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Every compact manifold carries a completely hyperbolic diffeomorphism

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Abstract. We show that a smooth compact Riemannian manifold of dimension greater than or equal to 2 admits a Bernoulli diffeomorphism with non-zero Lyapunov exponents.

0. *Introduction*

In this paper we prove the following theorem that provides an affirmative solution of the problem posed in [**BFK**].

MAIN THEOREM. *Given a compact smooth Riemannian manifold $\mathcal{K} \neq \mathcal{S}^1$ there exists a C^∞ diffeomorphism f of $\mathcal{K}$ such that:*

(1) *f preserves the Riemannian volume m on $\mathcal{K}$;*
(2) *f has non-zero Lyapunov exponents at m-almost every point $x \in \mathcal{K}$;*
(3) *f is a Bernoulli diffeomorphism.*

For surface diffeomorphisms this theorem was proved by Katok in [**K**]. In [**B2**], for any compact smooth Riemannian manifold $\mathcal{K}$ of dimension ≥ 5, Brin constructed a C^∞ Bernoulli diffeomorphism which preserves the Riemannian volume and has all but one Lyapunov exponents non-zero. Thus, combining the results of [**B2, BFK, K**] one obtains that any manifold $\mathcal{K}$ admits a diffeomorphism with ℓ zero exponents, where

$$\ell = \begin{cases} 0, & \text{if } \dim \mathcal{K} = 2, \\ 2, & \text{if } \dim \mathcal{K} = 4, \\ 1, & \text{otherwise.} \end{cases}$$

In this paper we show how to perturb the diffeomorphism to remove zero exponents. Let us review some main ingredients in the construction of hyperbolic Bernoulli diffeomorphisms.

(1) Let f be a diffeomorphism of $\mathcal{K}$ preserving a smooth volume m and $T\mathcal{K} = E \oplus F$ the splitting of $T\mathcal{K}$ into two invariant subbundles. We say that F *dominates* E (and write $E < F$) if there exists $\theta < 1$ such that

$$\max_{v \in E, \|v\|=1} \|df(v)\| \leq \theta \min_{v \in F, \|v\|=1} \|df(v)\|.$$

If f admits a dominating splitting then so does any diffeomorphism which is sufficiently close to f. Shub and Wilkinson [**SW**] have shown that if $T\mathcal{K} = E_1 \oplus E_2 \oplus E_3$ where $E_1 < E_2 < E_3$ then the function

$$f \to \int \log \det(df|E_2)(x)\, dm(x)$$

is not locally constant (see also [**D**]).

(2) If for any sufficiently small perturbation of f the subspace E_2 does not admit further splitting, then using results of Mañé [**M1**] (see also [**M2**]) and Bochi [**Bo**] one can approximate f by a diffeomorphism g such that all Lyapunov exponents of g along E_2 are close to each other. We will use this observation in the case $\dim \mathcal{K} = 4$.

(3) The results in (1) and (2) can be used for constructing non-uniformly hyperbolic systems on manifolds carrying diffeomorphisms with dominated decomposition. However, not every manifold has this property. On the other hand, results in [**B2, BFK**] allow one to construct on any manifold a diffeomorphism which is partially hyperbolic away from a singularity set. In this paper we extend results in (1) and (2) above to diffeomorphisms with singular splitting.

(4) The above results allow us to construct systems having non-zero exponents on a set of positive measure. We then establish local ergodicity using the approach of [**P**] (see also [**BP**] for detailed exposition and extensions of this approach).

(5) Finally, we use some ideas from [**BrP**] concerning transitivity of foliations to pass from local to global ergodicity.

The structure of the paper is as follows. We begin with the case $\dim \mathcal{K} \geq 5$, since in the multi-dimensional case there is more room to perturb and so the proof is simpler. Then we describe modifications needed if $\dim \mathcal{K} = 3$ or 4. In §§1–3 we review constructions of Katok [**K**] and Brin [**B2**] and establish some additional properties of the corresponding diffeomorphisms which are used in our analysis. In §4 we explain how to get rid of the zero Lyapunov exponent while in §5 we establish some crucial properties of our perturbation including transitivity and absolute continuity. In §6 we observe the Bernoulli property of our diffeomorphism and thus complete the proof in the case $\dim \mathcal{K} \geq 5$. We then proceed in §7 with modifications needed in dimensions three and four. Section 8 reviews Mañẽ's work on the discontinuity of Lyapunov exponents which is needed for the four-dimensional case.

Finally, let us mention that open sets of hyperbolic Bernoulli diffeomorphisms on some manifolds are constructed in [**ABV, BV, D, SW**].

Preliminaries and notation. In this paper we deal with various partially (uniformly and non-uniformly) hyperbolic diffeomorphisms and we adopt the following notation (see [**BP**] for details). A diffeomorphism F of a compact smooth Riemannian manifold $\mathcal{K}$ is called

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non-uniformly partially hyperbolic on a set $X \subset \mathcal{K}$ if for every $x \in X$ the tangent space at x admits an invariant splitting

$$T_x\mathcal{K} = E^s_F(x) \oplus E^c_F(x) \oplus E^u_F(x) \tag{0.1}$$

into *stable*, *central*, and *unstable subspaces*. This means that there exist numbers $0 < \lambda^s < \lambda^c_1 \le 1 \le \lambda^c_2 < \lambda^u$ and Borel functions $C(x) > 0$ and $K(x) > 0$, $x \in X$, such that:

(1) for $n > 0$,

$$\begin{aligned} \|d_xF^n v\| &\le C(x)(\lambda^s)^n e^{\varepsilon n}\|v\|, \quad v \in E^s(x),\\ \|d_xF^{-n} v\| &\le C(x)(\lambda^u)^{-n} e^{-\varepsilon n}\|v\|, \quad v \in E^u(x),\\ C(x)^{-1}(\lambda^c_1)^n e^{-\varepsilon n}\|v\| \le \|d_xF^n v\| &\le C(x)(\lambda^c_2)^n e^{\varepsilon n}\|v\|, \quad v \in E^c(x); \end{aligned}$$

(2)

$$\angle(E^s(x), E^u(x)) \ge K(x), \quad \angle(E^s(x), E^c(x)) \ge K(x), \quad \angle(E^u(x), E^c(x)) \ge K(x);$$

(3) for $m \in \mathbb{Z}$,

$$C(F^m(x)) \le C(x)e^{\varepsilon|m|}, \quad K(F^m(x)) \ge K(x)e^{-\varepsilon|m|}.$$

Throughout the paper we deal with the case

$$\lambda^c_2 - \lambda^c_1 \le \varepsilon$$

for sufficiently small $\varepsilon > 0$. We denote by

$$\chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|dF^n v\| \tag{0.2}$$

the *Lyapunov exponent* of v at x and by $\chi^i_F(x)$ the values of the Lyapunov exponents at x. We also adopt the notation $\chi^c_F(x)$ for the Lyapunov exponent along the central direction in the one-dimensional case and $\chi^c_1(x, F) \ge \chi^c_2(x, F)$ for the two Lyapunov exponents along the central direction in two-dimensional case (only these two cases will be considered). Given $\varepsilon > 0$, set

$$\Lambda^+(x, F, \varepsilon) = \sum_{\chi^i_F(x)>\varepsilon} \chi^i_F(x), \quad \Lambda^-(x, F, \varepsilon) = \sum_{\chi^i_F(x)<\varepsilon} \chi^i_F(x). \tag{0.3}$$

Denote by $V^s_F(x)$ and $V^u_F(x)$ the *local stable* and *unstable manifolds* at x. They can be characterized as follows: there is a neighborhood $U(x)$ of the point x such that for any $n > 0$,

$$\begin{aligned} V^u_F(x) &= \{y \in U(x) : d(F^{-n}(x), F^{-n}(y)) \le C(x)(\lambda^u)^{-n}e^{-\varepsilon n}\, d(x, y)\},\\ V^s_F(x) &= \{y \in U(x) : d(F^n(x), F^n(y)) \le C(x)(\lambda^s)^n e^{\varepsilon n}\, d(x, y)\}. \end{aligned} \tag{0.4}$$

Finally, we define the *global stable* and *unstable manifolds* at x by

$$\begin{aligned} W^u_F(x) &= \bigcup_{n\ge 0} F^n(V^u_F(F^{-n}(x))),\\ W^s_F(x) &= \bigcup_{n\ge 0} F^{-n}(V^s_F(F^n(x))). \end{aligned} \tag{0.5}$$

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Given a subset $X \subset \mathcal{K}$ we call two points $p, q \in \mathcal{K}$ *accessible via* X, if there are points $z_0 = p, z_1, \ldots, z_{\ell-1}, z_\ell = q, z_i \in X$, such that $z_i \in V^\alpha_F(z_{i-1})$ for $i = 1, \ldots, \ell$ and $\alpha \in \{s, u\}$. The collection of points $z_0, z_1, \ldots, z_\ell$ is called the *path* connecting p and q and is denoted by $[p, q]_F = [z_0, z_1, \ldots, z_\ell]_F$. The diffeomorphism F is said to have the accessibility property on X if any two points $p, q \in X$ are accessible.

Recall that a partition ξ of a Borel subset $X \subset \mathcal{K}$ is called a *foliation of* X *with* C^1 *leaves* if there exist continuous functions $\delta : X \to (0, \infty)$ and $q : X \to (0, \infty)$ and an integer $k > 0$ such that for each $x \in X$:

(1) there exists a smooth immersed k-dimensional manifold $W(x)$ containing x for which $\xi(x) = W(x) \cap X$ where $\xi(x)$ is the element of the partition ξ containing x; the manifold $W(x)$ is called the *(global) leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta(x))$ that contains x is called the *local leaf* at x and is denoted by $V(x)$; the number $\delta(x)$ is called *the size* of $V(x)$;

(2) there exists a continuous map $\phi_x : X \cap B(x, q(x)) \to C^1(D, M)$ (where $D \subset \mathbb{R}^k$ is the unit ball) such that $V(y)$, $y \in X \cap B(x, q(x))$ is the image of the map $\phi_x(y) : D \to \mathcal{K}$.

In this paper we will only consider foliations with C^1 leaves and for simplicity we will call them foliations.

1. *Katok's example*

Consider the two-dimensional unit disk $\mathcal{D}^2 = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$. Any diffeomorphism $g : \mathcal{D}^2 \to \mathcal{D}^2$ can be written in the form $g(u_1, u_2) = (g_1(u_1, u_2), g_2(u_1, u_2))$. We describe classes of functions and diffeomorphisms which are 'sufficiently flat' near the boundary $\partial\mathcal{D}^2$. The sequence $\rho = (\rho_0, \rho_1, \ldots)$ of real-valued continuous functions on $\mathcal{D}^2$ is called *admissible* if every function ρ_n is non-negative and is strictly positive inside the disk. We denote by $C^\infty_\rho(\mathcal{D}^2)$ the class of functions $\phi \in C^\infty(\mathcal{D}^2)$ which satisfy the following property: for every $n \geq 0$ there exists $\varepsilon_n > 0$ such that for every $(u_1, u_2) \in \mathcal{D}^2$ with $u_1^2 + u_2^2 \geq (1 - \varepsilon_n)^2$ we have

$$\left| \frac{\partial^n \phi(u_1, u_2)}{\partial^{i_1} u_1 \partial^{i_2} u_2} \right| < \rho_n(u_1, u_2)$$

for all non-negative integers $i_1, i_2, i_1 + i_2 = n$. We also denote

$$\mathrm{Diff}^\infty_\rho(\mathcal{D}^2) = \{g \in \mathrm{Diff}^\infty(\mathcal{D}^2) : g_i(u_1, u_2) - u_i \in C^\infty_\rho(\mathcal{D}^2), i = 1, 2\}.$$

PROPOSITION 1.1. (See [**K**]) *For every admissible sequence of functions* ρ *on* $\mathcal{D}^2$ *there exists a diffeomorphism* $g \in \mathrm{Diff}^\infty_\rho(\mathcal{D}^2)$ *which satisfies statements (1) and (2) of the Main Theorem.*

