$$

From (2.7) and (2.9) we get that

$$\omega(f)\leqslant(K_1\mu_2^{n_0}+\eta\mu_2^{\alpha n_0}C)\rho(x, y)^\alpha\leqslant C\rho(x, y)^\alpha$$

for $\alpha < n_0^{-1} \ln \eta^{-1}/\ln \mu_2$ (i.e. $\eta\mu_2^{\alpha n_0} < 1$), and

$$C > \frac{K_1\mu_2^{n_0}}{1-\eta\mu_2^{\alpha n_0}}.$$

In case b) we have

$$\omega_2(f) \leqslant \eta \cdot 1 = \eta\delta^{-1}\rho(f^{n_0}(x),\ f^{n_0}(y)) < \frac{\eta}{\delta}\mu_2^{n_0}\rho(x, y). \tag{2.10}$$

From (2.7) and (2.10) it follows that

$$\omega(f) \leqslant \left(K_1\mu_2^{n_0} + \frac{\eta}{\delta}\mu_2^{n_0}\right)\rho(x, y)^\alpha \leqslant C\rho(x, y)^\alpha$$

for any α and $C > (K_1 + \eta/\delta)\mu_2^{n_0}$. The lemma is proved.

We go over to the proof of the theorem. Let E be the distribution constructed in Lemma 3. Since this distribution satisfies the condition of Lemma 1, as $m \to \infty$ the sequence of distributions $f_*^{-mn_0}E$ converges in the metric d to the distribution E_1. By making use of Lemma 3 and proceeding by induction on m, we get that this sequence of distributions converges to the distribution E_1 in a space of distributions satisfying a Hölder δ-condition with fixed constant and exponent. Thus the distribution E_1 satisfies a Hölder δ-condition. The theorem is proved.

4. **Corollary 2.1.** *Let f be a partially hyperbolic dynamical system. Then the distributions E^s, E^u, E^0, $E^s \oplus E^0$ and $E^u \oplus E^0$ satisfy a Hölder δ-condition with constant and exponent depending only on the numbers λ_i and μ_i.*

5. We note that, as can be seen from the proof of Theorem 2.1,

$$\alpha < \frac{1}{n_0}\ln\eta^{-1}/\ln\mu_2 = \frac{1}{n_0}\ln K^{-1} + \ln\left(\frac{\mu_1}{\lambda_2}\right)^{-1}\Big/\ln\mu_2. \tag{2.11}$$

Inasmuch as K does not depend on n_0, which can be chosen sufficiently large, inequality (2.11) implies

$$\alpha < \ln\left(\frac{\mu_1}{\lambda_2}\right)^{-1}\Big/\ln\mu_2. \tag{2.12}$$

Inequality (2.12) provides an estimate of the Hölder exponent for the distribution E_1 in terms of the boundary of the spectrum of the diffeomorphism f. This implies

Corollary 2.2. *If, under the conditions of Theorem 2.1, $\mu_1\mu_2/\lambda_2 < 1$ (or $\mu_1/\lambda_2\lambda_1 < 1$), then the distribution E_1 (E_2) satisfies a Lipschitz condition.*

As was pointed out by Anosov, this result can be strengthened somewhat.

Corollary 2.3. *If, under the conditions of Theorem 2.1, $\mu_1\mu_2/\lambda_2 < 1$ (or $\mu_1/\lambda_2\lambda_1 < 1$), then the distribution E_1 (E_2) and the foliation $\mathfrak{S}_1$ ($\mathfrak{S}_2$) are smooth.*

This result was also announced in [16]. It immediately implies

Corollary 2.4. *If f is an Anosov diffeomorphism with a "point" spectrum, i.e. $0 < \lambda_1 = \mu_1 < 1 < \lambda_2 = \mu_2$, then the distributions E_1 and E_2 are smooth and the contracting and expanding foliations of f are smooth foliations. Moreover, there exists a neighborhood $\eta \subset \mathrm{Diff}^2(M^n)$ of the diffeomorphism f such that any diffeomorphism $g \in \eta$ is an Anosov diffeomorphism with smooth contracting and expanding foliations.*

6. Consider a fiber bundle (M, f) over an Anosov system g. This means that the dynamical system f acts in a bundle $\pi: M \to P$ with fiber Q (P and Q are compact manifolds) and the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \pi \downarrow & & \downarrow \pi \\ P & \xrightarrow{g} & P \end{array}$$

Suppose the spectrum of the Anosov system g is contained in two annuli with radii λ_1, μ_1 and λ_2, μ_2 (if g is an Anosov flow, one must add the unit circle) such that

$$0 < \lambda_1 \leqslant \mu_1 < 1 < \lambda_2 \leqslant \mu_2.$$

The compact sets $\pi^{-1}(z)$, $z \in P$, clearly form an invariant (relative to f) smooth foliation of the manifold M, which we denote by $\mathfrak{S}^0$.

Theorem 2.2. *Let $\Gamma_0 \subset \Gamma^0(TM^n)$ be the invariant submodule of continuous vector fields v such that $v(x) \in T_x\mathfrak{S}^0(x)$, $x \in M$, and suppose that the spectrum of the operator $f_*|_{\Gamma_0}$ is contained in an annulus with radii λ_0 and μ_0 such that $0 < \lambda_1 \leq \mu_1 < \lambda_0 \leq \mu_0 < \lambda_2 \leq \mu_2$. Then the following assertions are true:*

1) *f is a partially hyperbolic dynamical system whose spectrum is contained in the three annuli with radii $\lambda_1, \mu_1, \lambda_2, \mu_2$ and λ_0, μ_0.*

2) *The distribution E_0 is integrable and its integral manifolds form the foliation $\mathfrak{S}^0$.*

3) *There exist contracting and expanding foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ respectively corresponding to the distributions E_1 and E_2 having the property that $\pi(\mathfrak{S}^-(x)) = \mathfrak{S}^s(\pi(x))$, $\pi(\mathfrak{S}^+(x)) = \mathfrak{S}^u(\pi(x))$, where $x \in M$ and $\mathfrak{S}^s$ and $\mathfrak{S}^u$ are the contracting and expanding foliations of the Anosov action g.*

4) *The distributions $E_0 \oplus E_1$ and $E_0 \oplus E_2$ are integrable, i.e. the pairs of foliations $(\mathfrak{S}^-, \mathfrak{S}^0)$ and $(\mathfrak{S}^+, \mathfrak{S}^0)$ are integrable.*

5) *If $\mu_1\mu_0/\lambda_0 < 1$ ($\mu_0/\lambda_2\lambda_0 < 1$), the foliation $\mathfrak{S}^-$ ($\mathfrak{S}^+$) is smooth among the leaves of the foliation $\mathfrak{S}^0$.*

Proof. With the use of arguments analogous to those presented in [1], [2] and [8], it can be shown that the dynamical system f has contracting and expanding foliations respectively projecting into the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ of the Anosov system g. This implies assertions 1) and 3), inasmuch as

$$T_xM = T_x\mathfrak{S}^0(x) \oplus T_x\mathfrak{S}^-(x) \oplus T_x\mathfrak{S}^+(x)$$

(if f is a flow, one must add a one-dimensional direction of motion).

For the proof of assertion 4) we consider the continuous partition of M into the sets $\pi^{-1}(\mathfrak{S}^-(z))$, which locally are submanifolds of M. Thus we have a continuous foliation on M, which we denote by W. Clearly, if $x \in W(x_0)$, then $\mathfrak{S}^0(x) \subset W(x_0)$. Assertion 4) will follow if we show that $\mathfrak{S}^-(x) \subset W(x_0)$ when $x \in W(x_0)$. Let $y \in \mathfrak{S}(x_0)$. Inasmuch as $\mathfrak{S}^0$ is a smooth foliation and its leaves are compact, it follows from (2.1) that

$$\rho(f^n(\mathfrak{S}^0(x)), f^n(\mathfrak{S}^0(y))) \leqslant C\mu_1^n\rho(x, y), \tag{2.13}$$

where C is a certain constant. Hence $\rho(g^n(\pi(x)), g^n(\pi(y))) \leq C\mu_1^n\rho(x, y)$. It is known (see [1]) that a leaf $\mathfrak{S}^s(z)$ is the geometric locus of points z_1 such that

$$\rho(g^n(z), g^n(z_1)) \to 0$$

as $n \to \infty$. Therefore $\pi(y) \in \mathfrak{S}^s(\pi(x))$ and consequently $y \in W(x_0)$.

Assertion 5) follows from assertion 4) and Corollary 2.3 (inasmuch as the proof of Corollary 2.3 involves not the compactness of the manifold but the existence of uniform estimates). The theorem is proved.

Theorem 2.3. *Let f be a partially hyperbolic dynamical system and let E_0 be a smooth and integrable distribution whose integral manifolds form a smooth foliation $\mathfrak{S}^0$ with compact leaves. Suppose there exists for each $\epsilon > 0$ a $\delta > 0$ such that if $\rho(x, y) < \delta$, then $\mathfrak{S}^0(x) = h(\mathfrak{S}^0(y))$, where h is a diffeomorphism of the manifold lying in an ϵ-neighborhood of the identity diffeomorphism in the C^0 topology. Then the distributions $E_1 \oplus E_0$ and $E_2 \oplus E_0$ are integrable. If $\mu_1\mu_0/\lambda_0 < 1$ ($\mu_0/\lambda^2\lambda_0 < 1$), the foliation $\mathfrak{S}^-$ ($\mathfrak{S}^+$) is smooth along the leaves of the foliation $\mathfrak{S}^0$.*

Proof. The assertion of the theorem follows from Theorem 2.2 and the following lemma.

Lemma. *Under the conditions of the theorem, a dynamical system f can be represented as a fiber bundle over an Anosov system.*

Proof. Consider the quotient set $M/\mathfrak{S}^0$ ($\mathfrak{S}^0$ is a partition of M) and introduce the structure of a smooth manifold in it. We denote by π the natural projection $\pi: M \to M/\mathfrak{S}^0$. Let $x \in M$, let $W(x)$ ($\ni x$) be a smooth $(n - k)$-dimensional submanifold ($n = \dim M$, $k = \dim \mathfrak{S}^0(x)$) that is transversal to the leaves of the foliation $\mathfrak{S}^0$, and let $U(x)$ ($\subset W(x)$) be a neighborhood of the point x. We put

$$\overline{U}(x) = \bigcup_{z \in U(x)} \mathfrak{S}^0(z).$$

Consider the mapping $\phi(x)$: $\overline{U}(x) \to U(x)$, $\phi(x)(y) = z$, where $y \in \mathfrak{S}^0(z)$ and $z \in U(x)$. Clearly, it induces a bijective mapping $\overline{\phi}(x)$: $\pi(\overline{U}(x)) \to U(x)$. Therefore, since $\mathfrak{S}^0$ is a smooth foliation with compact leaves, the introduction in $M/\mathfrak{S}^0$ of the system of charts $(\pi(\overline{U}(x)), \phi(x))$ (relative to all possible $x \in M$ and neighborhoods $U(x)$) converts the set $M/\mathfrak{S}^0$ into a smooth compact manifold. The manifold M can then be considered as a bundle space over $M/\mathfrak{S}^0$ with fibers $\pi^{-1}(z) = \mathfrak{S}^0(x)$, $z = \pi(x) \in M/\mathfrak{S}^0$. By virtue of the invariance of the foliation $\mathfrak{S}^0$, the dynamical system f induces a smooth dynamical system g on the manifold $M/\mathfrak{S}^0$ such that $\pi \circ f = g \circ \pi$. It remains to show that g is an Anosov system. Since for every $x \in M$

$$T_xM = T_x\mathfrak{S}^+(x) \oplus T_x\mathfrak{S}^0(x) \oplus T_x\mathfrak{S}^-(x),$$

we have

$$T_zM/\mathfrak{S}^0 = \pi^*T_x\mathfrak{S}^+(x) \oplus \pi^*T_x\mathfrak{S}^-(x),$$

where $z = \pi(x)$. If $v \in \pi^*T_x\mathfrak{S}^-(x)$, then $v = \pi^*w$ and

$$\|dg^nv\| = \|dg^n\pi^*w\| = \|\pi^*df^nw\| \leqslant c_1\mu_1^n\|\pi^*w\| \leqslant c_1\mu_1^n\|v\|.$$

If $v \in \pi^*T_x\mathfrak{S}^+(x)$, then, as above, it can be shown that $\|dg^{-n}v\| \leq c_2\mu_2^{-n}\|v\|$. Hence g is an Anosov system whose spectrum is contained in the two annuli with radii λ_1, μ_1 and λ_2, μ_2 (if g is a flow, one must add the unit circle), while the corresponding distributions E_1 and E_2 are such that $E_1(z) = \pi^*T_x\mathfrak{S}^-(x)$ and $E_2(z) = \pi^*T_x\mathfrak{S}^+(x)$, $z = \pi(x)$. The lemma is proved.

§3. Absolute continuity of the foliations

1. We consider an arbitrary continuous k-foliation $\mathfrak{S}$ on a compact Riemannian manifold M^n (see [11]). Let $x \in M^n$ and $y \in \mathfrak{S}(x)$, and consider a pair of open smooth $(n-k)$-submanifolds $W_1 \ni x$ and $W_2 \ni y$ that are diffeomorphic to an open ball in $\mathbf{R}^{n-k}$ and transversal to the leaves of the foliation $\mathfrak{S}$. If $\epsilon > 0$ is sufficiently small and $\rho_{\mathfrak{S}(x)}(x, y) < \epsilon$, then there exist submanifolds $(x\epsilon)\ W_x \subset W_1$ and $(y\epsilon)\ W_y \subset W_2$ and a bijective continuous mapping $p: W_x \to W_y$ taking an arbitrary point $x_1 \in W_x$ into a point $y_1 = p(x_1) \in W_y \cap \mathfrak{S}(x_1)$ such that $\rho_{\mathfrak{S}(x)}(x_1, y_1) < 2\epsilon$.

We denote by μ_x and μ_y the internal measures induced on W_x and W_y by the Riemannian metric.

Definition 3.1. A foliation $\mathfrak{S}$ is said to be *absolutely continuous* if, for any two submanifolds W_x and W_y of the type described above, the measures μ_x and $p^*\mu_y$ are equivalent.

