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The essential coexistence phenomenon in Hamiltonian dynamics

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Dedicated to the memory of Anatole Katok

Abstract. We construct an example of a Hamiltonian flow f^t on a four-dimensional smooth manifold $\mathcal{M}$ which after being restricted to an energy surface $\mathcal{M}_e$ demonstrates essential coexistence of regular and chaotic dynamics, that is, there is an open and dense f^t-invariant subset $U \subset \mathcal{M}_e$ such that the restriction $f^t|U$ has non-zero Lyapunov exponents in all directions (except for the direction of the flow) and is a Bernoulli flow while, on the boundary ∂U, which has positive volume, all Lyapunov exponents of the system are zero.

Key words: essential coexistence, Hamiltonian flows, Lyapunov exponents, Bernoulli maps
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1. *Introduction*

The problem of essential coexistence of regular and chaotic behavior lies in the core of the theory of smooth dynamical systems. The early development of the theory of dynamical systems was focused on the study of regular behavior in dynamics such as presence and stability of periodic motions, translations on surfaces etc. The first examples of systems with highly complicated 'chaotic' behavior—so-called homoclinic tangles—were discovered by Poincaré in 1889 in conjunction with his work on the three-body problem. However, a rigorous study of chaotic behavior in smooth dynamical systems began in the second part of the last century due to the pioneering work of Anosov, Sinai, Smale and others, which has led to the development of hyperbolicity theory. It was therefore natural to ask whether the two types of dynamical behavior—regular and chaotic—can coexist in an essential way.

While regular and chaotic dynamics can coexist in various ways, in this paper we will consider one of the most interesting situations that can be described as follows. Let f^t be a volume-preserving smooth dynamical system with discrete or continuous time t acting on a compact smooth manifold M. We say that f^t exhibits *essential coexistence* if there is an open and dense f^t-invariant subset $U \subset M$ such that the restriction $f^t|U$ has non-zero Lyapunov exponents in all directions (except for the direction of the flow in the case when t is continuous) and is a Bernoulli system while, on the boundary ∂U, which has *positive volume*, all Lyapunov exponents of the system are zero. Note that it is the requirement that the boundary of U has positive volume that makes coexistence essential. It follows that the Kolmogorov–Sinai entropy of $f^t|U$ is positive while the topological entropy of $f^t|\partial U$ is zero; see the survey [**CHP13a**] for more information on essential coexistence, some results, examples and open problems. (We remark that in [**CHP13a**] essential coexistence described above was called type I as opposed to essential coexistence of type II in which U is allowed not to be dense (in particular, its complement U^c can contain an open set). We will not consider the type II phenomenon in this paper.)

The study of essential coexistence was inspired by discovery of the Kolmogorov–Arnold–Moser (KAM) phenomenon in Hamiltonian dynamics and a similar phenomenon in the space of volume-preserving systems. The latter was shown in the work of Cheng and Sun [**CS90**], Herman [**Her92**], Xia [**Xia92**] and Yoccoz [**Yoc92**], who proved that on any manifold M and for any sufficiently large r there are open sets of volume-preserving C^r diffeomorphisms of M all of which possess positive-volume sets of codimension-1 invariant tori; on each such torus the diffeomorphism is C^1 conjugate to a Diophantine translation; all of the Lyapunov exponents are zero on the invariant tori. The set of invariant tori is nowhere dense and it is expected that it is surrounded by a 'chaotic sea', that is, outside this set the Lyapunov exponents are non-zero almost everywhere and, hence, the system has at most countably many ergodic components. It has since been an open problem to find out to what extent this picture is true.

First examples of systems with discrete and continuous time demonstrating essential coexistence, which are volume preserving, were constructed in [**HPT13**, **CHP13b**, **Che12**] (see also [**CHP13a**] for a survey of recent results). Naturally, one would like to construct examples of systems with essential coexistence which are Hamiltonian. This is what we do in this paper: we present an example of a Hamiltonian flow on a four-dimensional manifold which demonstrates essential coexistence; see Theorem A. In the course of our construction we also obtain an area-preserving C^∞ diffeomorphism of a two-dimensional torus as well as a volume-preserving C^∞ flow on a three-dimensional manifold both demonstrating essential coexistence; see Theorems C and B, respectively. These examples are simpler than the ones in [HPT13, CHP13b], where the corresponding constructions require five-dimensional manifolds.