We outline the proof of Proposition 1.1. Let g_0 be a hyperbolic automorphism of the 2-torus $\mathcal{T}^2$ which has four fixed points $x_1 = (0, 0)$, $x_2 = (1/2, 0)$, $x_3 = (0, 1/2)$, $x_4 = (1/2, 1/2)$ (for example, the automorphism generated by the matrix $\left|\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right|$ is appropriate). The desired diffeomorphism g is constructed via the following commutative

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diagram

$$\begin{array}{ccccccccc}
\mathcal{T}^2 & \xrightarrow{\varphi_0} & \mathcal{T}^2 & \xrightarrow{\varphi_1} & \mathcal{T}^2 & \xrightarrow{\varphi_2} & \mathcal{S}^2 & \xrightarrow{\varphi_3} & \mathcal{D}^2 \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g} \\
\mathcal{T}^2 & \xrightarrow{\varphi_0} & \mathcal{T}^2 & \xrightarrow{\varphi_1} & \mathcal{T}^2 & \xrightarrow{\varphi_2} & \mathcal{S}^2 & \xrightarrow{\varphi_3} & \mathcal{D}^2
\end{array}$$

where $\mathcal{S}^2$ is the unit sphere. The map g_1 is obtained by slowing down g_0 near the points x_i. Its construction depends upon a real-valued function ψ which is defined on the unit interval $[0, 1]$ and has the following properties:

(1.1) ψ is C^∞ except for the point 0;

(1.2) $\psi(0) = 0$ and $\psi(u) = 1$ for $u \geq r$ where $0 < r < 1$ is a number;

(1.3) $\psi'(u) \geq 0$;

(1.4)
$$\int_0^1 \frac{du}{\psi(u)} < \infty.$$

The next condition on the function ψ expresses a 'very slow' rate of convergence of the integral $\int_0^1 (du/\psi(u))$ near zero. More precisely, for $i = 1, 2, 3, 4$ consider the disk $\mathcal{D}_r^i$ centered at x_i of radius r and endowed with the coordinate system (s_1, s_2), i.e.

$$\mathcal{D}_r^i = \{(s_1, s_2) : s_1^2 + s_2^2 \leq r\}.$$

Choose numbers $r_0 > r_1 > r > 0$ such that

$$\mathcal{D}_{r_0}^i \cap \mathcal{D}_{r_0}^j = \emptyset,\ i \neq j, \quad (g_0(\mathcal{D}_{r_1}^i) \cup g_0^{-1}(\mathcal{D}_{r_1}^i)) \subset \mathcal{D}_{r_0}^i, \quad \mathcal{D}_r^i \subset \operatorname{Int}(g_0(\mathcal{D}_{r_1}^i)).$$

We also set $\mathcal{D} = \bigcup_{i=1}^4 \mathcal{D}_{r_1}^i$. Let $\beta(u)$ be the inverse of the function

$$\gamma(u) = \sqrt{\int_0^u \frac{d\tau}{\psi(\tau)}}.$$

Consider the following two functions defined near the origin,

$$H_1(s_1, s_2) = (\log \alpha)\beta\Big(\sqrt{s_1^2 + s_2^2}\Big)\frac{s_1 s_2}{s_1^2 + s_2^2},$$

and

$$H_2(s_1, s_2) = (\log \alpha)\beta\Big(\sqrt{s_1^2 + s_2^2}\Big)\frac{s_2}{\sqrt{s_1^2 + s_2^2}},$$

as well as the function H defined near $\partial\mathcal{D}^2$ by

$$H(x_1, x_2) = (\log \alpha)\beta\Big(\sqrt{1 - x_1^2 - x_2^2}\Big)\frac{x_2}{\sqrt{x_1^2 + x_2^2}},$$

where α is the largest eigenvalue of the matrix generating g_0. We assume that the function ψ is chosen such that the following condition holds:

(1.5) for any sequence κ of admissible germs near the origin in $\mathbb{R}^2$ and any sequence ρ of admissible functions on $\mathcal{D}^2$ there is a sequence θ of admissible germs near $0 \in \mathbb{R}^+$ such that if $\beta \in C_\theta^\infty(\mathbb{R}^+, 0)$ then $H_1, H_2 \in C_\kappa^\infty(\mathbb{R}^+, 0)$ and $H \in C_\rho^\infty(\mathcal{D}^2)$.

Denote by $\tilde{g}^i_\psi$ the time-one map generated by the vector field v_ψ in $\mathcal{D}^i_{r_0}$, $i = 1, 2, 3, 4$, given as follows:

$$\dot{s}_1 = (\log \alpha) s_1 \psi(s_1^2 + s_2^2), \quad \dot{s}_2 = -(\log \alpha) s_2 \psi(s_1^2 + s_2^2).$$

One can show that $\tilde{g}^i_\psi(\mathcal{D}^i_{r_1}) \subset \mathcal{D}^i_{r_0}$ and $\tilde{g}^i_\psi$ coincides with g_0 in some neighborhood of the boundary $\partial\mathcal{D}^i_{r_0}$. Therefore, the map

$$g_1(x) = \begin{cases} g_0(x), & \text{if } x \in \mathcal{T}^2 \setminus \mathcal{D}, \\ \tilde{g}^i_\psi(x), & \text{if } x \in \mathcal{D}, \end{cases}$$

defines a homeomorphism of the torus $\mathcal{T}^2$ which is a C^∞ diffeomorphism everywhere except for the points x_i, $i = 1, 2, 3, 4$. The map g_1 leaves invariant a smooth probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$ where the density κ is a positive C^∞ function except for infinities at x_i. It is defined by the formula

$$\kappa(x) = \begin{cases} \psi^{-1}(s_1^2(x) + s_2^2(x)), & \text{if } x \in \mathcal{D}, \\ 1, & \text{otherwise}, \end{cases}$$

and

$$\kappa_0 = \int_{\mathcal{T}^2} \kappa\, dm.$$

We summarize the properties of the map g_1 in the following lemma.

LEMMA 1.2. (See [**K**])

(1) *The map g_1 is topologically conjugate to g_0 via a homeomorphism φ_0 which transfers the stable $W^s_{g_0}(x)$ and unstable $W^u_{g_0}(x)$ (global) curves of g_0 into smooth curves which are stable $W^-_{g_1}(x)$ and unstable $W^+_{g_1}(x)$ curves of g_1.*

(2) *There exist continuous families of stable cones $K^-_{g_1}(x)$ and unstable cones $K^+_{g_1}(x)$, $x \in \mathcal{T}^2 \setminus \{x_1, x_2, x_3, x_4\}$, such that*

$$g_1^{-1}(K^-_{g_1}(x)) \subset K^-_{g_1}(g_1^{-1}(x)), \quad g_1(K^+_{g_1}(x)) \subset K^+_{g_1}(g_1(x))$$

and the inclusions are strict on the closure of the set $\mathcal{T}^2 \setminus \mathcal{D}$.

(3) *The Lyapunov exponents of g_1 are non-zero almost everywhere with respect to the measure ν (and, indeed, with respect to any Borel invariant measure μ for which $\mu(\{x_i\}) = 0$, $i = 1, 2, 3, 4$).*

For every $x \in \mathcal{T}^2 \setminus \{x_1, x_2, x_3, x_4\}$ we define the stable and unstable one-dimensional subspaces at x by

$$E^-_{g_1}(x) = \bigcap_j g_1^{-j}(K^-_{g_1}(g_1^j(x))), \quad E^+_{g_1}(x) = \bigcap_j g_1^j(K^+_{g_1}(g_1^{-j}(x))).$$

LEMMA 1.3. (See [**K**])

(1) *The subspaces $E^-_{g_1}(x)$ and $E^+_{g_1}(x)$ depend continuously on x.*

(2) *The map g_1 is uniformly hyperbolic on $\mathcal{T}^2 \setminus \mathcal{D}$; more precisely, there is a number $\lambda > 1$ such that for every $x \in \mathcal{T}^2 \setminus \mathcal{D}$,*

$$\|dg_1 | E^-_{g_1}(x)\| \le \frac{1}{\lambda}, \quad \|dg_1^{-1} | E^+_{g_1}(x)\| \le \frac{1}{\lambda}.$$

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Once the maps φ_1, φ_2, and φ_3 are constructed the maps g_2, g_3, and g are defined to make the above diagram commutative. We follow [**K**] and describe a particular choice of maps φ_1, φ_2, and φ_3.

In a neighborhood of each point x_i, $i = 1, 2, 3, 4$, the map φ_1 is given by

$$\varphi_1(s_1, s_2) = \frac{1}{\sqrt{\kappa_0(s_1^2 + s_2^2)}}\left(\int_0^{s_1^2+s_2^2} \frac{du}{\psi(u)}\right)^{1/2}(s_1, s_2)$$

and it is the identity in $\mathcal{T}^2\setminus\mathcal{D}$. Thus, it is a homeomorphism which is a C^∞ diffeomorphism except for the points x_i; it carries the measure ν into the Lebesgue measure and it commutes with the involution $J(t_1, t_2) = (1 - t_1, 1 - t_2)$.

The map $\varphi_2 : \mathcal{T}^2 \to \mathcal{S}^2$ is a double branched covering and is regular and C^∞ everywhere except for the points x_i, $i = 1, 2, 3, 4$, where it branches; it commutes with the involution J and preserves the Lebesgue measure; there is a local coordinate system (τ_1, τ_2) in a neighborhood of each point $p_i = \varphi_2(x_i)$ such that

$$\varphi_2(s_1, s_2) = \left(\frac{s_1^2 - s_2^2}{\sqrt{s_1^2 + s_2^2}}, \frac{2s_1s_2}{\sqrt{s_1^2 + s_2^2}}\right).$$

In a neighborhood of the point p_4 the map φ_3 is given by

$$\varphi_3(\tau_1, \tau_2) = \left(\frac{\tau_1\sqrt{1 - \tau_1^2 - \tau_2^2}}{\sqrt{\tau_1^2 + \tau_2^2}}, \frac{\tau_2\sqrt{1 - \tau_1^2 - \tau_2^2}}{\sqrt{\tau_1^2 + \tau_2^2}}\right),$$

and it is extended to a C^∞ diffeomorphism φ_3 between $\mathcal{S}^2 \setminus \{p_4\}$ and Int $\mathcal{D}^2$ which preserves the Lebesgue measure.

This concludes the construction of the diffeomorphism g in Proposition 1.1.

2. *Some additional properties of the diffeomorphism in Katok's example*

We first observe the following crucial properties of the map g_1.

PROPOSITION 2.1. *There are constants $\gamma_0 > 0$ and $C > 0$ such that for every $\gamma_0 \geq \gamma > 0$ one can find a point $x_0 \in \mathcal{T}^2 \setminus \mathcal{D}$ for which*

$$g_1^j(B(x_0, \gamma)) \bigcap B(x_0, \gamma) = \emptyset, \quad -N < j < N, \ j \neq 0,$$
$$g_1^j(B(x_0, \gamma)) \bigcap \mathcal{D} = \emptyset, \quad -N < j < N,$$

where $N = N(\gamma) = -(\log\gamma/\log\lambda) - C$.

Proof. Note that the statement holds true for the linear hyperbolic automorphism g_0 and the desired result now follows from Lemma 1.2. □

We now describe some additional properties of the map g.

Let $\mathcal{U}$ be a sufficiently small neighborhood of the *singularity* set $\mathcal{Q} = \{q_1, q_2, q_3\}\cup\partial\mathcal{D}^2$ where $q_i = \varphi_3(p_i)$, $i = 1, 2, 3$.

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PROPOSITION 2.2.

(1) *The Lyapunov exponents of g are non-zero almost everywhere with respect to the Lebesgue measure m.*

(2) *There exist continuous families of stable cones* $K_g^-(x)$ *and unstable cones* $K_g^+(x)$, $x \in \mathcal{D}^2 \setminus \mathcal{Q}$, *such that*

$$g^{-1}(K_g^-(x)) \subset K_g^-(g^{-1}(x)), \quad g(K_g^+(x)) \subset K_g^+(g(x))$$

and the inclusions are strict on the closure of the set $\mathcal{D}^2 \setminus \mathcal{U}$.