2. **Theorem 3.1.** *Let f^t be a dynamical system of class C^2 on a compact smooth Riemannian manifold M^n (t is a continuous or discrete parameter). Suppose that the spectrum of f^t consists of two nonintersecting components S_1 and S_2 respectively lying in annuli with radii λ_1, μ_1 and λ_2, μ_2 such that $0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2$, $\mu_1 < 1$. Then the foliation $\mathfrak{S}$ corresponding to the distribution E_1 (see Proposition 2.3) is absolutely continuous.*

For the proof of the theorem we require a series of lemmas.

Lemma 1. *There exists a constant* $C_3 > 0$ *such that for any* $x_1 \in W_x$ *and* $y_1 \in W_y$ *(see Definition 3.1) and any* $t > 0$

$$\tilde{\rho}(T_{f^t(x_1)}f^t(W_x), E_2(f^t(x_1))) < C_3\left(\frac{\mu_1}{\lambda_2}\right)^t, \tag{3.1}$$

$$\tilde{\rho}(T_{f^t(y_1)}f^t(W_y), E_2(f^t(y_1))) < C_3\left(\frac{\mu_1}{\lambda_2}\right)^t. \tag{3.2}$$

Proof. Let $v \in T_{x_1}W_x$, $\|v\| = 1$. We represent the vector v in the form $v = \gamma_1 v_1 + \gamma_2 v_2$, where $v_1 \in E_1(x_1)$, $v_2 \in E_2(x_1)$, $\|v_1\| = \|v_2\| = 1$, with $\gamma_2 \neq 0$ inasmuch as $T_{x_1}W_x \cap E_1(x_1) = 0$. By making use of the invariance of the distributions E_1 and E_2, we obtain an analogous decomposition for the vector $df^t v$: $df^t v = \gamma_1 df^t v_1 + \gamma_2 df^t v_2$. Hence

$$\tilde{\rho}(df^t v, E_2(f^t(x_1))) \leq K\gamma_1 \| df^t v_1 \|, \tag{3.3}$$

where K is a certain constant. From (3.3) and (2.3) (where n is replaced by t) it follows that

$$\tilde{\rho}\left(\frac{df^t v}{\| df^t v \|}, E_2(f^t(x_1))\right) < \frac{K\gamma_1 q_1}{\gamma_2 q_2'}\left(\frac{\mu_1}{\lambda_2}\right)^t. \tag{3.4}$$

Inasmuch as the set W_x is compact while the angles between W_x and the leaves of the foliation $\mathfrak{S}$ are different from zero, we have

$$\max_{v \in T_{x_1}W_x, \|v\|=1} \frac{\gamma_1}{\gamma_2} \geqslant 0.$$

Inequality (3.1) now follows directly from (3.4). Inequality (3.2) is proved analogously. The lemma is proved.

Lemma 2. *There exist constants* $\eta_1 \in (0, 1)$ *and* $C_4 > 0$ *such that for any* $x_1 \in W_x$ *and* $y_1 \in W_y$, $p(x_1) = y_1$,

$$\tilde{\rho}(T_{f^t(x_1)}f^t(W_x), T_{f^t(y_1)}f^t(W_y)) < C_4\eta_1^t. \tag{3.5}$$

Proof. The assertion of the lemma immediately follows from Lemma 1 and inequalities (2.1) (where n is replaced by t) and (2.2).

Lemma 3. *For each* $\delta \in (0, \lambda_2 - \mu_1)$ *there exists a constant* $C_5 > 0$ *such that for* $x_1, x_2 \in W_x$ *and* $y_1, y_2 \in W_y$

$$\rho_{f^t(W_x)}(f^t(x_1), f^t(x_2)) > C_5(\lambda_2 - \delta)^t \rho(x_1, x_2), \tag{3.6}$$

$$\rho_{f^t(W_y)}(f^t(y_1), f^t(y_2)) > C_5(\lambda_2 - \delta)^t \rho(y_1, y_2). \tag{3.7}$$

Proof. From (3.1) and the compactness of W_x it follows that for any $\epsilon > 0$ there exists a $t_\epsilon > 0$ such that for all $t \geq t_\epsilon$ and for any $x_1 \in W_x$

$$\tilde{\rho}(T_{f^t(x_1)}f^t(W_x), E_2(f^t(x_1))) < \varepsilon.$$

We choose any $\delta \in (0, \lambda_2 - \mu_1)$. By virtue of Proposition 2.2 (see also (2.3)) for the chosen δ there exists an $\epsilon = \epsilon(\delta)$ such that for all $t \geq t_\epsilon$, any $x_1 \in W_x$ and $v \in T_{f^{t_\epsilon}(x_1)} f^t \omega(W_x)$, $\|v\| = 1$,

$$df^t_{f^{t_\varepsilon}(x_1)} \|v\| \geqslant (\lambda_2 - \delta)^{t-t_\varepsilon} \|v\|. \tag{3.8}$$

Consider an arbitrary smooth curve

$$\psi : I \to f^{t_\varepsilon}(W_x), \quad \psi(0) = f^{t_\varepsilon}(x_1), \quad \psi(1) = f^{t_\varepsilon}(x_2).$$

The length $|\psi|$ of this curve is given by the formula

$$|\psi| = \int_0^1 \|\psi'(s)\| \, ds.$$

The length of the image of ψ under the action of the mapping f^{t-t_ϵ}, $t \geq t_\epsilon$, is equal to

$$|f^{t-t_\varepsilon}(\psi)| = \int_0^1 \|(f^{t-t_\varepsilon}\psi(s))'\| \, ds.$$

From (3.8) it follows that

$$\|(f^{t-t_\varepsilon}\psi(s))'\| \geqslant (\lambda_2 - \delta)^{t-t_\varepsilon} \|\psi'(s)\|.$$

This immediately implies the first assertion of the lemma. The second assertion is proved analogously.

Lemma 4. *There exists a number $\gamma > 0$ such that if $x_1, x_2 \in f^t(W_x)$, $y_1, y_2 \in f^t(W_y)$, $\rho(x_1, x_2) < \gamma$ and $\rho(y_1, y_2) < \gamma$, then*

$$\tilde{\rho}(T_{x_1} f^t(W_x), T_{x_2} f^t(W_x)) < C\rho(x, y)^\alpha + 2C_3\left(\frac{\mu_1}{\lambda_2}\right), \tag{3.9}$$

$$\tilde{\rho}(T_{y_1} f^t(W_y), T_{y_2} f^t(W_y)) < C\rho(x, y)^\alpha + 2C_3\left(\frac{\mu_1}{\lambda_2}\right)^t. \tag{3.10}$$

The proof immediately follows from (3.1), (3.2), (2.2) and the triangle inequality.

Let $\mu_x(t)$ and $\mu_y(t)$ denote the internal measures induced on the submanifolds $f^t(W_x)$ and $f^t(W_y)$ by the Riemannian metric, let $p_t = f^t \circ p \circ f^{-t}$, and let

$$B_t = \{x' \in f^t(W_x) \mid \rho_{f^t(W_x)}(x', x_0) < r(t), \ x_0 \in f^t(W_x)\}.$$

Lemma 5. *If $r(t) \to 0$, $c_2\mu_1^t / r(t) \to 0$ and*

$$\frac{\rho(\partial B_t, \partial f^t(W_x))}{r(t)} \to \infty$$

as $t \to \infty$, then

$$\left|\frac{\mu_x(t)(B_t)}{\mu_y(t)(p_t(B_t))}-1\right|<\varepsilon(t), \tag{3.11}$$

with $\epsilon(t) \to 0$ *as* $t \to \infty$.

Proof. We cover the manifold M^n by a standard finite system of coordinate neighborhoods (U_i, ϕ_i), where $\phi_i(U_i)$ is a ball in $\mathbf{R}^n$. Most of the subsequent assertions should, whenever necessary, be referred not to the objects themselves but to their images under the action of the appropriate mappings ϕ_i.

We consider for sufficiently large t pieces of the submanifolds $f^t(W_x)$ and $f^t(W_y)$ that lie in a coordinate neighborhood of x_0 and contain the balls B_t and $p_t(B_t)$ respectively. These pieces of submanifolds are represented in this neighborhood by the graphs of certain smooth mappings

$$g_1, g_2 : T_{x_0}B_t \to T_{x_0}\mathfrak{S}(x_0).$$

Let $x_1 \in B_t$, $y_1 = p_t(x_1)$ and $y_2 = \Pi \cap f^t(W_{y_1})$, where Π is the k-plane passing through x that is parallel to the plane $T_{x_0}\mathfrak{S}(x_0)$. It is easily seen that there exists a constant $C_6 > 0$ (depending only on the distance between W_x and W_y) such that $\rho(y_1, y_2) < C_6\mu_1^t$. It therefore follows from (3.1), (3.2) and (2.2) that for certain $\eta_2 \in (0, 1)$ and $C_7 > 0$

$$\tilde{\rho}(T_{y_1}p_t(B_t),\ T_{y_2}p_t(B_t)) < C_7\eta_2^t. \tag{3.12}$$

Inequality (3.5) implies

$$\tilde{\rho}(T_{x_1}B_t,\ T_{y_1}p_t(B_t)) < C_4\eta_1^t. \tag{3.13}$$

From (3.12), (3.13) and the triangle inequality it follows that

$$\tilde{\rho}(T_{x_1}B_t,\ T_{y_2}p_t(B_t)) < C_8\eta_3^t, \tag{3.14}$$

where $C_8 = C_7 + C_5$ and $\eta_3 = \max\{\eta_1, \eta_2\}$.

Let z_i, $i = 1, \ldots, n-k$, denote coordinates on the plane $T_{x_0}B_t$. Inequality (3.14) implies

$$\sum_{i=1}^{n-k}\left|\frac{\partial g_1}{\partial z_i}-\frac{\partial g_2}{\partial z_i}\right| < C_9\eta_3^t, \tag{3.15}$$

where C_9 is a certain constant. We put

$$\beta(t) = \inf_{x \in B_t} \rho(x, p_t(x))$$

and denote by $B_{-\beta(t)}$ the $\beta(t)$-interior, and by $B_{\beta(t)}$ the $\beta(t)$-inflation, of the set B_t on the submanifold $f^t(W_x)$. From (2.1) it follows that $\beta(t) < C_2\mu_1^t\beta(0)$ and consequently, by virtue of the conditions of the lemma, that $\beta(t)/r(t) \to 0$. There therefore exists a function $\gamma(t)$, $\gamma(t) \to 0$ as $t \to \infty$, for which

$$\frac{\mu_x(t)(B_t)}{\mu_x(t)(B_{-\beta(t)})} - 1 < \gamma(t), \tag{3.16}$$

$$\frac{\mu_x(t)(B_{\beta(t)})}{\mu_x(t)(B_{\beta(t)})} - 1 < \gamma(t). \tag{3.17}$$

Let P denote the operator of projection onto the plane T_{x_0} along the plane $T_{x_0}\mathfrak{S}(x_0)$. Clearly,

$$P(B_{-\beta(t)}) \subset P(p_t(B_t)) \subset P(B_{\beta(t)}). \tag{3.18}$$

The measure of any set $A \subset f^t(W_x)$ $(A \subset f^t(W_y))$ can be calculated according to the formula

$$\mu(A) = \int\limits_{P(A)} \sqrt{1 + \left|\left(\frac{\partial g_i}{\partial z}\right)\left(\frac{\partial g_i}{\partial z}\right)'\right|}\, dz, \tag{3.19}$$

where $i = 1$ and $\mu = \mu_x(t)$ $(i = 2$ and $\mu = \mu_y(t))$. Inequalities (3.18) and (3.19) imply

$$\begin{aligned}\int\limits_{P(B_{-\beta(t)})} \sqrt{1 + \left|\left(\frac{\partial g_i}{\partial z}\right)\left(\frac{\partial g_i}{\partial z}\right)'\right|}\, dz &< \mu(p_t(B_t)) \\ &< \int\limits_{P(B_{\beta(t)})} \sqrt{1 + \left|\left(\frac{\partial g_i}{\partial z}\right)\left(\frac{\partial g_i}{\partial z}\right)'\right|}\, dz,\end{aligned} \tag{3.20}$$

where $i = 1$ and $\mu = \mu_x(t)$ $(i = 2$ and $\mu = \mu_y(t))$. The assertion of the lemma now follows from (3.15)–(3.17) and (3.20).

Let $\mu_1 < \eta_4 < \min\{1, \lambda_2\}$. For given area elements W_x and W_y there exist a point $x_0 \in W_x$ and a constant $C_{10} > 0$ such that if

$$r(t) = C_{10}\eta_4^t, \tag{3.21}$$

then the ball B_t of radius $r(t)$ with center at $f^t(x_0)$ satisfies the conditions of Lemma 5.

We fix a sufficiently large $t_0 > 0$ and consider the preimages $f^{-t}(B_{t_0})$ for $t \leq t_0$. Inequality (3.6) implies

$$\operatorname{diam} f^{-t}(B_{t_0}) < \frac{2r(t_0)}{C_5(\lambda_2 - \delta)^{t_0}}.$$

It therefore follows from (3.21) that

$$\operatorname{diam} f^{-t}(B_{t_0}) < C_{11}\eta_5^t, \tag{3.22}$$

where $C_{11} = 2C_{10}/C_5$ and $\eta_5 = \eta_4/(\lambda_2 - \delta)$. We choose the number δ in Lemma 3 so that $\eta_5 < 1$.

Analogous arguments (using (3.7) and (3.21)) show that

$$\operatorname{diam} f^{-t}(p_{t_0}(B_{t_0})) < C_{12}\eta_5^t, \tag{3.23}$$

where C_{12} is a certain constant.