In §2, we give a brief introduction to Hamiltonian dynamics recalling some basic facts and we state our main results, Theorems A, B and C, that lead to the desired example of a Hamiltonian flow with essential coexistence. In the following three sections we present the proofs of these results.

In particular, in §3, we construct a specific area-preserving embedding of the closed unit disk in $\mathbb{R}^2$ onto a two-dimensional torus, which maps the interior of the disk onto an open and dense subset of the torus whose complement has positive area.

This technically involved geometric construction lies at the heart of our proof of Theorem C, which is presented, along with the proof of Theorem B, in §4. Another important ingredient of the proofs of these two theorems is the map of the two-dimensional unit disk known as the Katok map. This is a C^∞ area-preserving diffeomorphism, which is the identity on the boundary of the disk and is arbitrarily flat near the boundary. It is ergodic and, in fact, is Bernoulli and has non-zero Lyapunov exponents almost everywhere. It was introduced by Katok in [Kat79] as the first basic step in his construction of C^∞ area-preserving Bernoulli diffeomorphisms with non-zero Lyapunov exponents on any surface. We will also use in a crucial way a result from [HPT04] showing that the Katok map is smoothly isotopic to the identity map of the disk. This isotopy allows us to construct a flow on the three-dimensional torus with essential coexistence. Finally, in §5 we give the proof of Theorem A, which provides the desired Hamiltonian flow with essential coexistence.

We note that while the Hamiltonian flow we construct in this paper displays essential coexistence, it does not quite exhibit the KAM phenomenon, since the Cantor set of invariant tori are just unions of circles, on all of which the flow has the same linear speed. One can modify our construction to obtain a Hamiltonian flow on a six-dimensional manifold whose restriction on any energy level exhibits the essential coexistence phenomenon; moreover, the set of positive measure where all Lyapunov exponents are zero consists of invariant tori and the flow on each such torus has a Diophantine velocity vector.

2. *Statement of results*

A standard reference to Hamiltonian systems is [AKN06]. A *symplectic manifold* is a smooth manifold $\mathcal{M}$ equipped with a *symplectic form* that is a closed non-degenerate 2-form ω. Necessarily, $\mathcal{M}$ must be even dimensional (say $2n$) and ω^n defines a volume form on $\mathcal{M}$. In the particular case $\mathcal{M} = T^*N$, the cotangent bundle of a smooth manifold N, there is a natural symplectic form $\omega = dp \wedge dq := \sum_{i=1}^n dp_i \wedge dq_i$, where (q, p) are the local coordinates in T^*N induced by the local coordinate q in N. In particular, the associated volume form is the standard one.

Let H be a C^2 function on a symplectic manifold $\mathcal{M}$ called a *Hamiltonian function*. The (autonomous) *Hamiltonian vector field* X_H is the unique vector field such that $\omega(X_H, \cdot) = dH(\cdot)$. For the standard symplectic form $\omega = dp \wedge dq$ one has $X_H = (\partial_p H(q, p), -\partial_q H(q, p))$. Let f_H^t denote the *Hamiltonian flow* generated by X_H. One can show that f_H^t preserves the symplectic form, the volume form ω^n and the Hamiltonian function H. As a result, each regular (non-degenerate) level surface $\mathcal{M}_e = \{H = e\}$, called an *energy surface*, is invariant under the flow. One can show that this flow preserves a smooth

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measure supported on the energy surface. As such, ergodic properties of an (autonomous) Hamiltonian flow are often discussed by restricting the flow to an energy surface.