(3) *The distributions*

$$E_g^-(x) = \bigcap_j g^{-j}(K_g^-(g^j(x))), \quad E_g^+(x) = \bigcap_j g^j(K_g^+(g^{-j}(x)))$$

are one-dimensional dg*-invariant and continuous on* $\mathcal{D}^2 \setminus \mathcal{Q}$*; moreover, the map g is uniformly hyperbolic on* $\mathcal{D}^2 \setminus \mathcal{U}$*: for* $x \in \mathcal{D}^2 \setminus \mathcal{U}$,

$$\|dg|E_g^-(x)\| \leq \frac{1}{\lambda}, \quad \|dg^{-1}|E_g^+(x)\| \leq \frac{1}{\lambda};$$

furthermore, there is an invariant set X of full measure such that for every $x \in X$,

$$E_g^s(x) = E_g^-(x), \quad E_g^u(x) = E_g^+(x),$$

where $E_g^s(x)$ *and* $E_g^u(x)$ *are given by (0.1) (with* $F = g$*;* $E_g^c(x) = 0$ *in this case).*

(4) *The map g possesses two one-dimensional foliations,* W_g^- *and* W_g^+*, of the set* $\mathcal{D}^2 \setminus \mathcal{Q}$ *such that*

$$T_x W_s^-(x) = E_g^-(x), \quad T_x W_u^-(x) = E_g^+(x), \quad x \in \mathcal{D}^2 \setminus \mathcal{Q};$$

the sizes of local leaves $V_g^-(x)$ *and* $V_g^+(x)$ *are bounded away from zero on the set* $\mathcal{D}^2 \setminus \mathcal{U}$*; moreover, for every* $x \in X$,

$$W_g^s(x) = W_g^-(x), \quad W_g^u(x) = W_g^+(x),$$

where $W_g^s(x)$ *and* $W_g^u(x)$ *are given by (0.5) (with* $F = g$*).*

(5) *There exists* $\gamma_0 > 0$ *such that for every* $\gamma_0 > \gamma > 0$ *one can find a point* $x_0 \in D^2 \setminus \mathcal{U}$ *such that*

$$g^j(B(x_0, \gamma)) \bigcap B(x_0, \gamma) = \emptyset, \quad -N < j < N, j \neq 0,$$
$$g^j(B(x_0, \gamma)) \bigcap \mathcal{U} = \emptyset, \quad -N < j < N,$$

where $N = N(\gamma) = -(\log \gamma / \log \lambda) - C$ *and* $C > 0$ *is a constant.*

Proof. The result follows immediately from Lemmas 1.2 and 1.3, and Proposition 2.1. □

Remarks. (1) Katok pointed out to us that the leaves $W_g^-(x)$ and $W_g^+(x)$ depend Lipschitz continuously over $x \in \mathcal{D}^2 \setminus \mathcal{Q}$ (private communication).

(2) One can show that the set $\mathcal{T}^2 \setminus (\varphi_1 \circ \varphi_2 \circ \varphi_3)^{-1}(X)$ is the union of the stable and unstable separatrices of the fixed points x_1, x_2, x_3, and x_4.

3. *The description of Brin's example*

We outline Brin's construction from [**B2**].

Given a positive integer $n \geq 5$ set $k = [(n-3)/2]$ and consider the $(n-3) \times (n-3)$ block diagonal matrix $A = (A_i)$, where $A_i = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ for $i < k$ and

$$A_k = \begin{cases} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, & \text{if } n \text{ is odd}, \\ \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}, & \text{if } n \text{ is even}. \end{cases}$$

It is easy to see that $\det A = 1$ and that A generates a volume-preserving hyperbolic automorphism of the torus $\mathcal{T}^{n-3}$. Let T^t be the suspension flow over A with the roof function

$$H = H_0 + \varepsilon H(x),$$

where H_0 is a constant and the function $H(x)$ is such that $|H(x)| \leq 1$. The flow T^t is an Anosov flow on the phase space $\mathcal{Y}^{n-2}$ which is diffeomorphic to the product $\mathcal{T}^{n-3} \times [0, 1]$, where the tori $\mathcal{T}^{n-3} \times 0$ and $\mathcal{T}^{n-3} \times 1$ are identified by the action of A.

One can choose the function $H(x)$ such that the flow T^t has the accessibility property.

Consider the following skew product map R of the manifold $\mathcal{M} = \mathcal{D}^2 \times \mathcal{Y}^{n-2}$:

$$R(z) = R(x, y) = (g(x), T^{\alpha(x)}(y)), \quad z = (x, y), \tag{3.1}$$

where the diffeomorphism g is constructed in Proposition 1.1 and $\alpha : \mathcal{D}^2 \to \mathbb{R}$ is a non-negative C^∞ function which is equal to zero in the neighborhood $\mathcal{U}$ of the singularity set $\mathcal{Q}$ and is strictly positive otherwise.

We define the *singularity* set for the map R by $\mathcal{S} = \mathcal{Q} \times \mathcal{Y}^{n-2}$, where $\mathcal{Q}$ is the singularity set of the map g (see Proposition 2.2). We also set $\mathcal{N} = (\mathcal{D}^2 \setminus \mathcal{U}) \times \mathcal{Y}^{n-2}$ and $Z = X \times \mathcal{Y}^{n-2}$, where the sets $\mathcal{U}$ and X are defined in Proposition 2.2.

PROPOSITION 3.1. *The following statements hold.*

(1) *The map R possesses four continuous cone families $K_R^-(z)$, $K_R^{-c}(z)$, $K_R^+(z)$, and $K_R^{+c}(z)$, $z \in \mathcal{M} \setminus \mathcal{S}$, such that*

$$\begin{aligned} R^{-1}(K_R^-(z)) \subset K_R^-(R^{-1}(z)), \quad & R(K_R^+(z)) \subset K_R^+(R(z)), \\ R^{-1}(K_R^{-c}(z)) \subset K_R^{-c}(R^{-1}(z)), \quad & R(K_R^{+c}(z)) \subset K_R^{+c}(R(z)), \end{aligned} \tag{3.2}$$

and inclusions are strict on the closure of the set $\mathcal{N}$; moreover, there exists $\mu > 1$ such that for all $z \in \mathcal{N}$,

$$\begin{aligned} \|dR(v)\| &> \mu\|v\| \quad \textit{for all } v \in K^+(z), \\ \|dR(v)\| &< \frac{1}{\mu}\|v\| \quad \textit{for all } v \in K^-(z). \end{aligned} \tag{3.3}$$

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(2) *For every $z \in Z$ the formulae*

$$E^s_R(z) = \bigcap_j R^{-j}(K^-_R(R^j(z))), \quad E^u_R(z) = \bigcap_j R^j(K^+_R(R^{-j}(z))),$$

determine dR-invariant stable and unstable continuous distributions such that

$$T_z\mathcal{M} = E^s_R(z) \oplus E^c_R(z) \oplus E^u_R(z),$$

where $E^c_R(z)$ is the one-dimensional central (flow) direction.

(3) *For every $z \in \mathcal{N} \cap Z$,*

$$\|dR|E^s_R(z)\| \le \frac{1}{\mu}, \quad \|dR^{-1}|E^u_R(z)\| \le \frac{1}{\mu}.$$

(4) *For every $z = (x, y) \in Z$,*

$$\pi_1 E^s_R(z) = E^s_g(x), \quad \pi_1 E^u_R(z) = E^u_g(x),$$
$$\pi_2 E^s_R(z) = E^s_{T'}(y), \quad \pi_2 E^u_R(z) = E^u_{T'}(y),$$

where $\pi_1 : T_z\mathcal{M} \to T_x\mathcal{D}^2$ and $\pi_2 : T_z\mathcal{M} \to T_y\mathcal{Y}^{n-2}$ are the natural projections.

(5)

$$m\{x \in \mathcal{M} : R^n(x) \in \mathcal{U} \text{ for all } n \in \mathbb{Z}\} = 0.$$

Proof. For every $z = (x, y) \in (\mathcal{U} \setminus \mathcal{S}) \times \mathcal{Y}^{n-2}$ we set

$$K^-_R(z) = K^-_g(x) \times K^s_{T'}(y), \quad K^+_R(z) = K^+_g(x) \times K^u_{T'}(y).$$

Now for every $z \in \mathcal{N}$ one can find numbers $n_1 = n_1(z)$ and $n_2 = n_2(z)$ such that

$$R^{n_1}(z), R^{-n_2}(z) \in (\mathcal{U} \setminus \mathcal{S}) \times \mathcal{Y}^{n-2}.$$

Set

$$K^+_R(z) = dR^{n_2} K^+_R(R^{-n_2}(z)), \quad K^-_R(z) = dR^{-n_1} K^-_R(R^{n_1}).$$

It is not difficult to show that $K^+_R(z)$ and $K^-_R(z)$ do not depend on the choice of numbers n_1 and n_2 and, by Proposition 2.2 (see statements (1)–(3)), have all the desired properties. We show that the distribution $E^u_R(z)$ is continuous over $z \in Z$. Indeed, let $z_n \in Z$ be a sequence of points which converges to a point $z \in Z$. By statements (2) and (3) of Proposition 2.2, given $\delta > 0$, one can find a number $m = m(z)$ such that the cone $R^m(K^+_R(R^{-m}(z)))$ is contained in the cone around $E^u_R(z)$ of angle δ. Therefore, for all sufficiently large n the cones $R^m(K^+_R(R^{-m}(z_n)))$ are contained in the cone around $E^u_R(z)$ of angle 2δ. Since $E^u_R(z_n) \subset R^m(K^+_R(R^{-m}(z_n)))$ the continuity of the distribution $E^u_R(z)$, $z \in Z$, follows. Similar arguments show the continuity of the distribution $E^s_R(z)$ over $z \in Z$. Statement (3) follows from statement (3) of Proposition 2.2 and statement (4) is obvious. The last statement is a consequence of statement (1) of Lemma 1.2 and the properties of the maps φ_1, φ_2, and φ_3 (see §1). □

PROPOSITION 3.2. *The distributions $E^s_R(z)$ and $E^u_R(z)$ generate two foliations, W^s_R and W^u_R, of Z; the sizes of local leaves $V^s_R(z)$ and $V^u_R(z)$ are bounded away from zero on the set $\mathcal{N} \cap Z$.*

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Proof. We follow arguments in [**B2**]. Let $z = (x, y) \in Z$. Set

$$W_R^s(z) = \bigcup_{\hat{x} \in W_g^s(x)} (\hat{x}, W_{T^t}^s(T^{t_s(\hat{x})}(y)),$$
$$W_R^u(z) = \bigcup_{\hat{x} \in W_g^u(x)} (\hat{x}, W_{T^t}^u(T^{t_u(\hat{x})}(y)),$$

where

$$\begin{aligned} t_s(\hat{x}) &= \sum_{n=0}^{\infty} \alpha(g^n(\hat{x}) - \alpha(g^n(x))), \\ t_u(\hat{x}) &= \sum_{n=0}^{\infty} \alpha(g^{-n}(\hat{x}) - \alpha(g^n(x))). \end{aligned} \tag{3.4}$$

Note that each series in (3.4) converges for every $x \in Z$. Indeed, since the point $(\varphi_1 \circ \varphi_2 \circ \varphi_3)^{-1}(x)$ does not lie on a separatrix of any of the fixed points x_1, x_2, x_3, and x_4 the series converges exponentially fast. The desired properties of the foliations W_R^s and W_R^u follow from Propositions 2.2 and 3.1. □

Remark. We shall show below (see Proposition 5.1) that the distributions $E_R^s(z)$ and $E_R^u(z)$ as well as foliations $W_R^s(z)$ and $W_R^u(z)$ can be extended to continuous distributions on and foliations of $\mathcal{M} \setminus \mathcal{S}$.

We proceed with Brin's construction.

LEMMA 3.3. (See [**B2**]) *There exists a smooth embedding of the manifold* $\mathcal{Y}^{n-2}$ *into* $\mathbb{R}^n$†.

We now state the main result in [**B2**].