Lemma 6. *Under the conditions of Lemma 5 (for example, if condition* (3.21) *is satisfied) there exists a constant* $C_{13} > 0$, *depending only on the original submanifolds* W_x *and* W_y, *such that*

$$\frac{1}{C_{13}} < \frac{\mu_x(f^{-t}(B_t))}{\mu_y(f^{-t}p_t(B_t))} < C_{13}. \tag{3.24}$$

Proof. Consider the open submanifolds $f^{-t}(B_t) \subset W_x$ and $f^{-t}(p_t(B_t))$. We have

$$\mu_x(f^{-t}(B_t)) = \int_{B_t} |df_z^{-t}|_{T_zB_t}|\, d\mu_x(t)(z)|, \tag{3.25}$$

$$\mu_y(f^{-t}(p_t(B_t)) = \int_{p_t(B_t)} |df_{z'}^{-t}|_{T_{z'}B_t}|\, d\mu_y(t)(z')|. \tag{3.26}$$

Here $z \in B_t$, $z' \in p_t(B_t)$ and $|\cdot|$ denotes the coefficient of volume expansion for a mapping $df_z^{-t}|_A$. Let us estimate the difference in the values of the integrand in (3.25) at two points z_1 and z_2. Inasmuch as $f^1 \in \mathrm{Diff}^2(M^n)$, there exists a constant $C_{14} > 0$ such that

$$\| df_{z_1}^{-1} - df_{z_2}^{-1} \| \leqslant C_{14}\rho(z_1, z_2)$$

(see (2.7)). There therefore exists a constant $C_{15} > 0$ such that for any subspace A

$$|\| df_{z_1}^{-1}|_A\| - \| df_{z_2}^{-1}|_A\|| < C_{15}\rho(z_1, z_2). \tag{3.27}$$

In addition, there exists a constant $C_{16} > 0$ such that for any subspaces A_1 and A_2 in T_zB_t

$$|\| df_z^{-t}|_{A_1}\| - \| df_z^{-t}|_{A_2}\|| \leqslant C_{16}\tilde{\rho}(A_1, A_2). \tag{3.28}$$

It follows from (3.27), (3.28) and (3.9) (see Lemma 4) that for sufficiently small $\rho(z_1, z_2)$

$$\begin{gathered} |\, \|df_{z_1}^{-t}|_{T_{z_1}B_t}\| - \| df_{z_2}^{-t}|_{T_{z_2}B_t}\||\\ \leqslant C_{15}\rho(z_1, z_2) + C_{16}\left(C\rho(z_1, z_2)^{\alpha} + 2C_3\left(\frac{\mu_1}{\lambda_2}\right)^t\right). \end{gathered} \tag{3.29}$$

Let

$$\mathcal{J}(\tau, z) = \| df_{f^{-\tau}(z)}^{-t}|_{T_{f^{-\tau}(z)}f^{-\tau}(B_t)}\|.$$

Replacing z_1 and z_2 by $f^{-\tau}(z_1)$ and $f^{-\tau}(z_2)$ respectively in (3.29) and making use of (3.22), we get

$$|\mathcal{J}(\tau, z_1) - \mathcal{J}(\tau, z_2)| < C_{15}\rho(f^{-\tau}(z_1), f^{-\tau}(z_2))$$
$$+ C_{16}\left(C\rho(f^{-\tau}(z_1), f^{-\tau}(z_2))^{\alpha} + 2C_3\left(\frac{\mu_1}{\lambda_2}\right)^{t-\tau}\right) \tag{3.30}$$
$$\leq C_{15}C_{12}\eta_5^t + C_{16}CC_{12}^{\alpha}\eta_5^{\alpha t} + 2C_3\left(\frac{\mu_1}{\lambda_2}\right)^{t-\tau} \leq C_{17}\eta_5^{t-\tau}.$$

Clearly, there exists a constant $\mathcal{J}_0 > 0$ such that

$$\mathcal{J}_0^{-1} < \mathcal{J}(\tau, z) < \mathcal{J}_0 \tag{3.31}$$

for all $0 \leq \tau \leq t$ and $z \in B_t$. Inasmuch as $\eta_5 < 1$, it follows from (3.30) and (3.31) that

$$C_{18}^{-1} < \prod_{\tau=0}^{t-1} [\mathcal{J}(\tau, z_1)\mathcal{J}^{-1}(\tau, z_2)] < C_{18}, \tag{3.32}$$

where C_{18} is a certain constant.

By considering points $z_1', z_2' \in p_t(B_t)$ and a funxtion $\mathcal{J}(\tau, z')$, which is defined on the submanifold $p_t(B_t)$ analogously to the function $\mathcal{J}(\tau, z)$, we can prove that

$$C_{18}^{-1} < \prod_{\tau=0}^{t-1} [\mathcal{J}(\tau, z_1')\mathcal{J}^{-1}(\tau, z_2')] < C_{18}. \tag{3.33}$$

The latter two inequalities show that if the integrands in (3.25) and (3.26) are replaced by their values at certain points z_1 and $p_t(z_1)$ respectively, the measures $\mu_x(f^{-t}(B_t))$ and $\mu_y(f^{-t}(p_t(B_t))$ themselves do not change by more than a factor C_{18}. For the completion of the proof of Lemma 6 it remains to estimate the product

$$\prod_{\tau=0}^{t-1} \mathcal{J}(\tau, z_1)\mathcal{J}^{-1}(\tau, p_t(z_1)).$$

Consider points z_1 and $p_t(z_1) \in \mathfrak{S}(z_1)$. Inasmuch as (3.29) is valid for any two sufficiently close points of the manifold M^n while the points under consideration satisfy (3.22) (with some other constant; see also (2.1)), we have

$$|\mathcal{J}(\tau, z_1) - \mathcal{J}(\tau, p_t(z_1))| < C_{19}\eta_5^{t-\tau},$$

where C_{19} is a certain constant. Therefore, since the functions $\mathcal{J}(\tau, z)$ and $\mathcal{J}^{-1}(\tau, z)$ are uniformly bounded with respect to τ and z and are different from zero and inasmuch as $\eta_5 < 1$, there exists a constant $C_{20} > 0$, not depending on t, such that

$$C_{20}^{-1} < \prod_{\tau=0}^{t-1} \mathcal{J}(\tau, z_1)[\mathcal{J}(\tau, p_t(z_1))]^{-1} < C_{20}. \tag{3.34}$$

It follows from (3.32)–(3.34) that the assertion of the lemma is valid for $C_{13} = C_{18}^2 C_{20}$.

The absolute continuity of the foliation $\mathfrak{S}$ is equivalent to the absolute continuity of the mapping p (see Definition 3.1). We will actually prove somewhat more.

Namely, we will show that the Jacobians of the mappings p are uniformly bounded and different from zero. For this purpose it suffices to prove (see [7]) that any ball $D_R(x_0) = \{x_1 \in W_x | \rho_{W_x}(x_1, x_0) < R,\ x_0 \in W_x\}$ satisfies the inequality

$$\frac{1}{C_{21}} \leqslant \frac{\mu_x(D_R(x_0))}{\mu_y(p(D_R(x_0)))} < C_{21}, \tag{3.35}$$

where C_{21} does not depend on the ball $D_R(x_0)$ in question. We note that Lemma 5 permits one to equate the measures of a ball B_t and its image $p_t(B_t)$; but these sets lie in "distant" images of the submanifolds W_x and W_y under the action of the dynamical system f^t. Lemma 6 makes it possible to equate the measures of the pre-images $f^{-t}(B_t)$ and $f^{-t}(p_t(B_t)) = p_t(f^{-t}(B_t))$, which lie on the submanifolds W_x and W_y but are not balls. In order to overcome this difficulty, we select a number ϵ_0 so that

$$\left| \frac{\mu_x(D_{-\varepsilon_0})}{\mu_x(D_R(x_0))} - 1 \right| < 0,1,$$

$$\left| \frac{\mu_y(p(D_{-\varepsilon_0}))}{\mu_y(p(D_R(x_0)))} - 1 \right| < 0,1, \tag{3.36}$$

where $D_{-\epsilon_0}$ denotes the ϵ_0-interior of a ball $D_R(x_0)$ on the submanifold W_x. By virtue of (3.6)

$$\rho_{f^t(W_x)}(f^t(D_{-\varepsilon_0}), \partial f^t(D_R(x_0))) > C_5(\lambda_2 - \delta)^t \varepsilon_0.$$

There therefore exists a $t_0 > 0$ such that for all $t > t_0$

$$r(t) \leqslant 2\rho_{f^t(W_x)}(f^t(D_{-\varepsilon_0}), \partial f^t(D_R(x_0))); \tag{3.37}$$

here $r(t) = C_{10}\eta_4^t$ (see (3.21)).

Lemma 7. *If $t \geq t_0$, then on the manifold $f^t(W_x)$ there exist $m = m(t)$ balls B_t^j of radius $r(t)$ that cover the set $f^t(D_{-\epsilon_0})$ and lie interior to $f^t(D_R(x_0))$, the multiplicity of the covering (i.e. the maximal number of balls covering one point) being finite and depending only on the distribution E_2 and the geometric properties of the manifold M^n.*

Proof. We note that the coordinate mappings ϕ_i and ϕ_i^{-1} (see Lemma 5) satisfy a Lipschitz condition with some constant C_{22}. Consider in the ball $\phi_i(u_i)$ a cubic lattice with edge length $r(t)/2\sqrt{n}C_{22}$. If a cube intersects $\phi_i(f^t(D_{-\epsilon_0}))$, we choose in each component of the intersection an arbitrary point u_j. Let $x_j = \phi_i^{-1}(u_j)$. The points x_j form an $r(t)/2$-net in the set $f^t(D_{-\epsilon_0})$. As the balls B_t^j, we consider the balls of radius $r(t)$ with centers at the points x_j. It is easily seen that these balls lie interior to $f^t(D_R(x_0))$ and cover the set $f^t(D_{-\epsilon_0})$, the multiplicity of the covering being not greater than $(16\sqrt{n}C_{22}^2)^n$ ($n = \dim M^n$). The lemma is proved.

The assertion of the theorem now follows from Lemmas 5–7 (see (3.11), (3.24), (3.36) and (3.37)), it being possible to put $C_{21} = 2C_{13}(16\sqrt{n}C_{22}^2)^n$ in (3.35).

3. **Corollary 3.1.** *The foliations $\mathfrak{S}^u$ and $\mathfrak{S}^s$ of a partially hyperbolic dynamical system f of class C^2 are absolutely continuous, the Jacobians of the corresponding mappings p being uniformly bounded and different from zero.*

The proof follows directly from Corollary 2.1 and Theorem 3.1 (see also §2.2).

4. **Definition 3.2.** A vector field $v \in \Gamma^0(TM^n)$ is *subordinate* to a continuous foliation $\mathfrak{S}$ with smooth leaves if $v(x) \in T_x\mathfrak{S}(x)$ and $v|_{\mathfrak{S}(x)}$ is a vector field of class C^1 that continuously depends on x in the C^1 topology.

Theorem 3.2. *Suppose that a vector field $v \in \Gamma^0(TM^n)$ is subordinate to a continuous k-foliation $\mathfrak{S}$ with smooth leaves, and suppose that the foliation $\mathfrak{S}$ is absolutely continuous, the Jacobians of the mappings appearing in Definition 3.1 being uniformly bounded and different from zero. If g is a translation by a certain time along the trajectories of the vector field $v(x)$, then the homeomorphism g is absolutely continuous and its Jacobian is bounded and different from zero.*

Proof. Let (U, ϕ) be a chart at a point $x_0 \in M^n$, let ξ denote the partition of U into the connected components of the intersection of the leaves of the foliation $\mathfrak{S}$ with the neighborhood U and let $C_\xi = C_\xi(x)$ denote the element of the partition ξ, viz. the local leaf, that passes through the point $x \in U$. The neighborhood U and the diffeomorphism ϕ can be chosen so that (see [2])

1) $\phi(U)$ is the unit cube I^n in $\mathbf{R}^n$;

2) $\phi(C_\xi)$ is a smooth surface that projects single-valuedly along an $(n-k)$-plane Q containing an $(n-k)$-face of the cube I^n onto a k-plane P containing a k-face of the cube.

We denote by η a partition of U into smooth submanifolds transversal to the leaves of $\mathfrak{S}$ such that $\phi(C_\eta)$ is an $(n-k)$-plane parallel to Q. Inasmuch as ξ is a measurable partition of U, the Lebesgue measure μ of any measurable set $A \subset U$ can be calculated according to the formula

$$\mu(A) = \int_U \chi_A(x)\,d\mu = \int_{U/\xi} d\nu_1(C_\xi) \int_{C_\xi(x)} \chi_A(x)\,d\nu_2(x). \tag{3.38}$$

Here $\nu_1(\cdot)$ is the measure in the quotient space U/ξ, $\nu_2(\cdot)$ is the conditional measure on an element $C_\xi(x)$ and χ_A is the characteristic function of the set A.

Lemma 1. *The measure $\nu_2(\cdot)$ on $C_\xi(x)$ is absolutely continuous with respect to Lebesgue measure on $C_\xi(x)$; the measure $\nu_1(\cdot)$ induces on any C_η a measure that is absolutely continuous with respect to Lebesgue measure on C_η.*

Proof. By making use of Theorem 3.1, the proof of this lemma can be carried out in the same way as the proof of the analogous assertion for Anosov systems in [2] (see §5).

Consider on some plane $\phi(C_\eta)$ a cube A_1 and consider through each point x of this cube the surface $\phi(C_\xi(x))$. Let $A_2(x)$ be a set such that $x \in A_2(x) \subset \phi(C_\xi(x))$ and $A_2(x)$ projects into a cube A_2 on the plane P with center at the point of intersection of P and $\phi(C_\eta)$. The set $\Pi = \bigcup_{x \in A_1} A_2(x) \subset I^n$ is called a *cylinder in* I^n; A_1 is called the *base*, and A_2 is called the *generator*, of the cylinder. By a *cylinder in* U is meant the preimage of a cylinder in I^n under the mapping ϕ.