A flow f^t on $\mathcal{M}$ preserving a measure μ is *ergodic* if, for every measurable set A which is invariant under the flow (that is, $f^{-t}(A) = A$ for every $t \in \mathbb{R}$), one has either $\mu(A) = 0$ or $\mu(A) = 1$. If the flow is ergodic, then the time-t map is ergodic for all but countable many $t \in \mathbb{R}$ and, if the time-t map of the flow is ergodic for some t, then the flow is ergodic; see [**FH20**]. The flow f^t is *Bernoulli* if, for any $t \neq 0$, the time-t map is Bernoulli (that is, it is isomorphic to a Bernoulli scheme endowed with a Bernoulli measure); see [**FH20**]. A flow f^t is *hyperbolic* if the flow has non-zero Lyapunov exponents almost everywhere, except for the flow direction. We say that an orbit of a flow has zero Lyapunov exponents if the flow has zero Lyapunov exponents at some (and, hence, any) point of the orbit. Denote by m the Lebesgue measure on $\mathcal{M}$.

THEOREM A. *There exist a four-dimensional manifold $\mathcal{M}$ and a non-degenerate Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$ such that restricted to any energy surface $\mathcal{M}_e := \{H = e\}$, the Hamiltonian flow $f_H^t : \mathcal{M}_e \to \mathcal{M}_e$ demonstrates essential coexistence, that is:*

(a) *there is an open and dense set $U = U_e \in \mathcal{M}_e$ such that $f_H^t(U) = U$ for any $t \in \mathbb{R}$ and $m(U^c) > 0$, where $U^c = \mathcal{M}_e \setminus U$ is the complement of U;*
(b) *restricted to U, $f_H^t|U$ is hyperbolic and ergodic; in fact, $f_H^t|U$ is Bernoulli;*
(c) *restricted to U^c, all orbits of $f_H^t|U^c$ are periodic with zero Lyapunov exponents.*

Remark 2.1. Actually our construction guarantees the following property of the flow $f_H^t|U^c$. For any $x \in U^c$, consider a small surface Σ through x which is transversal to the flow direction and let $\tilde{f}_H$ be the corresponding Poincaré map. Then we have $\tilde{f}_H|\Sigma = \mathrm{id}$ and $D^k \tilde{f}_H|T_x\Sigma = 0$ for any $k > 0$, where $T_x\Sigma$ is the tangent space to Σ at x.

We obtain a flow in Theorem A by constructing a C^∞ volume-preserving flow on $\mathbb{T}^3$ with essential coexistence.

THEOREM B. *There exists a volume-preserving C^∞ flow f^t on $\mathcal{M} = \mathbb{T}^3$ that demonstrates essential coexistence, that is, it has Properties (a)–(c) in Theorem A.*

In [**CHP13b**], the first three authors of this paper constructed a volume-preserving C^∞ flow f^t on a five-dimensional manifold with essential coexistence. Moreover, in that paper, U^c is a union of three-dimensional invariant submanifolds and f^t is a linear flow with a Diophantine frequency vector on each invariant submanifold. In the example given by Theorem A, U^c is a union of one-dimensional closed orbits since the center direction is only one dimensional.

The proof that Theorem B implies Theorem A is given in §5.

To obtain the flow in Theorem B, we construct a C^∞ area-preserving diffeomorphism on $\mathbb{T}^2$ that demonstrates essential coexistence.

THEOREM C. *There exists a C^∞ area-preserving diffeomorphism f on $\mathbb{T}^2$ such that:*

(a) *there is an open and dense set U such that $f(U) = U$ and $m(U^c) > 0$, where $U^c = \mathbb{T}^2 \setminus U$ is the complement of U;*

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(b) *restricted to* U, $f|U$ *is hyperbolic and ergodic; in fact,* $f|U$ *is Bernoulli;*
(c) *restricted to* U^c, $f|U^c = \mathrm{id}$ *(and, hence, its Lyapunov exponents are all zero).*

In [HPT13], the authors constructed a C^∞ volume-preserving diffeomorphism h of a five-dimensional manifold that also has Properties (a)–(c), where U^c is a union of three-dimensional invariant submanifolds and, restricted to U^c, h is the identity map (hence, with zero Lyapunov exponents).

Also, in [Che12], the first author constructed a C^∞ volume-preserving diffeomorphism h of a four-dimensional manifold with the same properties but the chaotic part has countably many ergodic components. There are other examples of dynamical systems that exhibit coexistence of chaotic and regular behavior though the regular part may not form a nowhere-dense set; see [CHP13a] for references and also [BT19]. (In this paper the authors showed that any area-preserving C^r diffeomorphism of a two-dimensional surface with an elliptic fixed point can be C^r-perturbed to the one exhibiting an elliptic island whose metric entropy is positive for every $1 \le r \le \infty$. Note that in this example the set with non-zero Lyapunov exponents is guaranteed to have positive but not necessarily full area and is not everywhere dense.)