PROPOSITION 3.4. *Given a compact smooth Riemannian manifold* $\mathcal{K}$ *of dimension* $n \geq 5$ *there exists a* C^∞ *diffeomorphism* h *of* $\mathcal{K}$ *such that:*

(1) *h preserves the Riemannian volume on $\mathcal{K}$;*

(2) *for almost every $z \in \mathcal{K}$ there exists a decomposition*

$$T_z\mathcal{K} = E_h^s(z) \oplus E_h^c(z) \oplus E_h^u(z)$$

into dh invariant stable, central, and unstable subspaces such that $\dim E_h^c(z) = 1$ *and the Lyapunov exponents at the point z of a vector $v \in T_z\mathcal{K}$*

$$\chi(z, v) \begin{cases} < 0, & \text{if } v \in E_h^s(z), \\ = 0, & \text{if } v \in E_h^c(z), \\ > 0, & \text{if } v \in E_h^u(z); \end{cases}$$

(3) *h satisfies the essential accessibility property and is a Bernoulli diffeomorphism.*

† The proof of this statement in [**B2**] needs some minor corrections. The manifold $\mathcal{Y}^{n-2}$ is of codimension two. Although not every codimension-two manifold has trivial normal bundle, $\mathcal{Y}^{n-2}$ does. This can easily be seen from its construction. Similar observation should be made wherever triviality of the normal bundle is used.

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Proof. Using Lemma 3.3 one can construct a smooth embedding $\chi_1 : \mathcal{K} \to \mathcal{B}^n$ (where $\mathcal{B}^n$ is the unit ball in $\mathbb{R}^n$) which is a diffeomorphism except for the boundary $\partial\mathcal{D}^2 \times \mathcal{Y}^{n-2}$. Then using results in [**K**] one can find a smooth embedding $\chi_2 : \mathcal{B}^n \to \mathcal{K}$ which is a diffeomorphism except for the boundary $\partial\mathcal{B}^n$. Since the map R is identity on the boundary $\partial\mathcal{D}^2 \times \mathcal{Y}^{n-2}$ the map $h = (\chi_1 \circ \chi_2) \circ R \circ (\chi_1 \circ \chi_2)^{-1}$ has all the properties stated in Proposition 3.4. □

4. *The perturbation of the diffeomorphism in Brin's example*

Fix a number $\gamma > 0$ and a point $y_0 \in \mathcal{Y}^{n-2}$ and set $\Delta = B(x_0, \gamma) \times B(y_0, \gamma)$ (where the point x_0 is chosen in Proposition 2.2, see statement (5)).

In this section we prove the following result.

PROPOSITION 4.1. *Given $\varepsilon > 0$, there is a C^∞ diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ such that*

(1) *P preserves the Riemannian volume m;*

(2) *$d_{C^1}(P, R) \leq \varepsilon$ where the map R is defined by (3.1); moreover, $P|(\mathcal{M} \setminus \Delta) = R|(\mathcal{M} \setminus \Delta)$;*

(3) *for almost every $z \in \mathcal{M}$ there exists a decomposition*

$$T_z\mathcal{M} = E^s_P(z) \oplus E^c_P(z) \oplus E^u_P(z)$$

into dP invariant subspaces such that $\dim E^c_P(z) = 1$ *and the Lyapunov exponent at the point z of a vector $v \in T_z\mathcal{M}$ is*

$$\chi(z, v) \begin{cases} <0, & \text{if } v \in E^s_P(z), \\ >0, & \text{if } v \in E^u_P(z); \end{cases}$$

(4) *the Lyapunov exponent $\chi^c_P(z)$ in the central direction satisfies*

$$\int_{\mathcal{M}} \chi^c_P(z)\, dm < 0.$$

Proof. Let $\varphi_x : \mathcal{Y}^{n-2} \to \mathcal{Y}^{n-2}, x \in \mathcal{M}$, be a family of volume-preserving C^∞ diffeomorphisms satisfying

$$d_{C^1}(\varphi_x, Id) \leq \varepsilon, \quad \varphi_x(y) = y \quad \text{for } (x, y) \in \mathcal{M} \setminus \Delta. \tag{4.1}$$

A particular choice of such a family of diffeomorphisms will be specified below (see Lemma 4.4). Set

$$\varphi(x, y) = (x, \varphi_x(y)), \quad P = \varphi \circ R. \tag{4.2}$$

It is easy to see that the map P is C^∞, volume preserving, and

$$P|(\mathcal{M} \setminus \Delta) = R|(\mathcal{M} \setminus \Delta), \quad d_{C^1}(P, R) \leq \varepsilon. \tag{4.3}$$

It follows from Proposition 3.1 and the first relation in (4.3) that, for every $z \in \mathcal{M} \setminus \mathcal{S}$,

$$\begin{aligned} P^{-1}(K^-_R(z)) \subset K^-_R(P^{-1}(z)), &\quad P(K^+_R(z)) \subset K^+_R(P(z)) \\ P^{-1}(K^{-c}_R(z)) \subset K^{-c}_R(P^{-1}(z)), &\quad P(K^{+c}_R(z)) \subset K^{+c}_R(P(z)) \end{aligned} \tag{4.4}$$

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and inclusions are strict on the set $\mathcal{M} \setminus \mathcal{S}$. Therefore, the formulae

$$E_P^s(z) = \bigcap_j P^{-j}(K_R^-(P^j(z))), \quad E_P^u(z) = \bigcap_j P^j(K_R^+(P^{-j}(z))) \tag{4.5}$$

define subspaces at every point $z \in Z$. Clearly, these subspaces are dP-invariant. Moreover, since the first coordinate of the point $P(x, y)$ depends only on x (see (4.2)) we obtain that

$$\pi_1 E_P^s(z) = E_g^s(x), \quad \pi_1 E_P^u(z) = E_g^u(x), \tag{4.6}$$

where $z = (x, y)$ (recall that $\pi_1 : T_z\mathcal{M} \to T_x\mathcal{D}^2$ is the natural projection).

Remark. We shall show below (see Proposition 5.1) that for any sufficiently small *gentle* perturbation P of the map R the distributions E_P^s and E_P^u can be extended to continuous distributions E_P^- and E_P^+ on the set $\mathcal{M} \setminus \mathcal{S}$ (but not just the set Z). However, the property (4.6) holds true only due to the special form of the perturbation (see (4.2)). This property is crucial for our further study (see Proposition 5.2).

LEMMA 4.2.
(1) *For every sufficiently small $\gamma > 0$ and $z = (x, y) \in Z$ with $x \in B(x_0, \gamma)$ we have that*

$$\begin{aligned} \angle(E_P^u(z), E_R^u(z)) &\le C\gamma^{\log\mu/\log\lambda}, \\ \angle(E_P^s(z), dP^{-1}E_R^s(P(z))) &\le C\gamma^{\log\mu/\log\lambda}. \end{aligned} \tag{4.7}$$

(2) *There is a number $\nu > 1$ such that for every $z \in \mathcal{N} \cap Y$,*

$$\|dP|E_P^s(z)\| \le \frac{1}{\nu}, \quad \|dP^{-1}|E_P^u(z)\| \le \frac{1}{\nu}. \tag{4.8}$$

Proof of the lemma. The second statement follows immediately from the first one and statement (3) of Proposition 3.1. We will prove the first inequality in (4.7), the proof of the second one is similar. Consider the point

$$z^* = (x^*, y^*) = R^{-(N-1)}(P^{-1}(z)),$$

where $N = N(\gamma)$ is defined in Proposition 2.2 (see statement (5)). By (4.3),

$$d(E_P^u(z^*), E_R^u(z^*)) \le \delta,$$

where d is the distance in the Grassmanian manifold and $\delta = \delta(\varepsilon) > 0$ is sufficiently small. Since

$$P^j(z^*) = R^j(z^*) \quad \text{for} \quad 0 \le j \le N-1 \tag{4.9}$$

we obtain using statement (3) of Proposition 3.1 that

$$d(dR^{N-1}E_P^u(z^*), dR^{N-1}E_R^u(z^*)) \le \frac{\delta}{\mu^{N-1}}.$$

Again using (4.9) we rewrite the last inequality as

$$d(E_P^u(P^{-1}(z)), E_R^u(P^{-1}(z))) \le \frac{\delta}{\mu^{N-1}} \le \delta\mu\gamma^{\log\mu/\log\lambda}.$$

Applying dP we obtain the desired result. □

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Since the maps R and P preserve the Riemannian volume we have, for every $z \in \mathcal{M} \setminus \mathcal{S}$,

$$\Lambda^+(z, R, \varepsilon) + \Lambda^-(z, R, \varepsilon) + \chi^c_R(z) = \Lambda^+(z, R, \varepsilon) + \Lambda^-(z, R, \varepsilon) = 0,$$
$$\Lambda^+(z, P, \varepsilon) + \Lambda^-(z, P, \varepsilon) + \chi^c_P(z) = 0,$$

(see (0.3) for the definition of the terms). It follows that

$$\int_{\mathcal{M}} \chi^c_P(z)\, dm = \int_{\mathcal{M}} \Lambda^+(z, R, \varepsilon)\, dm - \int_{\mathcal{M}} \Lambda^+(z, P, \varepsilon)\, dm + \int_{\mathcal{M}} \Lambda^-(z, R, \varepsilon)\, dm - \int_{\mathcal{M}} \Lambda^-(z, P, \varepsilon)\, dm. \tag{4.10}$$

LEMMA 4.3. *We have*

$$\int_{\mathcal{M}} \Lambda^+(z, P, \varepsilon)\, dm - \int_{\mathcal{M}} \Lambda^+(z, R, \varepsilon)\, dm = \int_{\Delta} (\log[\det(\Phi^u)(z)] + O(\varepsilon^{\log\mu/\log\lambda}))\, dm,$$

$$\int_{\mathcal{M}} \Lambda^-(z, P, \varepsilon)\, dm - \int_{\mathcal{M}} \Lambda^-(z, R, \varepsilon)\, dm = -\int_{\Delta} (\log[\det(\Phi^{-1})^s(z)] + O(\varepsilon^{\log\mu/\log\lambda}))\, dm,$$

where

$$\Phi^u(z) = d\varphi | E^u_R(z), \quad (\Phi^{-1})^s(z) = d\varphi | E^s_R(z). \tag{4.11}$$

Proof of the lemma. We will establish the first relation. The proof of the second one is similar. Consider the induced maps $\tilde{R}$ and $\tilde{P}$ generated by the maps R and P, respectively, on the set Δ. These maps are well defined for almost every $z \in \Delta$. Let $\tilde{\Delta}$ be the set of such points. By Kac's formula

$$\int_{\mathcal{M}} \Lambda^+(z, R, \varepsilon)\, dm = \int_{\tilde{\Delta}} \Lambda^+(z, \tilde{R}, \varepsilon)\, dm,$$
$$\int_{\mathcal{M}} \Lambda^+(z, P, \varepsilon)\, dm = \int_{\tilde{\Delta}} \Lambda^+(z, \tilde{P}, \varepsilon)\, dm.$$

It follows that

$$\int_{\mathcal{M}} [\Lambda^+(z, P, \varepsilon) - \Lambda^+(z, R, \varepsilon)]\, dm = \int_{\tilde{\Delta}} [\Lambda^+(z, \tilde{P}, \varepsilon) - \Lambda^+(z, \tilde{R}, \varepsilon)]\, dm.$$

Fix $z = (x, y) \in \tilde{\Delta}$. Every vector $v \in E^u_P(z)$ can be written in the form $v = v_R + w$ where $v_R \in E^u_R(z)$ and $w \in E^s_R(z) \oplus E^c_R(z)$. Denote by $N = N(z)$ the first return time of the point z to $\tilde{\Delta}$ under the map R. By (4.2) we have that the first return time of Z to $\tilde{\Delta}$ under the map P is also N. Moreover, by Lemma 4.2,

$$\begin{aligned} dP^N v &= d\varphi dR^N (v_R + w) \\ &= \|dR^N v_R\| d\varphi \left(\frac{dR^N v_R}{\|dR^N v_R\|} \right) (1 + \mathrm{O}(\mu^{-N})) \\ &= (1 + \mathrm{O}(\mu^{-N})) \|dR^N v_R\| \left[\Phi^u \frac{dR^N v_R}{\|dR^N v_R\|} + w^* \right], \end{aligned}$$