Lemma 2. *Let* $B = A^k \times A^{n-k}$ *denote a cube in* I^n *with center at* $\phi(x_0)$ *whose* k*-face* A^k *lies in a plane* P_1 *parallel to* P *and whose* $(n-k)$*-face* A^{n-k} *lies in a plane* Q_1 *parallel to* Q. *Then there exist positive numbers* r_0, C_1 *and* C_2 *not depending on the chosen cube such that, if* diam $B \le r_0$, *there exist cylinders* Π_1 *and* Π_2 *in* I^n *for which*

$$\Pi_2 \subset B \subset \Pi_1 \quad C_1\mu(\Pi_1) \leqslant \mu(B) \leqslant C_2\mu(\Pi_2). \tag{3.39}$$

Proof. Let $\alpha(x)$ be the magnitude of the angle between the plane $\phi(C_\eta(x))$ and the surface $\phi(C_\xi(x))$ at the point x. It is obvious that the function $\alpha(x)$ is continuous. Consequently there exist

$$\alpha_1 = \min_{x \in I^n} \alpha(x) > 0 \quad \text{and} \quad \alpha_2 = \max_{x \in I^n} \alpha(x) \leqslant \frac{\pi}{2}.$$

Consider on the plane Q_1 a cube A that is homothetic to the cube A^{n-k} and whose edge length is $2\tan\alpha_1 + 1$ times greater. If r_0 is sufficiently small, this cube lies in I^n. Let Π_1 denote the cylinder with k-generator A^k and $(n-k)$-base A_1. It is obvious that $B \subset \Pi_1$ and $\mu(B) > C_1\mu(\Pi_1)$, where $C_1 = (2\tan\alpha_1 + 1)^{n-k}$. The cylinder Π_2 is constructed analogously. The lemma is proved.

Lemma 3. *There exist numbers* r_1, C_3 *and* C_4 *such that for any cylinder whose diameter is less than* r_1

$$C_4\mu(\Pi) \leqslant \mu(g(\Pi)) \leqslant C_3\mu(\Pi). \tag{3.40}$$

Proof. Suppose the cylinder Π is contained in a local chart U at a point $x_0 \in$ int Π. If the diameter of Π is sufficiently small, there exists a $t_0 > 0$ such that $g(\Pi) \subset g(U) \subset V$, where g is a translation by a time $t \le t_0$ and V is another local chart at x_0 such that $U \subset V$. We identify these local charts with corresponding cubes in $\mathbf{R}^n$. Let W denote the $(n-k)$-base of the cylinder Π. We partition W into cubes W_i, $i = 1, \dots, m$, and consider the local leaves C_ξ through all of the points of each W_i. The resulting cylinders Π_i with bases W_i form a partition of Π. It suffices to prove (3.40) for each cylinder Π_i. The set $g(\Pi_i)$ cannot be a cylinder, since the set $g(W)$ is generally only a C^0 submanifold of M^n. But if one chooses the diameters of the cubes W_i sufficiently small, then, by virtue of the uniform continuity of g on W, the oscillation of g on each cube W_i can be made less than any preassigned ϵ. If ϵ is sufficiently small, the set $g(\Pi_i)$ can be approximated by cylinders Π_i' and

Π_i'' so that $\Pi_i' \subset g(\Pi_i) \subset \Pi_i''$ and $\frac{1}{2}\mu(\Pi_i') \le \mu(g(\Pi_i)) \le 2\mu(\Pi_i'')$, the $(n-k)$-bases W_i' and W_i'' of the cylinders Π_i' and Π_i'' being images under the successor mapping effected by means of the local leaves of the foliation $\mathfrak{S}$. Formula (3.38) implies

$$\begin{aligned}\mu(\Pi_i') &= \int\limits_{g(U)/g\xi} d\nu_1'(C_{g\xi}(x)) \int\limits_{C_{g\xi}(x)} \chi_{\Pi_i'}(x)\, d\nu_2'(x) \\ &= \int\limits_{M} \mathcal{J}(y)\, d\nu_1(x) \int\limits_{C_\xi(x)} \chi_{g^{-1}(\Pi_i')}(x)\, |\lambda(x)|\, d\nu_2(x).\end{aligned} \tag{3.41}$$

Here $g\xi$ denotes the partition of $g(U)$ into the sets $g(C_\xi) = C_{g\xi}$. (It is obvious that $g\xi$ is the partition into the connected components of the intersection of the leaves of $\mathfrak{S}$ and $g(U)$.) $\nu_1'(\cdot)$ is the measure in the quotient space $g(U)/g\xi$, which, by virtue of Lemma 1 and the absolute continuity of the mapping $p\colon W_i \to W_i'$, is absolutely continuous with respect to the measure $p_*\nu_1(\cdot)$ (the measure $\nu_1(\cdot)$ in (3.38) can be regarded as given on W). We have denoted the density of the measure ν_1' by T. In addition, there exist $K_1 > 0$ and $K_2 > 0$ such that $K_1 \le T(x) \le K_2$ for all $x \in M^n$. $\nu_2'(\cdot)$ is the conditional measure on $C_{g\xi}(x)$, which, according to Lemma 1, is absolutely continuous with respect to Lebesgue measure on $C_{g\xi}(x)$. $|\lambda(x)|$ is the coefficient of volume expansion of the mapping $g|_{\mathfrak{S}(x)}$ with respect to the measure $\nu_2(\cdot)$. From Definition 3.2 and the boundedness of the density of the measure ν_2 with respect to Lebesgue measure it follows that the function $|\lambda(x)|$ is equal to the product of a bounded function and a continuous function. There therefore exist numbers λ_1 and λ_2 depending only on g such that $\lambda_1 \le |\lambda(x)| \le \lambda_2$ for all $x \in M^n$. We also note that $0 \le \chi_{g^{-1}(\Pi_i')} \le \chi_{\Pi_i}$, since $\Pi_i' \subset g(\Pi_i)$. It consequently follows from (3.38) and (3.41) that $\mu(g(\Pi_i)) \le 2\mu(\Pi_i') \le 2K_2\lambda_2\mu(\Pi_i)$. Analogous arguments show that $\mu(g(\Pi_i)) \ge \frac{1}{2}K_1\lambda_1\mu(\Pi_i)$. Thus the lemma is proved for translations by a sufficiently small time. The proof in the general case is easily obtained by making use of the group properties of translations.

Let B be a cube with diam $B \le r_0$, and let Π_1 and Π_2 be the cylinders constructed in B in Lemma 2. If r_0 is sufficiently small, then diam $\Pi_1 \le r_1$. Inequalities (3.39) and (3.40) imply

$$\mu(g(B)) \leqslant \mu(g(\Pi_1)) \leqslant C_3\mu(\Pi_1) \leqslant C_3C_1^{-1}\mu(B), \tag{3.42}$$

$$\mu(g(B)) \geqslant \mu(g(\Pi_2)) \geqslant C_4\mu(\Pi_2) \geqslant C_4C_2^{-1}\mu(B). \tag{3.43}$$

The assertion of the theorem now follows directly from (3.42) and (3.43) and the Radon-Nikodým theorem (see [7]).

Corollary 3.2. *Under the conditions of Theorem 3.2, for every $T > 0$ the Jacobians of the transformations g^t, viz. the translations by a time t, $|t| \le T$, along the trajectories of the vector field $v(x)$, are uniformly bounded.*

Proof. We note that the coefficient of volume expansion of the mapping $g^t|_{\mathfrak{S}(x)}$ is bounded on the compactum $M^n \times [-T, T]$, i.e. there exist numbers λ_1 and λ_2

such that $\lambda_1 \leq |\lambda(t, x)| < \lambda_2$ for $-T \leq t \leq T$. The corollary is proved.

5. **Corollary 3.3.** *Let g^t be a translation by a time t along the trajectories of a vector field $v(x)$ subordinate to the foliation $\mathfrak{S}^u$ ($\mathfrak{S}^s$) of a partially hyperbolic dynamical system. Then the transformations g^t are absolutely continuous, and for every $T > 0$ the Jacobians of the transformations g^t, $|t| \leq T$, are uniformly bounded and different from zero.*

The proof follows directly from Corollaries 3.1 and 3.2.

§4. Transitivity of the foliations

1. Consider an arbitrary Anosov diffeomorphism g of a compact Riemannian manifold M^n, and let its contracting and expanding foliations be denoted by W^s and W^u respectively. It is easily seen that there exists a number $\beta > 0$ for which the following condition is satisfied: for any two points $x_1, x_2 \in M^n$ with $\rho(x_1, x_2) < \beta$ there exists a point $x_3 \in M^n$ such that

$$x_3 \in W^s(x_1) \cap W^u(x_2),$$
$$\rho_{W^s(x_1)}(x_1, x_3) < 3\beta, \quad \rho_{W^u(x_1)}(x_2, x_3) < 3\beta.$$

The pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ of a partially hyperbolic dynamical system f on a compact Riemannian manifold M^n can possess an analogous property.

Definition 4.1. A pair of continuous foliations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ with smooth leaves of dimensions $\dim \mathfrak{S}_1(x) = k_1$ and $\dim \mathfrak{S}_2(x) = k_2$ respectively is said to be *transitive* if there exist a natural number N and a positive number R for which the following condition is satisfied: for any two points $x, x' \in M^n$ there exist points $x_1, \ldots, x_N \in M^n$ such that $x_1 = x$, $x_N = x'$, and $x_{i+1} \in \mathfrak{S}_j(x_i)$, $i = 1, 2, \ldots, N-1$, $j = 1$ or 2, with

$$\rho_{\mathfrak{S}_j(x_i)}(x_i, x_{i+1}) < R.$$

2. **Theorem 4.1.** *Suppose that the pair of foliations $\mathfrak{S}_0^s$ and $\mathfrak{S}_0^u$ of a partially hyperbolic dynamical system f_0 on a manifold M^n is transitive, and suppose that the smoothness class r, the transitivity index N (see Definition 4.1) and the dimensions k_1 and k_2 of these foliations are connected with the dimension n of the manifold by the inequality*

$$r > \frac{N(k_1 + k_2)}{2} - n. \tag{4.1}$$

Then in the space of dynamical systems of class C^2 on the manifold M^n there exists a neighborhood $U \ni f_0$ such that any dynamical system $f \in U$ is partially hyperbolic and its pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ is transitive.

In the case of diffeomorphisms, U is a neighborhood of f_0 in the space $\mathrm{Diff}^2(M^n)$, and in the case of flows, U is a neighborhood of the corresponding vector field in the space $\Gamma^2(TM^n)$.)

Proof. Consider an arbitrary point $x \in M^n$, and let

$$B_R^s(x) = \{x_1 \in \mathfrak{S}_0^s(x) \mid \rho_{\mathfrak{S}_0^s(x)}(x, x_1) < R\} \subset \mathfrak{S}_0^s(x),$$
$$B_R^u(x) = \{x_2 \in \mathfrak{S}_0^u(x) \mid \rho_{\mathfrak{S}_0^u(x)}(x, x_2) < R\} \subset \mathfrak{S}_0^u(x). \tag{4.2}$$

The sets $B_R^s(x)$ and $B_R^u(x)$ are respectively diffeomorphic to open balls $B^{k_1} \subset \mathbf{R}^{k_1}$ and $B^{k_2} \subset \mathbf{R}^{k_2}$, i.e. there exist injective mappings $\phi_x: B^{k_1} \to M^n$ and $\psi_x: B^{k_2} \to M^n$ of class C^r with $\phi_x(B^{k_1}) = B_R^s(x)$ and $\psi_x(B^{k_2}) = B_R^u(x)$. We define the mappings $\phi(x, y_1): M^n \times B^{k_1} \to M^n$ and $\psi(x, y_2): M^n \times B^{k_2} \to M^n$ by the formulas

$$\varphi(x, y_1) = \varphi_x(y_1) \text{ and } \psi(x, y_2) = \psi_x(y_2).$$

Inasmuch as the foliations $\mathfrak{S}_0^s$ and $\mathfrak{S}_0^u$ are foliations of class C^r, the mappings ϕ_x and ψ_x can be chosen to be compatible in such a way that the mappings ϕ and ψ are also of class C^r. Let

$$(B^{k_1} \times B^{k_2})^k = \underbrace{(B^{k_1} \times B^{k_2}) \times \ldots \times (B^{k_1} \times B^{k_2})}_{k}.$$

We fix a point $x_0 \in M^n$ and put $\theta^1 = \phi_{x_0}$. A mapping $\theta^2: B^{k_1} \times B^{k_2} \to M^n$ is defined as follows:

$$\theta^2(\xi_1, \xi_2) = \psi_{\varphi_{x_0}(\xi_1)}(\xi_2).$$

Suppose that a mapping $\theta^{2k}: (B^{k_1} \times B^{k_2})^k \to M^n$ has already been defined. Then mappings $\theta^{2k+1}: (B^{k_1} \times B^{k_2})^k \times B^{k_1} \to M^n$ and $\theta^{2k+2}: (B^{k_1} \times B^{k_2})^{k+1} \to M^n$ are defined by the formulas

$$\theta^{2k+1}(\xi, \eta) = \varphi_{\theta^{2k}(\xi)}(\eta), \qquad \theta^{2k+2}(\xi, \eta, \zeta) = \psi_{\theta^{2k+1}(\xi, \eta)}(\zeta).$$

Here $\xi \in (B^{k_1} \times B^{k_2})^k$ and $\eta \in B^{k_1}$, $\zeta \in B^{k_2}$.

Thus a mapping $\theta^m: D^{nm} \to M^n$ of class C^r, where D^{nm} ($\subset \mathbf{R}^{nm}$) is the corresponding cartesian product of the balls B^{k_1} and B^{k_2}, is defined for any natural number m. From the transitivity of the pair of foliations $\mathfrak{S}_0^s$ and $\mathfrak{S}_0^u$ it follows that θ^N is a mapping of D^{nN} onto all of the manifold M^n. Inequality (4.1) connects the smoothness class of the mapping θ^N with the dimensions of D^{nN} and M^n. Under these conditions the set of regular values of the mapping θ^N is the set of second category and of full measure on the manifold M^n (see [18], Sard's theorem). Consider an arbitrary regular value $x_1 \in M^n$ and any preimage $y \in D^{nN}$ of it. The differential $d\theta_y^N$ maps the tangent space $T_y D^{nN}$ onto all of the tangent space $T_{x_1} M^n$. There therefore exists an n-subspace $E^n \subset T_y D^{nN}$ such that $d\theta_y^N(E^n) = T_{x_1} M^n$. Consider the sphere $S_\delta^{n-1} \subset E^n$ of radius δ with center at the origin. For a sufficiently small δ the sphere S_δ^{n-1} can be assumed to be imbedded in D^{nN}. If δ is sufficiently small, then the set $\theta^N(S^{n-1}_\delta)$ is diffeomorphic to S^{n-1}_δ, while a set bounded by it on M^n is diffeomorphic to a ball and contains the point x_1.