3. *Embedding the unit 2-disk into the 2-torus*

In this section we state our main technical result, which provides a C^∞ embedding from the open unit 2-disk $\mathbb{D}^2$ into the 2-torus $\mathbb{T}^2$ such that the image is open and dense but not of the full Lebesgue measure. We equip $\mathbb{D}^2$ and $\mathbb{T}^2$ with the standard Euclidean metric induced from $\mathbb{R}^2$ and we denote by $d = d(x, y)$ the standard distance.

Let M and N be manifolds and $h : M \to N$ a C^∞ diffeomorphism. The map h induces a map $h^* : \bigwedge^2(N) \to \bigwedge^2(M)$, where $\bigwedge^2$ denotes the set of 2-forms. Since 2-forms are volume (area) forms, we can regard that h^* sends smooth measures on N to those on M. By slightly abusing notation, we identity a 2-form with the smooth measure given by this form and with the density function of the smooth measure.

Let $m_{\mathbb{D}^2}$ denote the normalized Lebesgue measure on $\mathbb{D}^2$.

PROPOSITION 3.1. *There exists a* C^∞ *diffeomorphism* h *from* $\mathbb{D}^2$ *into* $\mathbb{T}^2$ *with the following properties:*

(1) *the image* $U = h(\mathbb{D}^2)$ *is an open and dense simply connected subset of* $\mathbb{T}^2$*; moreover,* $\partial U = \mathbb{T}^2 \setminus U = E \cup L$*, where* E *is a Cantor set of positive Lebesgue measure and* L *is a union of countably many line segments;*
(2) $h_* m_{\mathbb{D}^2} = m_U$*, where* m_U *is the normalized Lebesgue measure on* U*;*
(3) h *can be continuously extended to* $\partial\mathbb{D}^2$ *such that* $h(\partial\mathbb{D}^2) = \partial U$ *and, therefore, for any* $\varepsilon > 0$, $\mathcal{N}_\varepsilon = h^{-1}(V_\varepsilon)$ *is a neighborhood of* $\partial\mathbb{D}^2$*, where* $V_\varepsilon := \{x \in U : d(x, \partial U) < \varepsilon\}$.

In the proof of the proposition, we make an explicit construction of the map h. Note that h can be extended continuously to the boundary $\partial\mathbb{D}^2$ and, since $\dim_H(\partial\mathbb{D}^2) = 1 < 2 = \dim_H(\partial U)$ (here $\dim_H$ denotes the Hausdorff dimension), it cannot be Lipschitz on $\partial\mathbb{D}^2$.

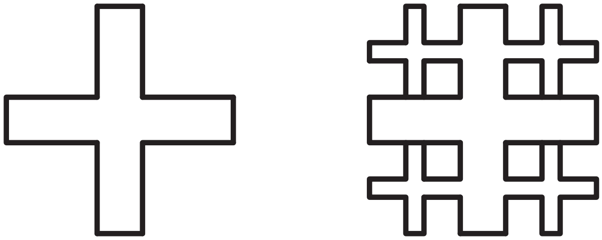

FIGURE 1. The first two steps of the construction: the sets U_1 and U_2.

We remark that one can use the Riemann mapping theorem to directly obtain a conformal C^∞ diffeomorphism h satisfying Statement 1 of Proposition 3.1. However, due to the Carathéodory's theorem, a conformal map can be extended to a homeomorphism of the closure of U if and only if ∂U is a Jordan curve, which is not the case here. Therefore, such a map does not satisfy Statement 3 of Proposition 3.1, which is crucial for our construction.