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where w^* is a vector in $E^s_R(z) \oplus E^c_R(z)$. Notice that

$$\int_{\tilde{\Delta}} \Lambda^+(z, \tilde{P}, \varepsilon)\, dm = \int_{\tilde{\Delta}} \log \det(d\tilde{P} | E^u_P(z))\, dm,$$
$$\int_{\tilde{\Delta}} \Lambda^+(z, \tilde{R}, \varepsilon)\, dm = \int_{\tilde{\Delta}} \log \det(d\tilde{R} | E^u_R(z))\, dm.$$

It follows that

$$\begin{aligned}\int_{\tilde{\Delta}} (\Lambda^+(z, \tilde{P}, \varepsilon) - \Lambda^+(z, \tilde{R}, \varepsilon))\, dm &= \int_{\tilde{\Delta}} \log \frac{\det \Phi^u(P^N | E^u_P(z))}{\det \Phi^u(R^N | E^u_R(z))}\, dm \\ &= \int_{\tilde{\Delta}} (\log \det \Phi^u(R^N(z)) + \mathrm{O}(\mu^{-N}))\, dm \\ &= \int_{\tilde{\Delta}} (\log \det \Phi^u(R^N(z)) + \mathrm{O}(\gamma^{\log \mu / \log \lambda}))\, dm.\end{aligned}$$

The desired result now follows. □

For $z = (x, y) \in Z$ we set

$$\tilde{\Phi}^u(z) = \frac{\partial \varphi_x}{\partial y}\bigg| (E^u_R(z) \cap T_z \mathcal{Y}^{n-2}), \quad (\tilde{\Phi}^{-1})^s(z) = \frac{\partial \varphi_x}{\partial y}\bigg| (E^s_R(z) \cap T_z \mathcal{Y}^{n-2}).$$

It follows from the definition of the map φ (see (4.2)) that

$$\det \Phi^u(z) = \det \tilde{\Phi}^u(z), \quad \det \Phi^s(z) = \det \tilde{\Phi}^s(z).$$

Therefore, using (4.10) and Lemma 4.3 we obtain that

$$\int_{\mathcal{M}} \chi^c_{\tilde{P}}(z)\, dm = \int_{\tilde{\Delta}} ([\log \det \tilde{\Phi}^u(z) - \log \det(\tilde{\Phi}^{-1})^s(z)] + \mathrm{O}(\gamma^{\log \mu / \log \lambda}))\, dm. \tag{4.12}$$

LEMMA 4.4. *There is a family of diffeomorphisms $\varphi_x : \mathcal{Y}^{n-2} \to \mathcal{Y}^{n-2}$ satisfying (4.1) and such that*

$$\int_{\tilde{\Delta}} [-\log \det \tilde{\Phi}^u(z) + \log \det(\tilde{\Phi}^{-1})^s(z)]\, dm \le -C\varepsilon^2 \gamma^{n-2} + \mathrm{O}(\varepsilon^3)\gamma^{n-2} + \mathrm{o}(1)\mathrm{O}(\gamma^n),$$

where $C > 0$ is a constant.

Proof of the lemma. Choose a coordinate system $\{x, y\} = \{x_1, x_2, y_1, y_2, \ldots, y_{n-2}\}$ in Δ such that:

(1) $dm = dx\, dy$;

(2)

$$E^c_{T'}(y_0) = \partial / \partial y_1, \quad E^s_{T'}(y_0) = \langle \partial / \partial y_2, \ldots, \partial / \partial y_k \rangle,$$
$$E^u_{T'}(y_0) = \langle \partial / \partial y_{k+1}, \ldots, \partial / \partial y_{i_{n-2}} \rangle$$

for some k, $2 \le k < n - 2$.

Let $\psi(t)$ be a C^∞ function with compact support. Set $\tau = \gamma^{-2}(\|x\|^2 + \|y\|^2)$ and define

$$\begin{aligned}\varphi_x^{-1}(y) = (x, y_1 \cos(\varepsilon\psi(\tau)) + y_2 \sin(\varepsilon\psi(\tau)), -y_1 \sin(\varepsilon\psi(\tau)) \\ + y_2 \cos(\varepsilon\psi(\tau)), y_3, \ldots, y_{n-2}).\end{aligned} \tag{4.13}$$

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Since the distributions $E^u_R(z)$ and $E^s_R(z)$ are continuous (see statement (2) of Proposition 2.2) by (4.11) we find that

$$\int_{\tilde{\Delta}} \log\det \tilde{\Phi}^u(z)\,dm = \mathrm{o}(1)m(\Delta) = \mathrm{o}(1)\mathrm{O}(\gamma^n) \tag{4.14}$$

and

$$\begin{aligned}\int_{\tilde{\Delta}} \log\det(\tilde{\Phi}^{-1})^s(z)\,dm &= \int_{\tilde{\Delta}} \log\det(d\varphi_x^{-1}|E^s_R)(z)\,dm \\ &= \int_{\tilde{\Delta}} \log\det\left(d\varphi_x^{-1}\middle|\left\langle\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_k}\right\rangle\right)(x,y)\,dx\,dy + \mathrm{o}(1)m(\Delta) \\ &= \int_{\tilde{\Delta}} \log\det\left(d\varphi_x^{-1}\middle|\left\langle\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_k}\right\rangle\right)(x,y)\,dx\,dy + \mathrm{o}(1)\mathrm{O}(\gamma^n).\end{aligned} \tag{4.15}$$

It is easy to see that

$$\begin{aligned}\det\left(d\varphi_x^{-1}\middle|\left\langle\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_k}\right\rangle\right)(x,y) &= -\frac{2y_1y_2}{\gamma^2}\varepsilon\psi'(\tau)\cos(\varepsilon\psi(\tau)) + \cos(\varepsilon\psi(\tau)) \\ &\quad - \frac{2y_2^2}{\gamma^2}\varepsilon\psi'(\tau)\cos(\varepsilon\psi(\tau)).\end{aligned}$$

It follows that

$$\begin{aligned}\log\det\left(d\varphi_x^{-1}\middle|\left\langle\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_k}\right\rangle\right)(x,y) &= -\frac{2y_1y_2}{\gamma^2}\varepsilon\psi'(\tau) - \frac{2y_1^2y_2^2}{\gamma^4}\varepsilon^2(\psi'(\tau))^2 \\ &\quad - \frac{1}{2}\varepsilon^2(\psi(\tau))^2 - \frac{2y_2^2}{\gamma^2}\varepsilon^2\psi(\tau)\psi'(\tau) + \mathrm{O}(\varepsilon^3).\end{aligned}$$

Making the coordinate change $\eta = y/\gamma$ we compute that

$$\begin{aligned}&\int_{\tilde{\Delta}} \log\det\left(d\varphi_x^{-1}\middle|\left\langle\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_k}\right\rangle\right)(x,y)\,dx\,dy \\ &\quad = \gamma^{n-2}\int_{B(x_0,\gamma)} dx \int_{\mathbb{R}^{n-2}} [-2\eta_1\eta_2\varepsilon\psi(\tau)']\,d\eta \\ &\qquad + \gamma^{n-2}\int_{B(x_0,\gamma)} dx \int_{\mathbb{R}^{n-2}} [-2\eta_1^2\eta_2^2\varepsilon^2(\psi(\tau)')^2]\,d\eta \\ &\qquad + \gamma^{n-2}\int_{B(x_0,\gamma)} dx \int_{\mathbb{R}^{n-2}} \left[-\frac{1}{2}\varepsilon^2(\psi(\tau))^2 - 2\varepsilon^2\psi(\tau)\psi(\tau)'\eta_2^2\right] d\eta + \mathrm{O}(\varepsilon^3)\gamma^{n-2}.\end{aligned} \tag{4.16}$$

Since the function ψ has compact support, the first integral in (4.16) is zero. Integrating by parts we obtain that

$$\int_{\mathbb{R}^{n-2}} \varepsilon^2\psi(\tau)\psi(\tau)'\eta_2^2\,d\eta = -\frac{1}{4}\int_{\mathbb{R}^{n-2}} \varepsilon^2(\psi(\tau))^2\,d\eta.$$

Hence, the third integral in (4.16) is also zero. The second integral is a strictly negative number of order $\mathrm{O}(\varepsilon^2\gamma^{n-2})$. The desired result follows. □

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Using Lemma 4.4 and (4.12) we obtain that

$$\int_{\mathcal{M}} \chi_P^c(z)\,dm = -C\varepsilon^2\gamma^{n-2} + \mathrm{O}(\varepsilon^3)\gamma^{n-2} + \mathrm{o}(1)\mathrm{O}(\gamma^n) + \mathrm{O}(\gamma^{\log\mu/\log\lambda+n}).$$

In order to complete the proof of the proposition we choose the number γ so small that $\gamma^2 \le \varepsilon^3$. □

5. *Absolute continuity and orbit density of the perturbation*

In this section we establish some additional crucial properties of the diffeomorphism P given by (4.2).

Definition. A perturbation P of the map R is called *gentle* if $P = R$ on $\mathcal{U} \times \mathcal{Y}^{n-2}$.

If P is a gentle perturbation of R which is sufficiently close to R then P satisfies (3.2) and (3.3). In what follows we assume that P has these properties. Set

$$E_P^+(z) = \bigcap_j dP^j(K_R^+(P^{-j}(z))), \quad E_P^-(z) = \bigcap_j dP^{-j}(K_R^-(P^j(z))),$$
$$E_P^{+c}(z) = \bigcap_j dP^j(K^{+c}(P^{-j}(z))), \quad E_P^{-c}(z) = \bigcap_j dP^{-j}(K^{-c}(P^j(z))), \tag{5.1}$$
$$E_P^c(z) = E_P^{+c}(z) \bigcap E_P^{c-}(z).$$

PROPOSITION 5.1. *The following statements hold:*

(1) $E_P^+(z)$, $E_P^-(z)$, $E_P^{+c}(z)$, $E_P^{-c}(z)$, *and* $E_P^c(z)$ *are* dP *invariant distributions which depend continuously over* $z \in \mathcal{M} \setminus \mathcal{S}$;

(2) *the distributions* $E_P^-(z)$ *and* $E_P^+(z)$ *are integrable and the corresponding global leaves* $W_P^-(z)$ *and* $W_P^+(z)$ *form foliations of the set* $\mathcal{M} \setminus \mathcal{S}$;

(3) *for every* $z \in Z$ *we have*

$$E_P^s(z) = E_P^-(z), \quad E_P^u(z) = E_P^+(z), \quad W_P^s(z) = W_P^-(z), \quad W_P^u(z) = W_P^+(z),$$

where the distributions $E_P^s(z)$, $E_P^u(z)$ *and the foliations* $W_P^s(z)$, $W_P^u(z)$ *are defined by (0.1) and (0.5), respectively; moreover, the sizes of the local leaves* $V_P^-(z)$ *and* $V_P^+(z)$ *are uniformly bounded away from zero on the set* $\mathcal{N}$;

(4) *the distributions and the foliations depend continuously on* P.

Proof. Consider the set

$$\mathcal{M}^+ = \{z \in \mathcal{M} \setminus \mathcal{S} : P^n(z) \to \mathcal{S} \text{ as } n \to +\infty\}.$$

Note that:

(a) for every $z \in \mathcal{M} \setminus \mathcal{M}^+$ there exists a sequence of numbers $n_k \to +\infty$ such that $P^{n_k}(z) \in \mathcal{N}$;

(b) for every $z \in \mathcal{M}^+$ there exists a number $n_0 = n_0(z)$ such that for every $n \ge n_0$ if we write $P_n(z) = (x_n, y_n)$ then $x_n = g^{n-n_0}x_{n_0}$.

It follows from (a) and (b) that $E_P^-(z)$ is a dP invariant distribution. We shall show that it is continuous. Fix $z \in \mathcal{M} \setminus \mathcal{S}$ and $\varepsilon > 0$. Let z_m be a sequence of points which converges to z. There exists $n > 0$ such that $dP^{-n}(K_R^-(P^n(z)))$ is contained in a cone

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around $E_P^-(z)$ of angle ε. By (a), (b), and the continuity of the cone family K_R^- one can find $M > 0$ such that for every $m \geq M$ the angle of the cone $dP^{-n}(K_R^-(P^n(z_m)))$ does not exceed 2ε. Since $E_P^-(z_m) \subset dP^{-n}(K_R^-(P^n(z_m)))$ we conclude that the Grassmanian distance between $E_P^-(z_m)$ and $E_P^-(z)$ does not exceed 3ε.