Lemma. *For each $\epsilon > 0$ there exists a neighborhood $U(x_0)$ of the dynamical system f_0 in the space $\mathrm{Diff}^2(M^n)$ (or in the space $\Gamma^2(TM^2)$) such that for any dynamical system $f \in U(x_0)$ the corresponding mappings $\tilde{\phi}_x$, $\tilde{\psi}_y$ and $\tilde{\theta}^N(y)$ can be chosen so that*

$$\sup_{y \in D^{nN}} \rho(\tilde{\theta}^N(y), \theta^N(y)) < \varepsilon.$$

Proof. The assertion of the lemma immediately follows from Proposition 2.4.
In the assertion of the lemma we put

$$\varepsilon = \frac{1}{2} \max_{x \in \theta^N(S^{n-1}_\delta)} \rho(x_1, x).$$

Then for any dynamical system $f \in U(x_0)$ the complete preimage $(\tilde{\theta}^N)^{-1}(x_1)$ is not empty and, moreover, for each point x with $\rho(x_1, x) < \epsilon/2$ the preimage $(\tilde{\theta}^N)^{-1}(x)$ is not empty. There therefore exists a collection of points $y_1, \ldots, y_N \in M^n$ such that $y_1 = x_0$, $y_N = x_1$, $y_{i+1} \in \mathfrak{S}^v(y_i)$, $i = 1, \ldots, N-1$, $v = s$ or u, with

$$\rho_{\mathfrak{S}^v(y_i)}(y_i, y_{i+1}) < 2R.$$

For a dynamical system $f \in U(x_0)$ consider the mapping $\tilde{\theta}^{2N}$ constructed for the balls $B^s_{2R}(x)$ and $B^u_{2R}(x)$ (see (4.2)), and let $M'(x_0)$ denote the range of this mapping. Inasmuch as $M'(x_0)$ contains the range of the mapping $\tilde{\theta}^N$ and hence a neighborhood of x_1, it also contains a neighborhood of y_{N-1}. By successively considering the points $y_{N-1}, y_{N-2}, \ldots, y_1 = x_1$, it is not difficult to verify that $M'(x_0)$ contains a neighborhood $M''(x_0)$ of x_0.

We construct the sets $U(x_0)$ and $M''(x_0)$ for each point $x_0 \in M^n$. The sets $M''(x_0)$ are open and cover all of M^n. We choose a finite covering $M''(x_0^1), \ldots, M''(x_0^m)$. The intersection $U = \bigcap_{i=1}^m U(x_0^i)$ can be taken as the neighborhood U in the assertion of Theorem 4.1. The pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ for any dynamical system $f \in U$ is transitive with indices $2Nm$ and $2R$. Theorem 4.1 is proved.

3. **Definition 4.2.** Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two foliations of class C^∞ (see [11]) and let Γ_1 and Γ_2 be the modules of vector fields of class C^∞ that are respectively tangent to them. The pair of foliations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is said to be *absolutely nonintegrable* if there exists a natural number N such that for any $x \in M^n$ the Lie brackets of order not exceeding N of the vector fields of Γ_1 or Γ_2 generate T_xM^n. (We note that Definition 4.2 makes sense, since the distributions E_i corresponding to submodules of Γ_1 and Γ_2 are infinitely smooth.)

Definition 4.3. A pair of foliations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ with smooth leaves is said to be *locally transitive* if there exists a natural number N such that for every $\epsilon > 0$ it is possible to indicate a $\delta > 0$ such that for any $x \in M^n$ and $x' \in B_\delta(x)$ (the ball of radius δ with center at x) there can be found points $x_1, \ldots, x_N \in M^n$, $x_1 = x$, $x_N =$

x', with $x_{i+1} \in \mathfrak{S}_j(x_i)$, $x_i \in B_\epsilon(x)$ and $\rho_{\mathfrak{S}_j(x_i)}(x_i, x_{i+1}) < 2\epsilon$, $i = 1, 2, \ldots, N-1$, $j = 1$ or 2.

If a pair of foliations is locally transitive, it is transitive.

Theorem 4.2. *If a pair of foliations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is absolutely nonintegrable, it is locally transitive.*

Proof. Let $x \in M^n$, $k_1 = \dim \mathfrak{S}_1(x)$ and $k_2 = \dim \mathfrak{S}_2(x)$. From the absolute nonintegrability of the pair of foliations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ it follows that there exist smooth vector fields (we call them basis vector fields) $v_1, \ldots, v_{k_1} \in E_1$ and $v_{k_1+1}, \ldots, v_{k_1+k_2} \in E_2$ for which the vectors $v_1(x), \ldots, v_{k_1}(x), v_{k_1+1}(x), \ldots, v_{k_1+k_2}(x), v_{k_1+k_2+1}(x), \ldots, v_n(x)$ form a basis of the space T_xM^n (here x is a fixed point while v_i, $i = k_1 + k_2 + 1, \ldots, n$, is a Lie bracket of the basis vector fields of order $m_i \le N$). We denote by $U(x)$ a neighborhood of x such that for any point $y \in U(x)$ the vectors $v_i(y)$, $i = 1, \ldots, n$, form a basis of the space T_yM^n. Let $P(t, x, v)$ denote the transformation of translation of a point x along a trajectory of a vector field v by a time t and let $T(t, x, v)$ denote the transformation of translation by a time t along the geodesic passing through a point x in the direction of the vector $v(x)$. Clearly

$$\rho(P(t, x, v),\ T(t, x, v)) < C_0t^2, \tag{4.3}$$

where C_0 is a certain constant not depending on x.

Let $w = [v_i, v_j]$. We define a mapping Q_1 by

$$Q_1(t, x, w) = P(\sqrt{t}, P(\sqrt{t}, P(\sqrt{t}, P(\sqrt{t}, x, v_i), v_j), -v_i), -v_j).$$

If the vector field w is a Lie bracket of order m of certain basis vector fields, so that $w = [v_i, w_1]$, where w_1 is a Lie bracket of basis vector fields of order $m-1$, we define a mapping Q_m by

$$Q_m(t, x, w) = Q_{m-1}\left(\sqrt[2m]{t}, P\left(\sqrt[2m]{t}, Q_{m-1}\left(\sqrt[2m]{t}, P\left(\sqrt[2m]{t}, x, v_1\right), w_1\right), -v_1\right), -w_1\right).$$

It is known (see [19]) that

$$\rho(P(t, x, w), Q_1(t, x, w)) \leqslant C_1t^2, \tag{4.4}$$

where C_1 is a certain constant not depending on x.

It follows from (4.4) and induction on m that

$$\rho(P(t, x, w), Q_m(t, x, w)) \leqslant C_mt^2, \tag{4.5}$$

where C_m is a certain constant not depending on x.

Let $B_\epsilon \subset T_xM^n$ be the ball of radius ϵ with center at the origin. If ϵ is sufficiently small, we define a mapping $\phi_x: B_\epsilon \to U(x)$ as follows: we decompose a vector $v \in B_\epsilon$ with respect to the basis v_i, $i = 1, \ldots, n$, in the space T_xM^n: $v = \sum_1^n a_iv_i$, and put

$$\varphi_x = Q_{m_n}(a_n, \ldots, v_n) \circ \ldots \circ Q_{m_{k_1+k_2+1}}(a_{k_1+k_2+1}, \ldots, v_{k_1+k_2+1})$$
$$\circ P(a_{k_1+k_2}, \ldots, v_{k_1+k_2}) \circ \ldots \circ P(a_1, \ldots, v_1).$$

We denote by $\exp_x: B_\epsilon \to U(x)$ the exponential map at x. From (4.3) and (4.5) it follows that for every $v \in B_\epsilon$

$$\rho(\exp_x v, \varphi_x(v)) < C\varepsilon^2. \tag{4.6}$$

Inasmuch as the mapping $\exp_x$ is a diffeomorphism (in particular, a mapping "onto") while the mapping ϕ is continuous, the assertion of the theorem follows from (4.6).

Theorem 4.3. *Suppose the pair of foliations $\mathfrak{S}_0^s$ and $\mathfrak{S}_0^u$ of class C^r of a partially hyperbolic dynamical system f_0 on a manifold M^n is locally transitive, and suppose that $r > N(k_1 + k_2)/2 - n$ (see Theorem 4.1). Then for every $\epsilon > 0$ there exist a number $\delta > 0$ and a neighborhood U of f_0 in the space of dynamical systems of class C^2 on the manifold M^n (see Theorem 4.1) such that any dynamical system $f \in U$ is partially hyperbolic and its pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ has the following property: for every $x, x' \in M^n$, $x' \in B_\delta(x)$, there exist points $x_1, \ldots, x_{2N}$, $x_1 = x$, $x_{2N} = x'$, such that $x_i \in \mathfrak{S}^\nu(x_{i-1})$ and $\rho_{\mathfrak{S}^\nu(x_i)}(x_i, x_{i+1}) < 2\epsilon$, $x_i \in B_\epsilon(x)$, $i = 1, \ldots, 2N$, $\nu = s$ or u (the almost local transitivity property).*

The proof of this theorem is a straightforward modification of the proof of Theorem 4.1.

§5. Metric properties of partially hyperbolic dynamical systems

1. We assume that a partially hyperbolic dynamical system f^t (t is a continuous or discrete parameter) preserves a smooth measure μ on a manifold M^n. Let $\nu(\mathfrak{S})$ denote the measurable hull of the partition of M^n into the leaves of a foliation $\mathfrak{S}$, and let $\pi(f^t)$ denote the largest partition of the measure space (M^n, μ) with zero entropy. The results of [5] (see Theorems 4.2 and 5.2) imply

Proposition 5.1. $\pi(f^t) \leq \nu(\mathfrak{S}^s) \wedge \nu(\mathfrak{S}^u)$.

This immediately implies

Proposition 5.2. *If $\nu(\mathfrak{S}^s) \wedge \nu(\mathfrak{S}^u) = \nu$ (ν is the trivial partition of (M^n, μ)), then any diffeomorphism f^t (for fixed t) is a K-automorphism of the measure space (M^n, μ).*

2. Let G^u and G^s denote the groups of homeomorphisms of M^n that are generated by translations along the trajectories of the vector fields subordinate to the foliations $\mathfrak{S}^u$ and $\mathfrak{S}^s$ respectively, and let G denote the group generated by G^u and G^s. We note that, by virtue of Corollary 3.3, each element $g \in G$ acts absolutely continuously on M^n.

Definition 5.1. The group G acts *metrically transitively* in the measure space

(M^n, μ) if, for any set A that is measurable and invariant mod 0 relative to any $g \in G$, either $\mu(A) = 0$ or $\mu(A) = 1$.

Proposition 5.3. *If the group G constructed above acts metrically transitively in (M^n, μ), then*

$$\nu(\mathfrak{S}^s) \wedge \nu(\mathfrak{S}^u) = \nu.$$

Proof. Let $A \subset M^n$ be a set that is measurable relative to the partition $\nu(\mathfrak{S}^s) \wedge \nu(\mathfrak{S}^u)$. It is easily seen that A is invariant mod 0 relative to any $g \in G$, so that either $\mu(A) = 0$ or $\mu(A) = 1$. Q.E.D.

Propositions 5.2 and 5.3 imply

Corollary 5.1. *If the group G acts metrically transitively in the space (M^n, μ), any diffeomorphism of a partially hyperbolic dynamical system f^t (t is a continuous or discrete parameter) is a K-automorphism of (M^n, μ).*

Theorem 5.1. *Suppose the pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ of a partially hyperbolic dynamical system f^t is transitive (see Definition 4.1). Then the group G acts transitively on the manifold M^n. Moreover, there exists a subset $K \subset G$ acting transitively on M^n such that all of the homeomorphisms $g \in K$ are equicontinuous and their Jacobians are uniformly bounded and different from zero.*

The proof is obvious.

4. **Theorem 5.2.** *Suppose that the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ of a partially hyperbolic dynamical system f^t on a compact Riemannian manifold M^n satisfy a Lipschitz condition. Suppose further that the pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ is transitive. Then the group G acts metrically transitively on M^n.*

Proof. Consider the subgroup G_0 of G generated by translations along the trajectories of the vector fields satisfying a Lipschitz condition and subordinate to the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$. It is easily seen that G_0 acts transitively on M^n and that any homeomorphism of G_0 satisfies a Lipschitz condition. Suppose that there exists a set A of nonzero Lebesgue measure that is invariant mod 0 relative to G_0 and let x_0 denote a point of density of A (see [7]): If $B_{r_m}(x_0)$ is a sequence of balls centered at x_0 of radii $r_m \to 0$, then

$$\frac{\mu(B_{r_m}(x_0) \cap A)}{\mu(B_{r_m}(x_0))} \to 1 \tag{5.1}$$

as $m \to \infty$.

We consider an arbitrary point $x_1 \in M^n$ and show that for any $\epsilon > 0$ there exists an $r > 0$ such that

$$\frac{\mu(B_r(x_1) \cap A)}{\mu(B_r(x_1))} > 1 - \varepsilon. \tag{5.2}$$

Inasmuch as the group G_0 acts transitively, there exists a homeomorphism $g_0 \in G_0$ such that $g_0(x_0) = x_1$. Let $C_0 > 1$ be the Lipschitz constant of the mapping g_0. There exists a $C_1 > 0$ such that for any measurable set $X \subset M^n$

$$(C_1C_0^n)^{-1}\mu(X) < \mu(g_0(X)) < C_1C_0^n\mu(X). \tag{5.3}$$

We also note that

$$B_{C_0^{-1}r_m}(x_1) \subset g(B_{r_m}(x_0)) \subset B_{C_0r_m}(x_1).$$

It therefore follows from (5.1) and (5.3) that

$$\frac{\mu(B_{C_0r_m}(x_1) \cap A)}{\mu(B_{C_0r_m}(x_1)\setminus A)} > \frac{(C_1C_0^n)^{-1}\mu(B_{r_m}(x_0) \cap A)}{C_1C_0^n\mu(B_{C_0^2r_m}(x_0)\setminus A)}$$

$$= C_2\frac{\mu(B_{r_m}(x_0) \cap A)}{\mu(B_{C_0^2r_m}(x_0)\setminus A)} > C_3\frac{\mu(B_{r_m}(x_0) \cap A)}{\mu(B_{r_m}(x_0)\setminus A)} \to \infty$$

as $m \to \infty$. Here C_2 and C_3 are certain constants. Thus the ball $B_{C_0r_m}(x_1)$ for sufficiently large m satisfies (5.2). Hence x_1 is a point of density of A, i.e. $A = M^n$ (mod 0). The theorem is proved.

