

D. V. Anosov and our road to partial hyperbolicity

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ABSTRACT. This paper provides a brief historical account of our interaction with our advisor and mentor Dmitry Victorovich Anosov and in particular, our scientific activity in the well-known Anosov–Katok seminar during the period from 1968 till 1979. We also comment on our joint work on partial hyperbolicity.

We met Dmitry Viktorovich Anosov in 1968 as third year Mekh-Mat (mathematics department of the Moscow State University) students and we started attending the Anosov–Katok seminar since its first meeting in the fall of 1969. For a long time since then our mathematical lives have been tied to and heavily influenced by these two great mathematicians whom we consider our mentors and advisers and not only in mathematics. In fact it would have been next to impossible for us to become professional mathematicians without their constant support and guidance.

In this note we will briefly describe our interaction with Anosov and comment on our joint work in mathematics during the decade from 1969 till 1979. M.B. emigrated from the former USSR in 1979 and while Ya.P. stayed in Russia for another decade his interaction with Anosov has gradually decreased.

Our lives in mathematics started rather well. In 1965 we both graduated with honors from elite mathematical high schools and successfully passed the entrance exams to Mekh-Mat which at the time was arguably the best mathematical center in the world. Indeed, the decade from the end of the 50s until the end of the 60s is generally considered the golden age of Moscow mathematics [9]. We were lucky to have our math courses taught by such distinguished mathematicians and scholars as Efimov (linear algebra), Manin (algebra), Arnold (ordinary differential equations), Vishik (partial differential equations), Shabat (complex analysis), Shilov (real analysis). In addition, during our 3rd through 5th years at Mekh-Mat we greatly benefited from many topics courses and special seminars offered by stellar faculty. Our undergraduate adviser was Yakov Grigorievich Sinai who was very popular among students. Naturally, we started to attend the Sinai–Alexeyev seminar which was then the central arena for those interested in dynamics. At this time our close interaction with Katok began. He was a graduate student of Sinai and got his PhD in 1968. On his own initiative he gave topics courses in dynamical systems and ergodic theory, and this was the only course work in dynamical systems

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we ever had. In the Fall of 1969 we started attending the Anosov–Katok seminar which became essentially our only connection to mathematics for a decade; for a more detailed description and history of this seminar see, [12, 13].

By the end of the 60s Mekh-Mat’s golden years were over. Foremost it manifested itself in increased anti-Semitism and general oppression against liberal thought [7, 8]. Almost no Jews were accepted as either undergraduate or graduate students at Mekh-Mat, and no Jewish faculty were hired. This was the reason Katok did not have a chance of getting a position at Mekh-Mat and was “lucky” to get a job at the Central Economics-Mathematics Institute where he was rather free to do research of his choice. However, he could not continue teaching or run seminars at Mekh-Mat, but Anosov could, and this is how the Anosov–Katok seminar started.

Anti-Semitism at Mekh-Mat affected both of us enormously. Although we graduated from Mekh-Mat with honors, and were recommended for the graduate school by our adviser Sinai and by the Mekh-Mat administration, the department communist party bureau rejected our applications. In the end M.B. got a job at the Research Economics Institute of the State Planning Committee and Ya.P. at the Research Institute of Optical-Physical Measurements. Here we faced a very hard choice – either to quit mathematics (as many of our classmates did) or to combine it with our meaningless full time jobs.

Since we were not affiliated with any mathematical institution, our resources to carry out research in mathematics were very limited and our mathematical future was quite uncertain. The Anosov–Katok seminar was the main reason and, in fact, the only possibility for us to stay in mathematics since it allowed us to be abreast of current developments in dynamical systems, helped us navigate our own research and discuss our results. The personal qualities of the seminar leaders created an open and democratic intellectual atmosphere which for us was a kind of escape from the unpleasant reality of our day-to-day duties at work.

Anosov was the official PhD adviser for both of us and played a vital role in our mathematical lives. Since he was a student of Pontryagin (who was at the very top of the Soviet mathematical hierarchy) and obtained spectacular results early in his career, he quickly advanced to the higher tiers of the Soviet mathematical establishment. Anosov was a professor of the Moscow State University, a member of the Steklov Mathematical Institute, a recipient of the prestigious State Prize, a member of the editorial boards of two top mathematical journals, and a member of the Higher Attestation Board (the state body charged with certifying higher scientific degrees). For many mathematicians to reach such a level and stay at it meant getting involved to a higher or lesser degree in some unethical activities. Anosov was one of the very few who never compromised on moral issues and, in fact, often used his influence to correct the wrong. He was one of the best representatives of the Russian intelligentsia with high self-imposed moral principles. It was his conscientious decision to always keep his “hands clean”; by the standards of the time this was a hard choice to make. There was not a drop of anti-Semitism in him and in fact, he helped quite a few Jewish mathematicians, us in particular. Although our research at the beginning of the 70s was rather successful, since we were Jewish and were not affiliated with any mathematical institution, if not for Anosov it would have been virtually impossible for us to publish our results in any major mathematical journal and get a PhD.