We shall show that the distribution $E_P^-(z)$ is integrable. Fix $z \in \mathcal{M} \setminus \mathcal{M}^+$. Consider a *u-admissible manifold* V^- at z, i.e. a local smooth submanifold passing through z and such that $T_w V^- \subset K_R^-(w)$ for every $w \in V^-$. We have for $z \in \mathcal{M}^+$,

$$W_P^-(z) = \bigcup_{n_i \geq 0} P^{-n_k}(V^-(P^{n_k}(z))) = W_P^s(z).$$

For $z \in \mathcal{M}^+$ the existence of the manifold $W_P^-(z)$ follows from property (a) and Proposition 2.2. The desired properties of the foliation W_P^- follow from continuity of the distribution $E_P^-(z)$, Lemma 4.2 (see equation (4.8)), and Proposition 2.2. Using similar arguments one can establish the desired properties of other distributions in (5.1) and the corresponding foliations. □

It is easy to see that the perturbation P given by (4.2) is gentle and hence Proposition 5.1 applies. Furthermore, due the special form of the perturbation we will obtain additional crucial information.

For every $z = (x, y) \in \mathcal{M} \setminus \mathcal{S}$ we define 'traces' of stable and unstable global leaves for the maps R and P on the fiber $(\mathcal{Y}^{n-2})_x$ by

$$\tilde{W}_R^s(y) = W_R^s(z) \cap (\mathcal{Y}^{n-2})_x, \quad \tilde{W}_P^-(y) = W_P^-(z) \cap (\mathcal{Y}^{n-2})_x,$$
$$\tilde{W}_R^u(y) = W_R^u(z) \cap (\mathcal{Y}^{n-2})_x, \quad \tilde{W}_P^+(y) = W_P^+(z) \cap (\mathcal{Y}^{n-2})_x.$$

PROPOSITION 5.2.

(1) *For every $z \in \mathcal{M} \setminus \mathcal{S}$ the collections of manifolds $\tilde{W}_R^s(y)$, $\tilde{W}_R^u(y)$, $\tilde{W}_P^-(y)$, $\tilde{W}_P^+(y)$ form four foliations of $(\mathcal{Y}^{n-2})_x$; for $x \in \mathcal{N}$, the sizes of local leaves $\tilde{V}_R^s(y)$, $\tilde{V}_R^u(y)$, $\tilde{V}_P^-(y)$, $\tilde{V}_P^+(y)$ are uniformly bounded away from zero.*

(2) *Given $\delta > 0$ there exists $\varepsilon > 0$ such that if $d_{C^1}(P, R) \leq \varepsilon$ then, for every $z = (x, y) \in \mathcal{N}$,*

$$\rho(\tilde{V}_R^s(y), \tilde{V}_P^-(y)) \leq \delta, \quad \rho(\tilde{V}_R^u(y), \tilde{V}_P^+(y)) \leq \delta.$$

Proof. The result follows from Propositions 3.1, 3.2, 5.1, and Lemma 4.2. □

We now establish the absolute continuity property. Choose a point $z_0 \in \mathcal{N}$ and consider the local manifolds $V_P^+(z)$, $z \in B(z_0, r) \cap Z$, for a sufficiently small number $r > 0$. Since the manifolds depend continuously on $z \in \mathcal{N} \cap Z$ there is a local submanifold W passing through z_0 and transversal to $V_P^+(z)$. Set

$$A = \bigcup_{z \in B(z_0, r) \cap Z} V_P^+(z). \tag{5.2}$$

Denote by ξ the partition of A by $V_P^+(z)$, $z \in B(z_0, r) \cap Z$. Note that the factor space A/ξ can be identified with $W \cap A$. Finally, we denote by m_z^+ and m_W respectively the Lebesgue measure on $V_P^+(z)$ and on W induced by the Riemannian metric. Since the set Z has full measure for almost every point $z_0 \in Z$, we have that $m_W(W \cap A) = 1$.

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PROPOSITION 5.3. *The foliation W_P^+ of the set $\mathcal{N} \cap Z$ is absolutely continuous: for almost every point $z \in \mathcal{N} \cap Z$,*

(1) *the conditional measure on the element $V_P^+(z)$ of this partition is absolutely continuous with respect to the measure m_z^+;*

(2) *the factor measure on the factor space A/ξ is absolutely continuous with respect to the measure m_W.*

A similar statement holds for the foliation W_P^- of $\mathcal{N} \cap Z$.

Proof. If the map P were (fully) non-uniformly hyperbolic the desired result would follow from [**BP**, Theorem 14.1] (see Lemma 14.4). It requires a simple and standard modification to generalize the arguments there to the partially non-uniform hyperbolic case. □

Our next statement establishes the essential accessibility property of the map P.

PROPOSITION 5.4. *If the perturbation P is sufficiently close to R then any two points $p, q \in Z \cap \mathcal{N}$ are accessible.*

Proof. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$. One can connect points p_1 and q_1 by a path $[x_0, \ldots, x_\ell]_g$ such that $x_0 = p_1$, $x_\ell = q_1$, and each point $x_i \in X$. Without loss of generality we may assume that $x_1 \in V_g^-(x_0)$. The local stable manifold $V_P^-(p)$ intersects the fiber $(\mathcal{Y}^{n-2})_{x_0}$ at a single point $y_1 \in Z$. Proceeding by induction we construct points $y_2, \ldots, y_\ell$, such that each point $z_i = (x_i, y_i) \in Z$, $i = 0, 1, \ldots, y_\ell$, and the path $[z_0, z_1, \ldots, z_\ell]_P$ connects the points p and z_ℓ. Note also that $y_\ell \in (\mathcal{Y}^{n-2})_{q_1}$. Fix a number $r > 0$ and consider the interval $[y^-, y^+]$ on the trajectory $T^t(q_2)$ centered at q_2 of radius r. Since the flow T^t has the accessibility property (see §3) for every $s \in [y^-, y^+]$ one can find a path $[y_\ell, s]_{T^t}$. Moreover, paths corresponding to different s are homotopic to each other. By Propositions 3.2 and 5.2 and statement (4) of Proposition 3.1, one can find a family of homotopic paths $[z_\ell, (q_1, s)]_P$ such that s runs over an interval on the trajectory $T^t(q_2)$. For sufficiently small ε, this interval contains a subinterval centered at q_2 of length $r - \delta > 0$. The desired result follows. □

We now show that the map P is topologically transitive; indeed, we prove a stronger statement.

PROPOSITION 5.5. *For almost every point $z \in \mathcal{N}$ the trajectory $\{P^n(z)\}$ is dense in $\mathcal{N}$ (i.e. $\overline{\{P^n(z)\}} \supset \mathcal{N}$).*

Proof. Consider a maximal set $E_0 \subset \mathcal{N}$ of points z for which:

z is topologically recurrent, i.e. for any $r > 0$ there exists $n \in \mathbb{Z}$ such that

$$P^n(z) \in B(z, r); \tag{5.3}$$

for any $w \in E_0$ the points z and w are accessible. (5.4)

LEMMA 5.6. $m(E_0) = 1$.

Proof of the lemma. Since the set of topologically recurrent points has full measure the desired result follows from Proposition 5.4. □

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LEMMA 5.7. *There exists a set E such that $m(E) = 1$, E satisfies (5.3), (5.4), and the following condition:*

for every $z \in E$ the sets $V_P^\alpha(z) \cap E$, $\alpha \in \{-, +\}$, have full measure with respect to the Riemannian volume on $V_P^\alpha(z)$. (5.5)

Proof of the lemma. Given a set $F \subset M$, set

$$F^* = \{z \in F : \text{the sets } F \cap V_P^\alpha(z), \alpha \in \{+, -\} \text{ have full measure}\}$$

(with respect to the Riemann volume on $V_P^\alpha(z)$). Starting with the set E_0, define inductively $E_n = E_{n-1}^*$. It follows from the absolute continuity of stable and unstable foliations (see Proposition 5.3) that $m(E_n) = 1$. Let $E = \bigcap_{n=0}^\infty E_n$. Then $m(E) = 1$ and (5.3) and (5.4) are satisfied since $E \subset E_0$. Also if $z \in E$ then $z \in E_n$ for each n. Therefore, the sets $V_P^\alpha(z) \cap E_n$, $\alpha \in \{+, -\}$ have full measure and hence, so do the sets $V_P^\alpha(z) \cap E$. □

Choose any two points $z, w \in E$ and let $[z_0, \dots, z_\ell]$ be a path connecting them.

LEMMA 5.8. *Given $\delta > 0$, there are points $z'_j \in E$, $j = 0, \dots, \ell$ such that $z'_0 = z$, and $d(z_j, z'_j) \le \delta$ for $j = 1, \dots, \ell$.*

Proof of the lemma. Without loss of generality we may assume that $z_1 \in V_P^+(z_0)$. If $z_1 \in E$ we set $z'_1 = z_1$. Otherwise, fix $0 < \delta_1 \le \delta$ and let $z'_1 \in E$ be a point such that $z'_1 \in V_P^+(z_0)$ and $d(z_1, z'_1) \le \delta_1$ (such a point exists for every δ_1 in view of (5.4)). If δ_1 is sufficiently small, for any $0 < \delta_2 \le \delta_1$ one can find a point $z'_2 \in E$ such that $z'_2 \in V_P^-(z'_1)$ and $d(z_2, z'_2) \le \delta_2$. Since the length of the path ℓ is uniformly bounded over z and w it remains to use induction to complete the proof. □

We proceed with the proof of the proposition. Choose $z, w \in E$ and let $z'_j \in E$, $j = 0, \dots, \ell$, be points constructed in Lemma 5.8. Fix $\delta > 0$ and the numbers $0 < \delta_1 < \cdots < \delta_\ell \le \delta$. There exists $m_1 > 0$ such that $d(P^n(z_0), P^n(z'_1)) \le \frac{1}{2}\delta_1$ for every $n \ge m_1$. By (5.2), there exists $n_1 \ge m_1$ for which $d(P^{n_1}(z_1), z_1) \le \frac{1}{2}\delta_1$. It follows that $d(P^{n_1}(z_0), z'_1) \le \delta_1$.

There exists $m_2 > 0$ such that, for every $n \ge m_2$, $d(P^{-n}(z'_1), P^{-n}(z'_2)) \le \frac{1}{3}\delta_2$. By (5.2), there is $n_2 \ge m_2$ for which $d(P^{-n_2}(z'_2), z'_2) \le \frac{1}{3}\delta_2$. It follows that $d(P^{-n_2}(z'_1), z'_2) \le \frac{2}{3}\delta_2$. Note that if δ_1 is chosen sufficiently small (depending only on n_2) and n_1 is chosen accordingly then $d(P^{n_1-n_2}(z_0), z'_2) \le \delta_2$. Proceeding by induction we find numbers n_i, $i = 1, \dots, \ell$, such that

$$d(P^{n_1-n_2+\cdots\pm n_\ell}(z_0), z'_\ell) \le \delta_\ell.$$

This implies that for almost every point $z \in \mathcal{N} \cap E$ the orbit $\{P^n(z)\}$ is everywhere dense. The desired result for almost every point $z \in \mathcal{M}$ follows from statement (2) of Proposition 4.1 and statement (5) of Proposition 3.1. □

6. *Proof of the Main Theorem: the case* $\dim \mathcal{K} \ge 5$

Consider the set $\mathcal{L}$ of points for which $\chi^c(z) < 0$ and, hence, all values of the Lyapunov exponent at z are non-zero. It is well known that ergodic components of $P|\mathcal{L}$ have positive

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measure. Let Q be such a component. In view of statement (5) of Proposition 3.1 the set $Q \cap \mathcal{N}$ has positive measure. Let z_0 be a Lebesgue point of the set $Q \cap \mathcal{N}$. Fix $r > 0$ and consider the set A defined by (5.2). Using Proposition 5.3 and applying the standard Hopf argument (see the proof of Theorem 13.1 in [**BP**]) one can show that $Q \supset A$ for sufficiently small r. This implies that Q is open $\pmod 0$ and so is the set $\mathcal{L}$. Applying Proposition 5.5 we conclude that $P|\mathcal{L}$ is ergodic. Note that the same arguments can be used to show that the map P^n is ergodic for all n. Hence, P is a Bernoulli diffeomorphism. It also follows from Proposition 5.5 that $m(\mathcal{L}) = 1$.