By a *cross* of size (α, β), where $\alpha < \beta$, we mean the image under a translation of the set

$$\left(-\frac{\beta}{2}, \frac{\beta}{2}\right) \times \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \bigcup \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \times \left(-\frac{\beta}{2}, \frac{\beta}{2}\right).$$

Roughly speaking, an (α, β)-cross is a union of two open rectangles with common center, one of size $\beta \times \alpha$ spaced vertically and the other of size $\alpha \times \beta$ spaced horizontally. The *left*, respectively, *right edge* of an (α, β)-cross is the image under a translation of the interval

$$I_l = I_{l,\alpha,\beta} = \left\{-\frac{\beta}{2}\right\} \times \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \quad \text{or} \quad I_r = I_{r,\alpha,\beta} = \left\{\frac{\beta}{2}\right\} \times \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right),$$

respectively. The *top* or *bottom edge* of a cross is understood similarly.

We say that an (α, β)-cross is inscribed in a square of size β if the cross is contained in the square.

Proof of Proposition 3.1. We split the proof into several steps.

Step 1. We give an explicit construction of the sets U, E and L. Consider the 2-torus $\mathbb{T}^2$, which we regard as the square $[0, 1]^2$ with opposite sides identified. Fix a number $\alpha \in (0, 0.05)$. We shall inductively construct a sequence of triples of disjoint sets (U_n, E_n, L_n) satisfying:

- U_n is a simply connected open subset in $\mathbb{T}^2$ and $U_n \supset U_{n-1}$;
- E_n is a disjoint union of 4^n identical closed squares and $E_n \subset E_{n-1}$;
- $L_n = \mathbb{T}^2 \backslash (U_n \cup E_n)$ consists of finitely many line segments and $L_n \supset L_{n-1}$.

Let U_1 be an $(\alpha, 1)$-cross inscribed in $\mathbb{T}^2 = [0, 1]^2$, E_1 the union of four closed squares of size $\beta_1 = (1 - \alpha)/2$ in the complement of U_1 and L_1 consist of four line segments, which are the left/right and top/bottom edges of the cross U_1. The four squares in E_1 are pairwise disjoint except on the boundary of $[0, 1]^2$; see Figure 1.

Suppose that U_i, E_i and L_i are all defined for $i = 1, \ldots, n$. By induction, E_n is a union of 4^n identical closed squares $\{E_{n,k}\}_{1 \le k \le 4^n}$ of size $\beta_n \times \beta_n$ that are pairwise disjoint except

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on the boundary of $[0, 1]^2$, where

$$\beta_n = 2^{-n}\left(1 - \sum_{k=1}^{n} 2^{k-1}\alpha^k\right). \tag{3.1}$$

Since $\alpha \in (0, 0.05)$, we have that

$$\beta_n > 2^{-n}\frac{1-3\alpha}{1-2\alpha} > 2^{-n-1}. \tag{3.2}$$

Let $U_{n+1,k}$ be the open (α^{n+1}, β_n)-cross inscribed in $\mathrm{int}(E_{n,k})$. Note that each $U_{n+1,k}$ touches a unique cross $U_{n,\ell}$ inside $U_n\backslash U_{n-1}$ either from the left or the right. Respectively, let $U'_{n+1,k}$ be $U_{n+1,k} \cup I_{r,\alpha^{n+1},\beta_n}$ or $U_{n+1,k} \cup I_{l,\alpha^{n+1},\beta_n}$, so that $U_n \cup U'_{n+1,k}$ is simply connected. Then we define

$$U_{n+1} = U_n \bigcup\left(\bigcup_{k=1}^{4^n} U'_{n+1,k}\right), \quad E_{n+1} = \overline{\mathrm{int}(E_n)\backslash U_{n+1}},$$
$$L_{n+1} = \mathbb{T}^2\backslash(U_{n+1} \cup E_{n+1}).$$

By construction:

- U_{n+1} is a simply connected open subset in $\mathbb{T}^2$ and $U_{n+1} \supset U_n$;
- E_{n+1} is a union of 4^{n+1} closed squares of size $\beta_{n+1} \times \beta_{n+1}$, which are disjoint except on the boundary of $[0, 1]^2$, and $E_{n+1} \subset E_n$;
- L_{n+1} consists of finitely many line segments, and $L_{n+1} \supset L_n$.