An analogous but somewhat weaker result is given in [14].

Corollary 5.2. *If a partially hyperbolic dynamical system f^t preserves a smooth measure μ and satisfies the conditions of Theorem 5.2, any diffeomorphism f^t (t fixed) is a K-automorphism of (M^n, μ).*

The proof follows from Theorem 5.2 and Corollary 5.1.

5. **Theorem 5.3.** *Suppose the distribution E_0 of a partially hyperbolic diffeomorphism f satisfies a Lipschitz condition and is integrable. Let $\mathfrak{S}^0$ denote the foliation formed by the integral manifolds of E_0. Suppose that the pairs of foliations $\mathfrak{S}^s$, $\mathfrak{S}^0$ and $\mathfrak{S}^u$, $\mathfrak{S}^0$ are integrable while the pair of foliations $\mathfrak{S}^s$, $\mathfrak{S}^u$ is locally transitive. Suppose further that the mapping p (see Definition 3.1) satisfies a Lipschitz condition along the integral manifolds of E_0. Then, if the diffeomorphism f preserves a smooth measure μ on the manifold M^n, it is a K-automorphism of the measure space (M^n, μ).*

Proof. Let $x_0 \in M^n$, let $B_\epsilon(x_0)$ denote the ball of radius ϵ with center at x_0, let $B_\epsilon^0(x_0)$ ($B_\epsilon^s(x_0)$ and $B_\epsilon^u(x_0)$) denote the ball on the leaf $\mathfrak{S}^0(x_0)$ ($\mathfrak{S}^s(x_0)$ and $\mathfrak{S}^u(x_0)$ respectively) of radius ϵ with center at x_0, and let ξ denote the measurable partition of $B_\epsilon(x_0)$ into local leaves of the foliation $\mathfrak{S}^0$, i.e. into the connected components $C_\xi(x)$ of the intersection of the leaves $\mathfrak{S}^0(x)$ and the ball $B_\epsilon(x_0)$.

Let $x_1 \in C_\xi(x_0)$. We call the set

$$\Pi_\varepsilon(x_1) = \bigcup_{x_2 \in B_\varepsilon^s(x_1)} B_\varepsilon^u(x_2)$$

an *area element*.

Lemma 1. *There exists an* $\epsilon > 0$ *such that for every point* $x_0 \in M^n$ *the sets* $\Pi_\epsilon(x_1)$ *form a continuous partition of a neighborhood of* x_0.

The proof of this lemma will be carried out without making use of the integrability of the pairs of foliations $\mathfrak{S}^s$, $\mathfrak{S}^0$ and $\mathfrak{S}^u$, $\mathfrak{S}^0$.

The following assertions hold.

1°. *There exist numbers* $\delta > 0$ *and* $C_1 > 1$ *such that for every point* $x_0 \in M^n$ *and any points* $x_1, x_2 \in B_\delta(x_0)$, $x_1 \in \mathfrak{S}^\theta(x_2)$, *where* $\theta = 0, s, u$,

$$\rho(x_1, x_2) \leqslant \rho_{\mathfrak{S}^\theta}(x_1, x_2) \leqslant C_1 \rho(x_1, x_2). \tag{5.4}$$

2°. *For every* $\nu > 0$ *there exists a number* $\alpha = \alpha(\nu)$ *such that for any* $x_0, x_1, x_2 \in M^n$, $x_1, x_2 \in B_\alpha(x_0)$,

$$\max_{y \in B^u_\delta(x_1)} \min_{z \in B^u_\delta(x_2)} \rho(y, z) \leqslant \nu. \tag{5.5}$$

3°. *There exist numbers* $\Delta_0 > 0$ *and* $C_2 > 0$ *such that for any* $\Delta \leq \Delta_0$ *and any* $x_1, x_2 \in M^n$, $x_2 \in \mathfrak{S}^0(x_1)$, $\Delta \leq \rho_{\mathfrak{S}^0}(x_1, x_2) \leq \Delta_0$,

$$\min_{y \in B^u_\delta(x_1)} \rho(x_2, y) \geqslant C_2 \Delta. \tag{5.6}$$

4°. *There exist numbers* $\alpha, \beta, \gamma, \nu$ *such that*

$$\alpha = \alpha(\nu) \quad (\textit{see } 2°), \tag{5.7}$$

$$\gamma \mu_1 C_1^2 < \min\left\{\alpha, \frac{\beta}{9}, C_2 \frac{\beta}{12} - \frac{\nu}{2}\right\}, \tag{5.8}$$

$$\beta < \frac{\delta}{\mu_2 C_1^2}. \tag{5.9}$$

5°. *If two elements* $\Pi_\epsilon(x_1')$ *and* $\Pi_\epsilon(x_1'')$ *intersect at a point* z, *their intersection entirely contains a ball on the leaf* $\mathfrak{S}^u(z)$.

Assertions 1°–3° immediately follow from the compactness of the manifold M^n and the continuity and transversality of the foliations $\mathfrak{S}^0$, $\mathfrak{S}^s$ and $\mathfrak{S}^u$. Assertion 5° is obvious. To prove assertion 4° we choose any $\beta > 0$ satisfying (5.9) and a $\gamma > 0$ such that $\gamma\mu_1 C_1^2 < \min\{\beta/9, C_2\beta/12\}$. There exists a $\nu > 0$ such that $\gamma\mu_1 C_1^2 < \min\{\beta/9, C_2\beta/12 - \nu/2\}$. Let $\alpha = \alpha(\nu)$ (see 2°). By decreasing the number γ, if necessary, we can achieve the fulfillment of (5.8).

Under the condition of assertion 5° we assign to the pair of intersecting area elements the collection of points $\{x_1', x_1'', y_1, y_2\}$, where y_1 is a point of the intersection of the leaves $\mathfrak{S}^u(z)$ and $\mathfrak{S}^s(x_1')$ and y_2 is a point of the intersection of the leaves $\mathfrak{S}^u(z)$ and $\mathfrak{S}^u(x_1'')$.

We proceed to the proof of the lemma. Suppose that the ball $B_\gamma(x_0)$ contains an area element $\Pi_\gamma(x_1)$ lying entirely within a chosen neighborhood and intersecting the area element $\Pi_\gamma(x_0)$. It can also be assumed without loss of generality that

$C_1\mu_2\delta < \Delta_0$. From (5.9) it follows that $f(B_\beta(x_0)) \subset B_\delta(f(x_0))$. Consider the collection of points $\{x_0, x_1, x_2, x_3\}$ assigned to the area elements $\Pi_\gamma(x_0)$ and $\Pi_\gamma(x_1)$. The proof of the lemma will be complete if we show that each of the following three exhaustive possibilities leads to a contradiction.

1) $\rho(f(x_0), f(x_1)) \geq \beta$. According to (5.6) we have

$$\min_{y \in B^u_\delta(f(x_0))} \rho(f(x_1), y) \geqslant C_2\beta. \tag{5.10}$$

(We note that the conditions of 3° are satisfied since

$$f(x_1) \in \mathfrak{S}^0(f(x_0)), \quad \beta \leqslant \rho(f(x_0), f(x_1)) \leqslant \rho_{\mathfrak{S}^0}(f(x_0), f(x_1))$$
$$\leqslant \mu_2\rho_{\mathfrak{S}^0}(x_0, x_1) \leqslant C_1\mu_2\rho(x_0, x_1) \leqslant C_1\mu_2\delta < \Delta_0.)$$

Inequality (5.8) implies $f(x_3) \in B_\alpha(f(x_0))$. Therefore, by virtue of assertion 2° and (5.4), (5.5) and (5.8), we get

$$\min_{y \in B^u_\delta(f(x_0))} \rho(f(x_1), y) \leqslant \min_{y \in B^u_\delta(f(x_0))} [\rho(f(x_1), f(x_2)) + \rho(f(x_2), y)]$$
$$\leqslant \rho(f(x_1), f(x_2)) + \max_{z \in B^u_\delta(f(x_3))} \min_{y \in B^u_\delta(f(x_0))} \rho(z, y) \leqslant 2\gamma\mu_1 C_1^2 + \nu.$$

It follows from (5.10) that $C_2\beta \leq \nu + 2\gamma\mu_1 C_1^2$. But this contradicts (5.8).

2) $\rho(f(x_0), f(x_1)) < \beta$, but $\rho(f(x_3), f(x_2)) \geq \beta/2$. From (5.8) we have

$$\rho(f(x_0), f(x_1)) \geqslant \rho(f(x_3), f(x_2)) - \rho(f(x_0), f(x_2))$$
$$- \rho(f(x_2), f(x_3)) \geqslant \frac{\beta}{2} - 3\gamma\mu_1 C_1^2 \geqslant \frac{\beta}{2} - \frac{3\beta}{9} \geqslant \frac{\beta}{6}.$$

This implies by virtue of the arguments used in the first case that

$$\min_{y \in B^u_\delta(f(x_0))} \rho(f(x_1), y) \geqslant C_2\frac{\beta}{6}$$

and

$$\min_{y \in B^u_\delta(f(x_0))} \rho(f(x_1), y) \leqslant 2\gamma\mu_1 C_1^2 + \nu.$$

Therefore $C_2\beta/6 < \nu + 2\gamma\mu_1 C_1^2$, which contradicts (5.8).

3) $\rho(f(x_0), f(x_1)) < \beta$ and $\rho(f(x_3), f(x_4)) < \beta/2$. There exists an $n_0 > 1$ such that (i) these inequalities remain valid when f is replaced by f^{n_0-1}, and (ii) at least one of them is violated when f is replaced by f^{n_0}. From (5.9) it follows that the points $f^{n_0}(x_1)$, $f^{n_0}(x_2)$ and $f^{n_0}(x_3)$ lie in the ball $B_\delta(f^{n_0}(x_0))$. Inasmuch as

$$\rho(f^k(x_0), f^k(x_3)) < \rho(x_0, x_3) \text{ and } \rho(f^k(x_1), f^k(x_2)) < \rho(x_1, x_2), \ 0 < k < n_0,$$

a repetition of the arguments presented above for cases 1) and 2) (with f replaced by f^{n_0}) produces the required contradiction. The lemma is proved.

Let $U(x_0)$ be the neighborhood of x_0 constructed in Lemma 1 and let $x \in U(x_0)$. We define a mapping q: $C_\xi(x_0) \to C_\xi(x)$ by the formula

$$q(x_1) = \Pi_\varepsilon(x_1) \cap C_\xi(x), \quad x_1 \in C_\xi(x).$$

Lemma 2. *The mapping q satisfies a Lipschitz condition and hence is absolutely continuous.*

Proof. Let $y \in \mathfrak{S}^s(x_1) \cap \mathfrak{S}^u(x)$. Since the pairs of foliations $\mathfrak{S}^s$, $\mathfrak{S}^0$ and $\mathfrak{S}^u$, $\mathfrak{S}^0$ are integrable, the mapping q is the composition $q = p_1 \circ p_2$. Here the mappings p_2: $C_\xi(x_1) \to C_\xi(y)$ and p_1: $C_\xi(y) \to C_\xi(x)$ are the restrictions to the local leaves $C_\xi(x_1)$ and $C_\xi(y)$ respectively of the successor mapping effected by means of the leaves of the foliation $\mathfrak{S}^u$. By virtue of the conditions of the theorem each of these mappings, and hence the mapping q, satisfies a Lipschitz condition. The lemma is proved.

Let A be a set of positive measure consisting mod 0 of the leaves of the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$, and let

$$P_\varepsilon(x_0) = \bigcup_{x_1 \in C_\xi(x_0)} B^s_\varepsilon(x_0).$$

We call a point x_0 a *typical point* if it is simultaneously a point of density of the three sets A, $A \cap P_\epsilon(x_0)$ and $A \cap C_\xi(x_0)$.

Lemma 3. *Let C be the set of typical points. Then* $C = A$ (mod 0).

The proof of the lemma follows from the absolute continuity of the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ (see [1, 7]).

Lemma 4. *There exists a set* $D = A$ (mod 0) *consisting in a neighborhood $U(x_0)$ of a typical point x_0 of the area elements $\Pi_\epsilon(x_1)$, $x_1 \in C_\xi(x_0)$.*

Proof. Let

$$Q = \bigcup_{x \in A \cap C_\xi(x_0)} B^s_\varepsilon(x)$$

(ϵ is chosen in accordance with Lemma 1). It is obvious that $Q = A \cap P_\epsilon(x_0)$ (mod 0) and that Q consists in a neighborhood $U(x_0)$ (see Lemma 1) entirely of the leaves of the foliation $\mathfrak{S}^s$. Let

$$\overline{Q} = \{x \in Q \mid \text{ there exists a } Z(x) \subset \mathfrak{S}^u(x),\ \mu(Z(x)) = 0,\ \mathfrak{S}^u(x) \setminus Z(x) \subset A\}.$$

Clearly, $\overline{Q} = Q$ (mod 0). The proof of the lemma now follows upon putting $D = \bigcup_{x \in \overline{Q}} \mathfrak{S}^u(x)$.

Consider a set W on an area element $\Pi_\epsilon(x_1)$. The set $\Pi = \bigcup_{x_2 \in W} B^0_r(x_2)$ is called a *cylinder in M^n with base W.*

Lemma 5. *There exist numbers r_0, C_1 and C_2 such that for any ball $B_r(x_0)$, $r \leq r_0$, there can be found cylinders Π_1 and Π_2 for which*

$$\Pi_2 \subset B_r(x_0) \subset \Pi_1, \quad C_1\mu(\Pi_1) \leqslant \mu(B_r(x_0)) \leqslant C_2\mu(\Pi_2). \tag{5.11}$$

The proof is analogous to the proof of Lemma 2 to Theorem 3.2.