Set $f = (\chi_1 \circ \chi_2) \circ P \circ (\chi_1 \circ \chi_2)^{-1}$ where the maps χ_1 and χ_2 are constructed in Proposition 3.4. It follows that the map f satisfies all the desired properties.

Remark. Let us mention another approach for establishing ergodicity of P. Using the theory of invariant foliations one can show that if P is sufficiently close to R then $\tilde{W}^\alpha_P(z)$ are uniformly close to $\tilde{W}^\alpha_R(z)$ for all $z \in Z$, $\alpha = u, s$. Let $\Omega \subset \mathcal{N}$ be such that there exist sets Ω^α which consist of the whole leaves of $\tilde{W}^\alpha_P$ such that $m_{\mathcal{N}}(\Omega \triangle \Omega^\alpha) = 0$ (where $m_{\mathcal{N}}$ is the restriction of the Riemannian volume to $\mathcal{N}$). It follows from [**PS**] that $m_{\mathcal{N}}(\Omega) = 0$ or $m_{\mathcal{N}}(\Omega) = 1$. Hence, if Λ is a P-invariant set then $m(\Lambda \bigcap \mathcal{N}_z) = 0$ or $m(\Lambda \bigcap \mathcal{N}_z) = 1$ for almost all $z \in \mathcal{M}$. It follows that Λ factors down to a g-invariant set. This implies that P is ergodic. In this paper we choose to present another proof since it extends to the case $\dim \mathcal{K} = 3$ or 4 as we show below.

7. *Proof of the Main Theorem: the case* $\dim \mathcal{K} = 3$ *and* 4

Consider the manifold $\mathcal{M} = \mathcal{D}^2 \times \mathcal{T}^\ell$, where $\ell = 1$ if $\dim \mathcal{K} = 3$ and $\ell = 2$ if $\dim \mathcal{K} = 4$, and the skew product map R

$$R(z) = R(x, y) = (g(x), R_{\alpha(x)}(y)), \quad z = (x, y), \tag{7.1}$$

where the diffeomorphism g is constructed in Proposition 1.1, $R_{\alpha(x)}$ is the translation by $\alpha(x)$, and $\alpha : \mathcal{D}^2 \to \mathbb{R}$ is a non-negative C^∞ function which is equal to zero on the set $\mathcal{U}$ (defined in Proposition 2.2) and is strictly positive otherwise.

We define the *singularity* set for the map R by $\mathcal{S} = \mathcal{Q} \times \mathcal{T}^\ell$, where $\mathcal{Q}$ is the singularity set of the map g, and we also set $\mathcal{N} = (\mathcal{D}^2 \setminus \mathcal{U}) \times \mathcal{T}^\ell$ and $Z = X \times \mathcal{T}^\ell$ (see Proposition 2.2).

As before we have four cone families $K^+_R(z)$, $K^{+c}_R(z)$, $K^-_R(z)$, and $K^{-c}_R(z)$ which satisfy (3.2) and (3.3).

We say that the map R is *robustly accessible* if for all $p, q \in \mathcal{N}$ and any pair of foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ which are close to W^+_R and W^-_R respectively, there exists a path $[p, q] = [z_0 z_1 \dots z_\ell]$ such that $z_{j+1} \in \mathcal{F}^\alpha(z_j)$, $\alpha \in \{+, -\}$.

PROPOSITION 7.1. *The function $\alpha(x)$ (see (7.1)) can be chosen such that the map R is robustly accessible.*

Proof. By [**B1**] (see also [**BW**]), a generic skew product over multiplication by the map $\left|\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right|$ of $\mathcal{T}^2$ is robustly accessible. Now the statement follows from statement (1) of Lemma 1.2. □

Choose the function $\alpha(x)$ such that R is robustly accessible. Then any gentle perturbation of R has the accessibility property. Repeating the proof of Proposition 5.5 we obtain the following result.

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COROLLARY 7.2. *Any gentle perturbation P of R which is sufficiently close to R has no open invariant sets.*

We consider a gentle perturbation P of R in the form $P = \varphi \circ R$. We wish to choose φ such that

$$\int_{\mathcal{M}} \log \det(dP|E^c_P)(z)\, dm(z) = -\rho < 0. \tag{7.2}$$

Indeed, in the case $\mathcal{M} = \mathcal{D}^2 \times \mathcal{S}^1$, consider a coordinate system $\xi = \{\xi_1, \xi_2, \xi_3\}$ in a small neighborhood of a point z_0 such that:

(1) $dm = d\xi$;

(2) $E^c_R(z_0) = \partial/\partial\xi_1$, $E^s_R(z_0) = \partial/\partial\xi_2$, $E^u_R(z_0) = \partial/\partial\xi_3$.

Let $\psi(t)$ be a C^∞ function with compact support. Set $\tau = \|\xi\|^2/\gamma^2$ and define

$$\varphi^{-1}(\xi) = (\xi_1 \cos(\varepsilon\psi(\tau)) + \xi_2 \sin(\varepsilon\psi(\tau)), -\xi_1 \sin(\varepsilon\psi(\tau)) + \xi_2 \cos(\varepsilon\psi(\tau)), \xi_3).$$

The proof of (7.2) is similar to the proof of Lemma 4.4 (with γ chosen such that $\gamma \le \varepsilon^3$). In the case $\mathcal{M} = \mathcal{D}^2 \times \mathcal{T}^2$ write $\mathcal{M} = (\mathcal{D}^2 \times S^1) \times S^1$ and let $\varphi_1 = \varphi \times \mathrm{Id}$ where φ is the above map (note that the distributions E^s_R, E^u_R, and E^c_R are translation invariant).

In the case $\dim \mathcal{K} = 3$ the remaining part of the proof repeats the arguments in the case $\dim \mathcal{K} \ge 5$ (see Propositions 5.1, 5.3, 5.4 and 5.5 and §5). Note that the embeddings $\chi_1 : \mathcal{M} \to \mathcal{B}^3$ and $\chi_2 : \mathcal{B}^3 \to \mathcal{K}$ should be chosen according to [**BFK**].

We now proceed with the case $\dim \mathcal{K} = 4$. We further perturb the map P to $\bar{P}$ to obtain a set of positive measure on which $\bar{P}$ has three negative Lyapunov exponents.

PROPOSITION 7.3. *Suppose that the support of the map φ is sufficiently small. Then for all positive $\varepsilon_1, \varepsilon_2$ there exists a gentle perturbation $\bar{P}$ of P such that $d_{C^1}(P, \bar{P}) \le \varepsilon_1$ and*

$$\int_{\mathcal{M}} [\chi^c_1(z, \bar{P}) - \chi^c_2(z, \bar{P})]\, dm(z) \le \varepsilon_2,$$

where $\chi^c_1(z, \bar{P}) \ge \chi^c_2(z, \bar{P})$ are the Lyapunov exponents of $\bar{P}$ along the subspace $E^c_{\bar{P}}(z)$.

Proof. See §8. □

If ε_1 and ε_2 are sufficiently small then $\chi^c_1(z, \bar{P}) < 0$ and $\chi^c_2(z, \bar{P}) < 0$ on a set of positive measure. Indeed, by (7.2) there exist $\varepsilon_1 > 0$ and $C > 0$ such that for any gentle perturbation $\bar{P}$ of P with $d_{C^1}(P, \bar{P}) \le \varepsilon_1$ we have

$$\int_{\mathcal{M}} (\chi^c_1(z, \bar{P}) + \chi^c_2(z, \bar{P}))\, dm \le -\frac{\rho}{2}$$

and $|\chi^c_1(z, \bar{P}) \pm \chi^c_2(z, \bar{P})| \le C$. Hence, $\chi^c_1(z, \bar{P}) + \chi^c_2(z, \bar{P}) < -\rho/4$ on a set of measure at least $\rho/4C$ and $\chi^c_1(z, \bar{P}) - \chi^c_2(z, \bar{P}) > \rho/8$ on a set of measure at most $8\varepsilon_2/C$.

To complete the proof one now proceeds as in the case $\dim \mathcal{K} \ge 5$.

8. *Almost conformality*

We will prove Proposition 7.3. We follow the arguments in [**M1, Bo**] and split the proof into several steps. In what follows we adopt the following agreement: if at some step we use a statement of the type:

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'for any positive $\varepsilon_{\ell_1}, \dots, \varepsilon_{\ell_p}$ there exist positive $\varepsilon_{k_1}, \dots, \varepsilon_{k_q}$ such that ...'

then each time thereafter we assume that ε_{k_j} $(j = 1, \dots, q)$ are functions of ε_{ℓ_i} $(i = 1, \dots, p)$ satisfying the condition above.

Consider the set $D = \{z \in \mathcal{M} \setminus \mathcal{S} : \chi_1^c(z, P) \neq \chi_2^c(z, P)\}$. If $m(D) = 0$ the desired result follows (it suffices to choose $\bar{P} = P$). From now on we assume that $m(D) > 0$. Let $E_1^c(z)$ and $E_2^c(z)$ be the one-dimensional Lyapunov directions corresponding to $\chi_1^c(z, P)$ and $\chi_2^c(z, P)$. They are defined for almost every $z \in D$.

LEMMA 8.1. *For every $\varepsilon_3 > 0$ there is a measurable function $n_0 : \mathcal{M} \setminus \mathcal{S} \to \mathbb{N}$ such that for any $z \in \mathcal{M} \setminus \mathcal{S}$ and two one-dimensional subspaces $E', E'' \in E_P^c(z)$ one can find maps*

$$L_j(z, E', E'') : E_P^c(P^{j-1}(z)) \to E_P^c(P^j(z)), \quad 1 \leq j \leq n_0(z)$$

satisfying:

(1) $L_j(z, E', E'') = \mathbb{R}_{\beta_j(z,E',E'')}(dP|E_P^c(z))$ *where* $\mathbb{R}_\beta$ *denotes the rotation by angle* β *and* $\beta_j = \beta_j(z, E', E'')$ *is such that*

$$\|\beta_j\| \leq \varepsilon_3, \quad \beta_j = 0 \text{ on } \mathcal{U}; \tag{8.1}$$

(2) *if*

$$\hat{L}(z, E', E'') = L_{n_0(z)}(z, E', E'') \circ \cdots \circ L_1(z, E', E'')$$

then $\hat{L}(z, E', E'')E' = dP^{n_0(z)}E''$.

Proof. Let A be the set of points $z \in \mathcal{M} \setminus \mathcal{S}$ for which the statements of Lemma 8.1 hold. It is easy to see that A is invariant. Since the number $n_0(z)$ does not depend on the choice of subspaces E' and E'' by continuity of dP we find that the set A is open. In view of Corollary 7.2 if A is not empty it coincides with $\mathcal{M} \setminus \mathcal{S}$. We shall show that $A \neq \emptyset$.