We would like to emphasize that Anosov's advising style was not of the type that is common and expected in Western schools: he never proposed any problem to us to work on and we did not expect him to help us work out technical difficulties should we face some. Perhaps partly this may be due to the fact that we were not students at any graduate school. However, when we obtained some interesting results that we were eager to present and discuss he would be always willing to listen and express his opinion. From time to time he would provide us with some relevant recent papers or preprints which otherwise we would not be aware of or able to find. Most important he was instrumental in helping us publish our major papers and he did it purely because he considered our results to be a major achievement in dynamical systems.

Unfortunately many other people "in power" acted differently and did not feel embarrassed to ask for something in return. For example, publishing one of our papers with the help of a person "in power" was conditioned on explaining some results of the paper to a student of this person, so that the student could claim and publish these results "independently". Actually this was not considered outrageous at that time; or, as Anosov put it "An evil world begets evil morality."

Soon after we graduated from Mekh-Mat, the Anosov-Katok seminar was thrown out of the university and moved to the Steklov Mathematical Institute. We met once a week starting at 5 p.m. to accommodate many of the participants with full time jobs. It lasted for about 2 hours. At the beginning the entrance to the building and seminar participation were not controlled. Before long, however, as part of its anti-Semitic policy, the institute administration demanded that the list of participants of each seminar be submitted for approval. Anosov's problem was that many participants of his seminar were Jewish. He found a way around it by adding (to the seminar list) fictitious Russian sounding names to please the eye of the administration.

In 1971 after looking through papers [11, 18] Katok noted to us that it would be interesting to consider dynamical systems with stable and unstable directions of not complementary dimensions and pointed out that the frame flow on a manifold of negative curvature was a natural example of this situation. In about 2 years we obtained the results which are now considered the foundation of partial hyperbolicity. This was a rather bumpy road with progress often followed by setbacks. We could only work after hours or on weekends thus taking time away from our families which required understanding, sacrifice and strong support of our wives. On the positive side, we had frequent long discussions with Anosov and Katok who were genuinely interested in our work, and this kept us going. As an example, Brin visited Anosov at his dacha to present a "proof" of ergodicity for a system with accessibility. The argument was long and convoluted and took about half an hour. Sharp-minded Anosov thought for about 5 minutes and pointed out a subtle mistake – one of the sets considered did not need to be measurable which ruined the argument.

We would like to make a few comments on our work on partial hyperbolicity [5]. At the beginning we followed the path which Anosov and Sinai developed for hyperbolic systems [2]: establish the Hölder continuity of the stable and unstable distributions, their integrability, and the absolute continuity of the stable and unstable foliations. However, in the partially hyperbolic case this already required a substantial modification of the known techniques and introduction of some new

methods. For example, in proving integrability we used a proper version of Peron’s method which works better in the setting of partial hyperbolicity than more standard Hadamard’s method, and in proving the absolute continuity property we had to deal with a serious obstacle of possible expansions (albeit at a slower rate) in the direction transverse to the unstable.

Further, since the stable and unstable invariant foliations are not smooth in general, we realized that to prove ergodicity one needed a generalization of the notion of commutator of vector fields for the non-differentiable setting. Eventually this led us to the notion of accessibility which we called *transitivity of foliations*. Anosov systems obviously have this property. In modern language, the non-differentiable commutator leads to accessibility through the *Brin argument* [3].

We understood from the beginning that the classical Hopf argument needed a substantial modification to work in the settings of partially hyperbolic systems. Our main result states that accessibility implies ergodicity under the following additional assumptions: Lipschitz continuity of the central distribution and its integrability, dynamical coherence (i.e., integrability of center-stable and center-unstable distributions), and the Lipschitz continuity of the stable and unstable holonomy maps along the central leaves. While these requirements are strong, it was not difficult to verify them in the situations we were interested in – group extensions and, in particular, frame flows. Later major progress in partial hyperbolicity to a large extent involved removing and/or weakening these conditions [6, 10, 17].