We denote by G_1 and G_2 the groups of translations along the trajectories of the vector fields subordinate to the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ respectively and satisfying the following condition: in a neighborhood $U(x_0)$ the foliation $\mathfrak{S}^0$ is invariant relative to any translation along a trajectory of the vector field.

If the neighborhood $U(x_0)$ is chosen sufficiently small, then the nontriviality of the groups G_1 and G_2 is easily seen to follow from the integrability of the pairs of foliations $\mathfrak{S}^s$, $\mathfrak{S}^0$ and $\mathfrak{S}^u$, $\mathfrak{S}^0$ and the fact that the foliation $\mathfrak{S}^0$ satisfies a Lipschitz condition. Let G denote the group generated by the groups G_1 and G_2.

Lemma 6. *The group G acts transitively in the neighborhood $U(x_0)$. Moreover, there exists a subset $K \subset G$ such that $\bigcup_{g\in K} g(x) \supset U(x_0)$ for any $x \in U(x_0)$ while the mappings $g \in K$ are equicontinuous and their Jacobians are uniformly bounded and different from zero.*

Proof. The first two assertions of the lemma are obvious; the last follows from Corollary 3.3.

We proceed to the proof of the theorem. Let $B_\gamma(x_0) \subset U(x_0)$ and β be chosen with respect to γ from the condition of local transitivity of the pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$. By virtue of Proposition 5.3 and the compactness of the manifold M^n it suffices to show that any point of the ball $B_\beta(x_0)$ is a point of density of the set D. By virtue of Lemma 5 it suffices to prove that for every $\epsilon > 0$ there exists a cylinder Π for which

$$\frac{\mu(D \cap \Pi)}{\mu(\Pi)} > 1 - \varepsilon. \tag{5.12}$$

Let Π be a cylinder of small diameter containing a point $y \in B_\beta(x_0)$ as an interior point. We partition the base W of this cylinder into sets W_i and, taking them for bases, construct cylinders Π_i partitioning Π. Let y_i be an interior point of the set $\Pi_i \cap \Pi_\gamma(y)$. According to Lemma 6 there exist numbers $\mathcal{J}_1, \mathcal{J}_2$ and mappings $g_i \in G$ such that $g_i(y_i) = x_0$ and the Jacobians of the mappings g_i are uniformly bounded from below and above by the numbers $\mathcal{J}_1$ and $\mathcal{J}_2$. Inasmuch as the foliation $\mathfrak{S}^0$ is invariant relative to the mappings g_i, the set $g_i(\Pi_i)$ is composed in the neighborhood $U(x_0)$ of the leaves of the foliation $\mathfrak{S}^0$. Moreover, for any $\delta > 0$ there exist cylinders Π_i' and Π_i'' such that

$$\Pi_i' \subset g_i(\Pi_i) \subset \Pi_i'', \quad \frac{\mu(\Pi_i')}{\mu(\Pi_i'')} > 1 - \delta. \tag{5.13}$$

If the diameter of the cylinder Π is sufficiently small, then, by virtue of the equicontinuity of all of the homeomorphisms g_i (see Lemma 6), the diameter of each cylinder

Π_i' is also sufficiently small. Inasmuch as the point x_0 is a point of density of the set D, the cylinders Π_i and Π_i' can be chosen so that

$$\frac{\mu(D \cap \Pi_i')}{\mu(\Pi_i')} > 1 - \delta. \tag{5.14}$$

From (5.13) and (5.14) it follows that

$$\frac{\mu(D \cap g_i(\Pi_i))}{\mu(g_i(\Pi_i))} > 1 - 2\delta. \tag{5.15}$$

Let $E_i = g_i(\Pi_i) \setminus (D \cap g_i(\Pi_i))$. Then (5.15) implies

$$\frac{\mu(E_i)}{\mu(g_i(\Pi_i))} < 2\delta. \tag{5.16}$$

From this result we immediately get that for each i

$$\frac{\mu(D \cap \Pi_i)}{\mu(\Pi_i)} > 1 - 2\frac{\mathcal{J}_1}{\mathcal{J}_2}\delta.$$

Inasmuch as δ is arbitrary, this implies the fulfillment of (5.12). Theorem 5.3 is proved.

Theorem 5.4. *Suppose the distribution E_0 of a partially hyperbolic flow f^t satisfies a Lipschitz condition and is integrable, while the corresponding foliation $\mathfrak{S}^0$ is integrable with the foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$. Suppose that the mappings p (see Definition 3.1) satisfy a Lipschitz condition along the integral manifolds of E_0. Suppose further that the flow f^t preserves a smooth measure μ while the pair of foliations $\mathfrak{S}^s$ and $\mathfrak{S}^u$ is locally transitive. Then each diffeomorphism f^t (t fixed) is a K-automorphism of the measure space (M^n, μ).*

Proof. We note that the smooth foliation formed by the trajectories of the flow is integrable with each of the foliations $\mathfrak{S}^s$, $\mathfrak{S}^0$, $\mathfrak{S}^u$, and that each of these foliations is smooth along the trajectories of the flow. Therefore the proof of Theorem 5.4 is a straightforward modification of the proof of Theorem 5.3.

6. **Corollary 5.3.** *Suppose a dynamical system f is a fiber bundle over an Anosov system, and suppose that it satisfies the conditions of Theorem 2.2, with $\mu_1\mu_0/\lambda_0 < 1$ and $\mu_0/\lambda_2\lambda_0 < 1$. If the pair of foliations $\mathfrak{S}^+$ and $\mathfrak{S}^-$ is locally transitive and f preserves a smooth measure μ, then f is a K-automorphism of (M^n, μ) (if f is a flow, then each diffeomorphism of the flow is a K-automorphism of (M^n, μ)).*

The proof follows from Theorems 5.3, 5.4 and 2.2.

Corollary 5.4. *Suppose that a dynamical system f_0 satisfies the conditions of Corollary 5.3 and that the smoothness class of its pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ is sufficiently high (see the conditions of Theorem 4.1). If a dynamical system f sufficiently C^1-close to f_0 is a fiber bundle with a smooth invariant measure, then it is a K-automorphism of (M^n, μ) (if f is a flow, then each diffeomorphism of the flow is a K-automorphism).*

Proof. We note that according to Theorem 4.3 the pair of foliations of the dynamical system f has the almost local transitivity property, which was used in proving Theorems 5.3 and 5.4. Therefore the assertion being proved follows from Theorems 2.2, 5.3 and 5.4.

Corollary 5.5. *Suppose that the distribution E_0 of a partially hyperbolic dynamical system f satisfies the conditions of Theorem 2.3, with $\mu_1\mu_0/\lambda_0 < 1$ and $\mu_0/\lambda_2\lambda_0 < 1$. If the pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ is locally transitive and f preserves a smooth measure μ, then f is a K-automorphism of (M^n, μ) (if f is a flow, then each diffeomorphism f is a K-automorphism).*

The proof follows from Theorems 5.3, 5.4 and 2.3.

§6. Metric properties of a frame flow on a manifold of negative curvature

1. Consider the k-frame flow Φ^t on a smooth Riemannian manifold M^n (see §1). In this section some of the known properties of a frame flow will be applied without proof. The compact manifolds $V_{n-1,k-1}(\omega)$ form an invariant foliation relative to the group Φ^t, which we denote by $\mathfrak{S}^0$.

Proposition 6.1. *The foliation $\mathfrak{S}^0$ is a smooth (of class C^∞) foliation of the manifold Ω_k. The mapping $P^t(\omega, \xi^{k-1})$ is an isometry of the leaf $\mathfrak{S}^0(\omega)$ onto the leaf $\mathfrak{S}^0(S^t(\omega))$.*

It is known (see [1, 2]) that the geodesic flow S^t on a manifold of negative curvature is an Anosov system and has contracting and expanding foliations, which we respectively denote by $\mathfrak{S}^s$ and $\mathfrak{S}^u$.

Proposition 6.2. *A k-frame flow has in the space Ω_k a contracting foliation $\mathfrak{S}^-$ and an expanding foliation $\mathfrak{S}^+$, with $\pi(\mathfrak{S}^-(w)) = \mathfrak{S}^s(\pi(w))$ and $\pi(\mathfrak{S}^+(w)) = \mathfrak{S}^u(\pi(w))$. Moreover, the contraction coefficients λ_1 and μ_1 in (2.1) (the expansion coefficients λ_2 and μ_2) are one and the same for the foliations $\mathfrak{S}^-$ and $\mathfrak{S}^s$ ($\mathfrak{S}^+$ and $\mathfrak{S}^u$).*

Theorem 6.1. *A k-frame flow on a compact Riemannian manifold M^n of negative curvature is a partially hyperbolic dynamical system.*

Proof. For every $x \in M^n$ we consider the subspaces $E_i(x) \subset T_xM^n$, $i = 1, 2, 3, 4$, such that $E_1(x) = T_x\mathfrak{S}^-(x)$, $E_2(x) = T_x\mathfrak{S}^0(x)$, $E_3(x) = T_x\mathfrak{S}^+(x)$ and $E_4(x)$ is the distribution corresponding to the foliation by the trajectories of the flow Φ^t. Clearly, $T_xM^n = \bigoplus_1^4 E_i(x)$ and the distributions E_i, $i = 1, 2, 3, 4$, are continuous and generate submodules $\Gamma_i \subset \Gamma^0(TM^n)$, with $\Gamma^0(TM^n) = \bigoplus_1^4\Gamma_i$. If $S_i = \operatorname{sp}\Phi_*^1|_{\Gamma_i}$, $i = 1, 2, 3, 4$, then $\operatorname{sp}\Phi_*^1 = \bigcup_1^4 S_i$, $S_i \cap S_j = \varnothing$, $i \neq j$, $i, j = 1, 2, 3$. Inasmuch as the mapping $P(t, \xi^{k-1})$ is an isometry, S_2 lies on the unit circle. From Proposition 6.2 it follows that S_1 lies in the annulus with radii λ_1 and μ_1 while S_3 lies in the annulus with radii λ_2 and μ_2, with $0 < \lambda_1 \leq \mu_1 < 1 < \lambda_2 \leq \mu_2$. The theorem is proved.

Proposition 6.3. *The foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of a k-frame flow on a manifold of negative curvature are absolutely continuous.*

The proof follows from Theorems 3.1 and 6.1.

Proposition 6.4. *The pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of a k-frame flow on a manifold of constant negative curvature is transitive.*

Proof. We note that the universal covering of any n-dimensional manifold M^n of constant negative curvature is an n-dimensional Lobačevskiĭ space and that the k-frame flow on M^n is a factor of an n-frame flow. It therefore suffices to prove the assertion for the foliations of an n-frame flow on an n-dimensional Lobačevskiĭ space.

Consider the space $\mathbf{R}^n$ with a Euclidean system of coordinates $x_1, \ldots, x_n$ and the hyperplane $x_n = 0$ (the absolute). The Lobačevskiĭ space is isometric to the upper halfspace $\mathbf{L}^n = \{x \in \mathbf{R}^n | x_n > 0\}$ with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

Let us geometrically describe the leaves of the contracting and expanding foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$. It is obvious that the set of origins of the frames belonging to any leaf $\mathfrak{S}^-(w)$ or $\mathfrak{S}^+(w)$ forms a horosphere S in $\mathbf{L}^n$. Suppose S is a hyperplane $x_n = c$. It can be shown that a leaf $\mathfrak{S}^-(w)$ (or $\mathfrak{S}^+(w)$) is the set of frames $w = (x, \xi_1(x), \xi^{n-1}(x))$, with $x \in S$, $\xi_1(x) \perp S$ and the frames $\xi^{n-1}(x)$ being obtained from one another by parallel translations in the Euclidean metric on the hyperplane S. Suppose S is a sphere S^{n-1} tangent to the absolute and x_0 is the upper pole of the sphere. It can be shown that the corresponding leaf is the set of frames $w = (x, \xi_1(x), \xi^{n-1}(x))$, with $x \in S$, $\xi_1(x) \perp S$ and $\xi^{n-1}(x) = \Pi_{x_0 x}\xi^{n-1}(x_0)$, where $\Pi_{x_0 x}$ is the operator of parallel translation (in the Euclidean metric on the sphere S^{n-1}) of the frame $(x_0, \xi^{n-1}(x_0))$ along the great circle arc joining the points x_0 and x. In this connection, the direction of the first vector of the frame depends on whether one is considering the contracting or the expanding leaf.

Remark 6.1. The space Ω_n of n-frames has two connected components (n-frames can have different orientations). It is therefore natural to consider the transitivity of the pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of the n-frame flow within each of the connected components.