Let $x \in \mathcal{D}^2 \setminus \mathcal{Q}$ be a periodic point of the map g of period r whose trajectory does not intersect supp(φ) (such a point always exists if supp(φ) is sufficiently small). We have that $P^r\mathcal{T}^2(x) = \mathcal{T}^2(x)$ where $\mathcal{T}^2(x)$ is a fiber over x. Moreover, $P^r|\mathcal{T}^2(x)$ is a translation. Therefore, the desired result holds for any $z \in \mathcal{T}^2(x)$. □

Given positive ε_3, ε_4, and N define

$$D_1(\varepsilon_3, \varepsilon_4, N) = \bigg\{z \in \mathcal{M} : n_0(z, \varepsilon_3) \leq N, \left|\frac{1}{n}\log\|dP^n|E_\ell^c(z, P)\| - \chi_\ell^c(z, P)\right| \leq \varepsilon_4,$$
$$\ell = 1, 2,\ \angle(E_1^c(P^n(z), P), E_2^c(P^n(z), P)) \geq e^{-\varepsilon_4|n|} \text{ for any } |n| \geq N\bigg\}.$$

LEMMA 8.2. *For any positive $\varepsilon_3, \varepsilon_4, \varepsilon_5$ one can find $N_1 > 0$ such that for any $N \geq N_1$,*

$$m(D \setminus D_1(\varepsilon_3, \varepsilon_4, N)) \leq \varepsilon_5.$$

Proof. The result follows from the Birkhoff ergodic theorem and Oseledec' theorem. □

Fix $z \in D_1(\varepsilon_3, \varepsilon_4, N)$. Since $\chi_1^c(z, P) \geq \chi_2^c(z, P)$ we obtain from the definition of the set $D_1(\varepsilon_3, \varepsilon_4, N)$ that for every point z in this set, $v \in E_2^c(z, P)$, $\|v\| = 1$, and $|n| \geq N$,

$$\left|\frac{1}{n}\log\|dP^n v\| - \chi_2^c(z, P)\right| \leq \varepsilon_4 \tag{8.2}$$

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and for $v \in E_1^c(z, P)$, $\|v\| = 1$ such that $\angle(v, E_2^c(z, P)) \geq e^{-\varepsilon_4}$, and $|n| \geq N$,

$$\left|\frac{1}{n} \log \|dP^n v\| - \chi_1^c(z, P)\right| \leq 2\varepsilon_4. \tag{8.3}$$

LEMMA 8.3. *For any positive $\varepsilon_3, \varepsilon_4, \varepsilon_6, \varepsilon_7$, and N_2 there exist positive N_3 and ε_5 such that:*

(1) *for any $\varepsilon_8 > 0$ and $N \geq N_3$ one can find a set $\Omega = \Omega(N)$ for which $P^j(\Omega) \bigcap \Omega = \emptyset$, $0 < j \leq N$ and if $\bar{\Omega} = \bigcup_{j=0}^N P^j(\Omega)$ then $m(D \setminus \bar{\Omega}) \leq \varepsilon_8$;*

(2) *if*

$$\begin{aligned} D_2(\varepsilon_3, \varepsilon_4, \varepsilon_6, N, M) = \{z \in \bar{\Omega} : z = P^{j_0}(y),\ &\textit{for some } y \in \Omega, 0 < j_0 \leq N \textit{ and} \\ &\mathrm{Card}\{j : |(N-j)/j - 1| \leq \varepsilon_6 \textit{ and} \\ &f^j(y) \in D_1(\varepsilon_3, \varepsilon_4, N)\} \leq M\} \end{aligned}$$

then

$$m(D_2(\varepsilon_3, \varepsilon_4, \varepsilon_6, N, N_2)) \leq \varepsilon_7.$$

Proof. The first statement is just the Rokhlin–Halmos lemma. Note that the measure of each set $R^j(\Omega)$ is of order $1/N$ and that the number

$$\mathrm{Card}\left\{j : \left|\frac{N-j}{j} - 1\right| \leq \varepsilon_6\right\}$$

is of order $\varepsilon_6 N$. The second statement follows. □

The set $\Omega(N)$ is called a *tower of height N*.

LEMMA 8.4. *For any positive $\varepsilon_3, \varepsilon_7, \varepsilon_9$ there exist positive $\varepsilon_4, \varepsilon_6$ such that the following statement holds. Fix $z \in D_1(\varepsilon_3, \varepsilon_4, N_1)$, with positive n_1, n_2 satisfying*

$$\left|\frac{n_2}{n_1} - 1\right| \leq \varepsilon_6, \quad n = n_1 + n_2 \geq N_3,$$

and maps $L_j(z) = L_j(z, E_1^c(z, P), E_2^c(z, P))$, $j = 1, \dots, k \leq \varepsilon_6 N_3$, satisfying (8.1) such that $\hat{L}(z) = L_k(z) \circ \cdots \circ L_1(z)$ moves $E_1^c(z, P)$ into $E_2^c(P^k(z), P)$. Then

$$\begin{aligned} \exp\left[n\left(\frac{\chi_1^c(z, P) + \chi_2^c(z, P)}{2} - \varepsilon_9\right)\right] &\leq \|(dP^{n-k} \circ \hat{L}(z) \circ dP^{n_1} | E_P^c(P^{-n_1}(z)))\| \\ &\leq \exp\left[n\left(\frac{\chi_1^c(z, P) + \chi_2^c(z, P)}{2} + \varepsilon_9\right)\right]. \end{aligned}$$

Proof. Set

$$\mathcal{P} = dP^{n-k} \circ \hat{L}(z) \circ dP^{n_1} | E_P^c(z).$$

Let $e_1 \in E_1^c(z, P)$ and $e_2 \in E_2^c(z, P)$ be a normalized basis in $E_P^c(z)$. Then by (8.2) and (8.3),

$$\frac{1}{n} \log \|\mathcal{P} e_\ell\| = \chi_\ell^c(z, P) n_1 + \chi_{3-\ell}^c(z, P) n_2 + \mathrm{O}(\varepsilon_4 n)$$

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for $\ell = 1, 2$. Let $\Pi(z) : E^c(z) \to E^c(z)$ be a linear map satisfying $\det \Pi(z) = 1$ with the vectors $\Pi(z)e_1$ and $\Pi(z)e_2$ orthogonal. Then

$$\begin{aligned}
\log &\left\| \exp\left(n \frac{\chi_1^c(z,P) + \chi_2^c(z,P)}{2} \right) \mathcal{P} \right\| \\
&= \log \|\Pi^{-1}(P^n(z))\| \\
&\quad + \log \left\| \Pi(P^n(z)) \circ \exp\left(n \frac{\chi_1^c(z,P) + \chi_2^c(z,P)}{2} \right) \mathcal{P} \circ \Pi^{-1}(P^{n_1}(z)) \right\| \\
&\quad + \log \|\Pi(P^{-n_1}(z))\|
\end{aligned}$$

and each term is of order $O((\varepsilon_6 + \varepsilon_4)n)$. The desired result follows. □

LEMMA 8.5. *For any positive ε_{10}, ε_{11}, ε_{12}, ε_{13} there exist positive ε_3, ε_7, ε_9, and N_2 such that the following holds. Let $\Omega_1 = \Omega \setminus D_2(\varepsilon_3, \varepsilon_4, \varepsilon_6, N_2, N_3)$ where $\Omega = \Omega(N_3)$ is a tower of height N_3 and*

$$\Omega_2 = \left\{ f^j(z) : z \in \Omega_1 \text{ and } j \text{ is the smallest number for which } \left| \frac{N_3 - j}{j} - 1 \right| \le \varepsilon_6 \text{ and } f^j(z) \in D_1(\varepsilon_3, \varepsilon_4, N_2) \right\}.$$

Also let $k = \varepsilon_6 N_3$. Then:

(1) *there exists an open set Ω_3 satisfying $m(\Omega_3 \triangle \Omega_2) \le \varepsilon_{10}$ and a map $\hat{P} = P \circ \hat{\varphi}$ such that*
$$\operatorname{supp}(\hat{\varphi}) = \left(\bigcup_{j=0}^{k-1} \tilde{P}^j(\Omega_3) \right) \setminus (\mathcal{U} \times \mathcal{T}^2);$$

(2) $d_{C^1}(\hat{\varphi}, Id) \le \varepsilon_1$;

(3) *there exists $\Omega_4 \subset \Omega_2$ such that $m(\Omega_2 \setminus \Omega_4) \le \varepsilon_{11}$ and for all $z \in \Omega_4$,*
$$\|(d\hat{P}^n | E_P^c)(z) - \hat{L}(z)\| \le \varepsilon_{12} \quad \text{for some } n \le k, \tag{8.4}$$
where $\hat{L}(z) : E_P^c(z) \to E_P^c(P^n(z))$ moves $E_1^c(z, P)$ to $E_2^c(P^n(z), z)$ (see Lemma 8.4);

(4) *for any $z \in \Omega$,*
$$d(E_P^c(z), E_{\hat{P}}^c(z)) \le \varepsilon_{13}.$$

Proof. The proof is similar to [**Bo**]. Consider a finite atlas $\Phi = \{\Phi_1 \cdots \Phi_n\}$ such that in each chart Φ_i one can introduce a coordinate system $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ satisfying

$$dm = d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Approximate Ω_2 by the finite union of balls $\bigcup_j B(z_j, r_j)$, with $r_j \le \rho$ where ρ is sufficiently small. By coordinate rotation we may assume that $E_P^c(z_j) = \langle \partial/\partial\xi_1, \partial/\partial\xi_2 \rangle|_{z_j}$. We can apply Lemma 8.4 to each $z \in \Omega_2$ and construct the maps $L_1(z), \ldots, L_{N_1}(z)$ such that $\hat{L}(z) = L_{N_1}(z) \circ \cdots \circ L_1(z)$ moves $E_1^c(z, P)$ to $E_2^c(P^n(z), P)$.

By slightly shrinking the set Ω_2 if necessary we may assume that the maps $L_i(z)$ are continuous on Ω_2. Recall that each map $L_\ell(w)$ is a twist of the form

$$L_\ell(w) = \mathbb{R}_{\beta_\ell(w)}(dP|E^c_P(w)).$$

We define $\bar{\varphi}$ on each $B(z_j, r_j)$ to be

$$\hat{\varphi}(\xi_1, \xi_2, \xi_3, \xi_4) = (\mathbb{R}_{\psi(\|\xi\|/r_j)\beta_1(z_j)}(\eta_1, \eta_2), \xi_3, \xi_4),$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\} = \exp^{-1}_{z_j}(\xi_1, \xi_2, \xi_3, \xi_4)$ and the function $\psi(x)$ is supported on $[0, 1]$ and

$$\psi(x) = 1, \quad x \in [0, \tfrac{1}{2}]. \tag{8.5}$$

Continuing by induction for each $\ell \leq N_1$ we approximate the sets $P^\ell(B(z_j, r_j))$ by balls and define $\hat{\varphi}$ on each ball to be an appropriate twist generated by the maps $L_\ell(z)$. This construction allows us to define $\hat{\varphi}$ in such a way that (8.4) holds for $n = N_1$ on a set Δ_1 for which $m(\Delta_1) > c(N_1)m(\Omega_2)$. Here $c(N_1)$ is a constant which can be made arbitrarily close to $\left(\frac{1}{16}\right)^{N_1}$ if the approximation by balls is chosen appropriately; we exploit here the fact that in view of (8.5)

$$\frac{m(B(z, r/2))}{m(B(z, r))} = \frac{1}{16}.$$

Consider a point $z \in \Omega_2 \setminus \Delta_1$. Let $\bar{N}_1(z) > N_1$ be the first moment when the trajectory $\{P^j(z)\}$ visits the set D_1. Define $\hat{\varphi}$ along the orbit $\{f^{j+N}(z)\}$ with $\bar{N}_1(z) \leq j \leq \bar{N}_1(z) + N_1$ to be appropriate twists such that the map $dP^{\bar{N}_1(z)-N_1} \circ d\bar{P}^{N_1}$ moves $E^c_1(z, P)$ to $dP^{\bar{N}_1(z)} \circ d\bar{P}^{N_1} E^c_2(z, P)$. Thus, we obtain a set Δ_2 for which $m(\Delta_2) > m(\Omega_2 \setminus \Delta_1) \geq c$ and $n = N_1 + \bar{N}_1(z)$ on Δ_2. Repeating this procedure (N_2/N_1) times we obtain the required map $\hat{\varphi}$. All properties of the map $\hat{P}$ can now be verified by the arguments similar to those in Lemma 4.4. □

It remains to show that $\varepsilon_{10}, \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}$ can be chosen such that

$$\left| \frac{1}{N_3} \log \|\hat{P}^{N_3}(z)|E^c_{\hat{P}}(z)\| \, dm(z) - \frac{1}{2} \int_{\mathcal{M}} \log \det(d\hat{P}(z)|E^c_{\hat{P}}(z)) \, dm(z) \right| \leq \varepsilon_2.$$

This again is similar to the proof of Lemma 4.4 and we leave the details to the reader.

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