There are three lines at the end of the introduction in our joint paper [5] which an unprepared reader may consider strange: “The results of Sec. 3 as well as Theorems 4.1 and 4.2 belong to M. Brin. The results of Sec. 2 belong to Ya. Pesin. The rest of the results are joint.” The explanation is that when we discussed our PhD theses with Anosov, he recommended that we explicitly split the results between the two of us and then added that for two non-Jewish authors this joint paper would be more than enough for two dissertations but “two Jewish authors of one paper” must each write an additional separate paper on a different subject. Following his advice Brin started working on the genericity of ergodicity for frame flows resulted in [3, 4] and Pesin began working on what later has evolved as the non-uniform hyperbolicity theory [14–16].

To defend a PhD thesis, someone not in a graduate school needed to have an official scientific adviser as well as to find a mathematical institution which would accept his/her thesis for defense. In our case an obvious adviser would be either Katok or Anosov. Although Katok was more involved with our research, the chances of finding an institution which would agree to consider a thesis with a Jewish adviser and a Jewish student were zero. So Anosov was the only choice, and this was a huge commitment on his behalf. It still took several years after our theses were completed to get our PhDs. Brin eventually got his PhD in 1975 from the Kharkov State University thanks to great efforts by Naum Ilyich Akhiezer. Pesin got his PhD in 1979 from the Gorky State University with support from Leonid Pavlovich Shilnikov.

Here is the story of how it happened. After several unsuccessful attempts to find a place for the defense, Anosov made a deal with O, the head of the Scientific Council at Gorky. O had two graduate students with theses ready to be defended and he agreed to arrange the three defenses on the same day in exchange for Anosov ensuring a safe passage of the theses through the Higher Attestation Board. For

Anosov it was a rare and very serious compromise with his principles, but he decided it was worth it. As he put it “Yasha, I traded you for two”. As it turned out the thesis of one of O’s students claimed three theorems of which two were completely wrong and the third one needed very serious corrections. As a result that student dropped out and, as Anosov remarked: “it became a fair trade”. The deal notwithstanding, the positive outcome was not guaranteed. Unexpected help came from Evgeniya Aleksandrovna Leontovich-Andronova, the widow of a famous mathematician Andronov and a prominent member of the Scientific Council. At the end of the defense proceedings she said: “When my husband was alive, this work would result in a Doctor of Science degree¹, years later it would be considered an outstanding PhD thesis, and now we are thinking whether to vote yes or no”. Actually, there were still two negative votes.

We remember with pleasure the many hours we spent talking to Anosov at his home about mathematics as well as many other subjects. The atmosphere was very welcoming, and often his mother would bring out tea and cookies. Anosov was raised in a family of prominent scientists and had a large library at home. He was interested in and knew history very well, appreciated art and music. On occasion one could observe some elements of a “nobleman-among-peasants” in his demeanor which became more pronounced in later years. Anosov possessed a great sense of humor, and his remarks were often sharp and ironic. Some examples of this can be found in the introduction to his famous book on geodesic flows [1]. For instance, commenting on the Hadamard–Perron theorem he writes: “Every five years or so, if not more often, someone “discovers” the theorem of Hadamard and Perron, proving it either by Hadamard’s method or by Perron’s. I myself have been guilty of this”.

In 1991 the University of Maryland held a dynamical systems conference. Fresh from Russia Anosov entered the room in the middle of a talk, and after the talk was over many participants (quite a few of them Russian) rushed to greet the master. His immediate remark was: “let us rename the conference Anosov’s seminar and make Russian the conference language”. Anosov’s next stop was Penn State where he gave a colloquium talk. When he was running overtime, the colloquium chair interrupted him saying that his time was up. Anosov asked if there was time for questions. “Certainly”, replied the chair, “OK”, said Anosov, “then I will ask myself a question”. He then continued for another 15 minutes uninterrupted.

There was a common perception among quite a few Russian mathematicians in the 60s and 70s that a real mathematician should know if not all of mathematics then at least most of its major branches. Although this was hardly possible, some came rather close to this ideal. Anosov was one of them, his knowledge of mathematics was amazingly broad and deep. This manifested itself in his role as an editor. The books and papers he edited range from topology to geometry to dynamical systems and contain numerous footnotes, long and substantive introductions and remarks. For Anosov this was a way to express his mathematical views.

Anosov’s name is forever a part of the theory of dynamical systems. We are proud and fortunate that Dmitry Victorovich Anosov was our teacher and mentor.

¹The second degree after PhD, see [13].

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