Let us reason by induction on n. The transitivity of a 2-frame flow on the Lobačevskiĭ space $\mathbf{L}^2$ is obvious. To show that the transitivity of the foliations of an n-frame flow on $\mathbf{L}^n$ follows from the analogous fact for the $(n-1)$-dimensional case, we consider two arbitrary n-frames w_1 and w_2 on $\mathbf{L}^n$:

$$w_1 = (x_1, \xi_{1,1}(x), \xi_1^{n-1}(x)), \quad w_2 = (x_2, \xi_{2,1}(x), \xi_2^{n-1}(x)).$$

We denote by $w_3 \in \mathfrak{S}^-(w_2)$ the frame with origin at the pole of the corresponding horosphere (if the horosphere of the leaf $\mathfrak{S}^-(w_2)$ is a hyperplane parallel to the absolute, we put $w_3 = w_2$). Inasmuch as the first vector $\xi_{3,1}$ of w_3 is parallel to the x_n axis, the horosphere of one of the leaves $\mathfrak{S}^-(w_3)$ or $\mathfrak{S}^+(w_3)$ is a hyperplane parallel to the absolute. Let $w_4 = (x_4, \xi_{4,1}, \xi_4^{n-1})$ denote a frame on the chosen leaf for which the interval $[x_1, x_4]$ is parallel to the x_n axis. Analogous arguments can be carried out for the frames w_4 and w_1. Thus, in place of the frames w_1 and w_2 we now consider frames w_4 and w_5 for which the vectors $\xi_{4,1}$ and $\xi_{5,1}$ and the interval $[x_4, x_5]$ are parallel to the x_n axis. Let $\eta_{4,1}$ and $\eta_{5,1}$ respectively denote the second vectors of the frames w_4 and w_5, and let S denote the horosphere of the point x_4 that is a sphere in $\mathbf{R}^n$ with pole x_4. Also, let $\tilde{\eta}_{5,1}$ denote the result of a parallel translation of the vector $\eta_{5,1}$ at the point x_4 along the interval $[x_4, x_5]$. If $\tilde{\eta}_{5,1} = \eta_{4,1}$, we consider the space $\mathbf{L}^{n-1}$ generated by the $(n-1)$-dimensional hyperplane passing through the interval $[x_4, x_5]$ and perpendicular to the vector $\eta_{4,1}$. Thus, in this case the problem is reduced to the $(n-1)$-dimensional case. If $\tilde{\eta}_{5,1} \neq \eta_{4,1}$, we consider the vector

$$\gamma = \frac{\tilde{\eta}_{5,1} - \eta_{4,1}}{\| \eta_{5,1} - \eta_{4,1} \|}$$

and denote by S' the horosphere with pole $x_4 + 2R\gamma$ (R is the radius of the horosphere S) and by Π the hyperplane parallel to the absolute and passing through x_4. Any two of the three horospheres S, S' and Π have precisely one point in common. Let $\mathfrak{S}_1 \ni w_4$ denote the leaf whose origin is the horosphere S and let $\mathfrak{S}_2$ denote the leaf, having a point in common with $\mathfrak{S}_1$, whose origin is S'. Finally, let $\mathfrak{S}_3$ denote the leaf, having a point in common with $\mathfrak{S}_2$, whose origin is Π. Consider the frame $w_6 = (x_4, \xi_{6,1}, \xi_6^{n-1})$, lying on $\mathfrak{S}_3$, with origin x_4. It is not difficult to verify that the second vector $\eta_{6,1}$ of this frame coincides with the vector $\tilde{\eta}_{5,1}$. Consider the space $\mathbf{L}^{n-1}$ generated by the hyperplane passing through the point x_4 and perpendicular to the vector $\eta_{6,1}$. The frames (x_4, ξ_6^{n-1}) and (x_5, ξ_5^{n-1}) belong to the space of $(n-1)$ frames on $\mathbf{L}^{n-1}$.

Thus the problem is reduced to the $(n-1)$-dimensional case. Proposition 6.4 is proved.

Proposition 6.5. *The pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of k-frame flow on a manifold of constant negative curvature is absolutely nonintegrable.*

The proof follows directly from Proposition 6.4 and the geometric properties of the space $\mathbf{L}^n$.

2. **Theorem 6.2.** *Let M^n be a compact Riemannian manifold with a metric ρ of constant negative curvature. Then in the space of C^3 metrics on M^n there exists a neighborhood η of ρ such that for any metric ρ the k-frame flow Φ^t on M^n with metric $\bar{\rho}$ is a K-flow.*

For the proof of this theorem we require several lemmas.

Lemma 1. *Under the conditions of Theorem 6.2 there exists a neighborhood η of ρ such that for any metric $\bar{\rho} \in \eta$ the k-frame flow Φ^t on the manifold M^n with metric $\bar{\rho}$ has an almost locally transitive pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$.*

The proof follows from Theorems 4.2, 4.3 and Proposition 6.5.

Lemma 2. *Each diffeomorphism of the k-frame flow Φ^t on a manifold of constant negative curvature is a K-automorphism.*

The proof follows from Lemma 1 (for $\bar{\rho} = \rho$) and Corollary 5.2, inasmuch as $\mathfrak{S}^-$ and $\mathfrak{S}^+$ are smooth foliations under the conditions of Lemma 2.

Lemma 3. *Each diffeomorphism of the flow Φ^t in Lemma 1 is a K-automorphism.*

The proof follows from Lemma 1 and Theorem 5.4.

From Lemma 3 it follows that the flow Φ^t is ergodic, which permits one to apply Theorem 5.2 of [5]. By virtue of this theorem there exists a partition η of the space Ω_k such that

1) $\Phi^t\eta > \eta$ for all $t < 0$,

2) $\bigvee_t \Phi^t\eta = \epsilon$,

3) $\bigwedge_t \Phi^t\eta = \nu_{\mathfrak{S}^+}$, where $\nu_{\mathfrak{S}^+}$ is the measurable hull of the partition ξ into the leaves of the foliation $\mathfrak{S}^+$.

We note that from the method of construction of the partition η described in [5] it follows that the partition η can be chosen so that (i) C_η is an open set of small diameter on a leaf of the foliation $\mathfrak{S}^+$ (in particular, $\eta > \xi$), and (ii) it projects into a partition $\pi \circ \eta$ of the manifold W^{2n-1}. The latter means that, for any two elements C_η^1 and C_η^2 of the partition η, either $\pi(C_\eta^1) \cap \pi(C_\eta^2) = \varnothing$, or $\pi(C_\eta^1) = \pi(C_\eta^2)$, and hence that the sets $\pi(C_\eta)$ form a partition of the manifold W^{2n-1}, which we denote by $\pi \circ \eta$. Clearly, $C_{\pi\circ\eta} = \pi(C_\eta)$. It is easily seen that the partition $\pi \circ \eta$ satisfies conditions 1)–3) with Φ^t replaced by S^t and $\nu_{\mathfrak{S}^+}$ replaced by $\nu_{\mathfrak{S}^u}$; here $\nu_{\mathfrak{S}^u} = \nu$, inasmuch as S^t is a K-flow.

Let $\bar{\pi}$ denote the largest completely invariant partition with zero entropy. We have $\bar{\pi}(\Phi^t) = \nu$ for each t. Consequently the theorem will follow if we show that $h(\Phi^t, \eta) = h(\Phi^t)$ for some t (see [6], Theorems 12.1 and 12.3).

Lemma 4. *Let ζ be a measurable partition of the space Ω_k such that $\zeta > \xi$ and ζ projects into a partition of the manifold W^{2n-1} for which any $C_{\pi\circ\xi}$ is an open set of diameter δ on a leaf of the foliation $\mathfrak{S}^u$. Then $H(\zeta) = H(\pi \circ \zeta)$ if δ is sufficiently small.*

Proof. It can be shown that if δ is sufficiently small, there exists a finite partition of the space Ω_k into sets U_i of positive measure such that each U_i consists of elements of the partition ζ and lies in a local chart of the manifold Ω_k, with $U_i = \mathfrak{S}^0 \times V_i$ and the sets V_i having positive measure, forming a partition of the manifold W^{2n-1} and consisting of elements of the partition $\pi \circ \zeta$. Clearly $C_\xi \cap U_i = x \times C_{\pi\circ\zeta} \cap V_i$ for some $x \in \mathfrak{S}^0$. We have

$$H(\zeta) = -\int_{\Omega_k} \ln \tilde{\mu}(C_\zeta)\, d\Omega_k = -\sum_i \int_{U_i} \ln \tilde{\mu}(C_\zeta \cap U_i)\, d\Omega_k$$

$$= -\sum_i \int_{\mathfrak{S}^0 \times V_i} \ln \tilde{\mu}(x \times C_{\pi\circ\zeta} \cap V_i)\, d\Omega_k$$

$$= -\sum_i \int_{\mathfrak{S}^0} d\nu \int_{V_i} \ln \mu(C_{\pi\circ\zeta} \cap V_i)\, d\mu$$

$$= -\sum_i \int_{V_i} \ln \mu(C_{\pi\circ\zeta} \cap V_i)\, d\mu = -\int_{W^{2n-1}} \ln \mu(C_{\pi\circ\zeta})\, d\mu = H(\pi\circ\zeta).$$

The lemma is proved.

Lemma 5. *Suppose the partition ζ satisfies the conditions of Lemma 4 and δ is sufficiently small. Then for all t*

$$h(\Phi^t, \zeta) = h(S^t, \pi\circ\zeta).$$

Proof. We denote by ζ^m the partition

$$\zeta^m = \bigvee_{k=0}^{m-1} (\Phi^t)^{-k}\zeta$$

(t fixed) and by $(\pi\circ\zeta)^m$ the partition

$$(\pi\circ\zeta)^m = \bigvee_{k=0}^{m-1} (S^t)^{-k}(\pi\circ\zeta).$$

From Lemma 4 and the definition of $h(\Phi^t, \zeta)$ it follows that

$$h(\Phi^t, \zeta) = \lim_{m\to\infty} \frac{1}{m} H(\zeta^m) = \lim_{m\to\infty} \frac{1}{m} H(\pi\circ\zeta^m)$$
$$= \lim_{m\to\infty} \frac{1}{m} H((\pi\circ\zeta)^m) = h(S^t, \pi\circ\zeta).$$

The lemma is proved.

Lemma 6. *For all t, $h(\Phi^t) = h(S^t)$.*

Proof. There exists a sequence of partitions ζ_l of the space Ω_k satisfying the conditions of Lemma 4, with $\zeta_l \nearrow \epsilon$. Clearly, $\pi\circ\zeta_l \nearrow \epsilon$. According to Lemma 5, for all t we have $h(\Phi^t, \zeta_l) = h(S^t, \pi\circ\zeta_l)$ and hence (see [6])

$$h(\Phi^t) \quad \lim_{l\to\infty} h(\Phi^t, \zeta_l) = \lim_{l\to\infty} h(S^t, \pi\circ\zeta_l) = h(S^t).$$

The lemma is proved.

We pass to the proof of the theorem. Inasmuch as S^t is a K-flow, the partition $\pi\circ\eta$ satisfying conditions 1)–3) is a generating partition. It therefore follows from Lemmas 5 and 6 that $h(\Phi^t, \eta) = h(S^t, \pi\circ\eta) = h(S^t) = h(\Phi^t)$. The theorem is proved.

Remark 6.2. The assertion of Theorem 6.2 can be strengthened. Namely, in the space of smooth-measure-preserving dynamical systems of class C^2 on the manifold Ω_k that are fiber bundles over dynamical systems on the manifold W^{2n-1} there exists a neighborhood U of a frame flow Φ^t on M^n with a metric of constant negative curvature such that any diffeomorphism from a flow $f^t \in U$ is a K-automorphism of the space Ω_k. The proof of this fact follows from Theorem 5.4.

G. A. Margulis has communicated the following fact to us.

Proposition 6.6. *There exists a compact Riemannian manifold of negative curvature for which the k-frame flow Φ^t is nonergodic for $k \geq 2$.*

For the proof we require two lemmas.

Lemma 1. *There exists a compact Kähler manifold of negative curvature.*

Proof. Consider the unit ball

$$B = \left\{ z \mid \sum_{i=1}^{n} |z_i|^2 < 1 \right\}$$

in the n-dimensional complex space $\mathbf{C}^n$. The set B equipped with the Bergman metric is a Kähler manifold of negative curvature (see [17]). According to [15] the manifold B has a compact factor. The lemma is proved.

Lemma 2. *The k-frame flow Φ^t on a compact Kähler manifold M^n of negative curvature is nonergodic for $k \geq 2$.*

Proof. Let α denote the metric tensor on the manifold M^n and let $\mathcal{J}$ denote an almost complex structure that is invariant relative to parallel translations along the geodesics on M^n. Consider an arbitrary frame $w = (\xi_1, \ldots, \xi_k)$ and the frame $\Phi^t(w) = w_1 = (\eta_1, \ldots, \eta_l)$. Clearly,

$$\alpha(\xi_i, \mathcal{J}\xi_j) = \alpha(\eta_i, \mathcal{J}\eta_j), \quad i, j = 1, 2, \ldots, k.$$

Thus the functions $f_{i,j}(w) = \alpha(\xi_i, \mathcal{J}\xi_j)$ are invariant relative to the flow Φ^t, and hence the flow Φ^t is nonergodic. Lemma 2 is proved.

The assertion of Proposition 6.6 follows from Lemmas 1 and 2. We will describe below the ergodic components of the flow under consideration.

Let $\widetilde{\Omega}_n$ denote the space of n-frames $w = (x, \xi_1, \ldots, \xi_n)$ on the unit ball B in $\mathbf{C}^n$ with the Bergman metric that satisfy the condition

$$\alpha(\xi_i, \xi_j) = 0, \quad \alpha(\xi_i, \mathcal{J}\xi_j) = 0 \quad \text{for} \quad i \neq j. \tag{C}$$

Let D denote the compact factor introduced in Lemma 1 of the manifold (B, α), and let $\widetilde{\Omega}_n(D)$ denote the space of n-frames on D satisfying condition (C). It is obvious that the frame flows in the spaces $\widetilde{\Omega}_n$ and $\widetilde{\Omega}_n(D)$ are factors of the $2n$-frame flows on the manifolds (B, α) and (D, α) respectively.

Proposition 6.7. *A frame flow $\widetilde{\Phi}^t$ in the space $\widetilde{\Omega}_n(D)$ is a K-flow.*

Proof. Let us show that the pair of contracting and expanding foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of the flow $\widetilde{\Phi}^t$ is absolutely nonintegrable. For this purpose it suffices to prove the corresponding assertion in the space $\widetilde{\Omega}_n$. The unit ball B with the Bergman metric is a homogeneous space of the group $SU(n, 1)$. It is easily verified that the induced action of $SU(n, 1)$ in the space $\widetilde{\Omega}_n$ is transitive. A frame flow in the space $\widetilde{\Omega}_n$ is generated by an element x of the Lie algebra of $SU(n, 1)$. Inasmuch as the curvature of the manifold (B, α) is negative, the operator ad x has eigenvalues λ_+ and λ_- such that $\mathrm{Re}\,\lambda_+ > 0$ and $\mathrm{Re}\,\lambda_- < 0$. Therefore the horospherical subgroups H_+ and H_- of the element x are nontrivial, and hence the subgroup H generated by the horospherical subgroups is also nontrivial. Inasmuch as H is a normal divisor (see [21]), while the group $SU(n, 1)$ is simple, we conclude that $H = SU(n, 1)$. Thus the pair of foliations $\mathfrak{S}^+$ and $\mathfrak{S}^-$ is absolutely nonintegrable. It follows from Theorem 5.2 that any diffeomorphism of the flow $\widetilde{\Phi}^t$ is a K-automorphism. The concluding part of the proof of Proposition 6.7 repeats the corresponding arguments in the proof of Theorem 6.2.

It follows from Proposition 6.7 that the frame flow constructed in Proposition 6.6 is a K-flow on each ergodic component.

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