Now we define the open set U, the Cantor set E and the set L by

$$U = \bigcup_{n\geq 0} U_n, \quad E = \bigcap_{n\geq 0} E_n, \quad L = \bigcup_{n\geq 1} L_n.$$

It is clear that U is simply connected and $L = \mathbb{T}^2\backslash(U \cup E)$ consists of countably many line segments. Moreover,

$$\mathrm{Leb}_{\mathbb{T}^2}(E) = \lim_{n\to\infty} \mathrm{Leb}_{\mathbb{T}^2}(E_n) = \lim_{n\to\infty} 4^n\beta_n^2 = \left(\frac{1-3\alpha}{1-2\alpha}\right)^2 > 0.$$

That is, the Cantor set E has positive Lebesgue measure.

Step 2. We now construct a C^∞ diffeomorphism $\varphi : \mathbb{D}^2 \to U$, which may not be area preserving.

By the Riemann mapping theorem or, more precisely, using the Schwarz–Christoffel mapping from the unit disk to polygons, there is a C^ω diffeomorphism $\widehat{\varphi}_0 : \mathbb{D}^2 \to U_1$ which can be continuously extended to $\partial\mathbb{D}^2$.

For any $n \geq 1$, choose $\widehat{\varphi}_n : U_n \to U_{n+1}$ as in Lemma 3.3 below and define a sequence of C^∞ diffeomorphisms $\varphi_n : \mathbb{D}^2 \to U_n$ by

$$\varphi_n = \widehat{\varphi}_{n-1} \circ \cdots \circ \widehat{\varphi}_1 \circ \widehat{\varphi}_0. \tag{3.3}$$

For any $x \in \mathbb{D}^2$, we then let $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$. By Lemma 3.3(3), for any $x \in \overline{\mathbb{D}}^2$ and $n > j > 0$,

$$d(\varphi_j(x), \varphi_n(x)) \le \sum_{i=j}^{n-1} d(\varphi_i(x), \varphi_{i+1}(x)) \le \sum_{i=j}^{n-1} d(\varphi_i(x), \widehat{\varphi}_i(\varphi_i(x))) \le \sum_{i=j}^{n-1} 2\beta_i.$$

By (3.1), this implies that the sequence φ_n is uniformly Cauchy and, hence, φ is well defined and continuous on $\overline{\mathbb{D}}^2$.

To show that φ is a C^∞ diffeomorphism, it suffices to show that φ is a C^∞ local diffeomorphism, as well as that φ is a one-to-one map (see [**GA74**, §1.3]). The one-to-one property follows immediately from our construction. From Lemma 3.4 below, for any $x \in \mathbb{D}^2$ there exists $n = n(x) \ge 1$ such that $\varphi = \varphi_n$ in a neighborhood of x, which implies that φ is a C^∞ local diffeomorphism.

Step 3. Now we construct a C^∞ local diffeomorphism ψ of $\mathbb{D}^2$ such that $h := \varphi \circ \psi$ is area preserving (that is, $h_* m_{\mathbb{D}^2} = m_U$) and can be continuously extended to the closure $\overline{\mathbb{D}}^2$.

Denote $\mu = (\varphi^{-1})_* m_U$. Since both $m_{\mathbb{D}^2}$ and m_U are normalized Lebesgue measures, we have

$$\int_{\mathbb{D}^2} d\, m_{\mathbb{D}^2} = 1 = \int_U d\, m_U = \int_{\mathbb{D}^2} d\mu.$$

We show that there is a C^∞ diffeomorphism $\psi : \mathbb{D}^2 \to \mathbb{D}^2$ that can be continuously extended to $\partial\mathbb{D}^2$ such that $\psi_*\mu = m_{\mathbb{D}^2}$.

Set $\mu_1 = m_{\overline{\mathbb{D}}^2}$ and, for $n > 1$, define a sequence of measures μ_n such that:

(i) $\mu_n \in C^\infty(\overline{\mathbb{D}}^2)$, that is, the measure μ_n is absolutely continuous with respect to $m_{\overline{\mathbb{D}}^2}$ with density function of class C^∞;

(ii) $\mu_n = \mu$ on $\varphi^{-1}(U_{n-1})$;

(iii) $\int_{\varphi^{-1}(U'_{n,k})} d\mu_n = \int_{\varphi^{-1}(U'_{n,k})} d\mu$ for each $k = 1, \ldots, 4^n$.

It is clear that for any $n \ge 1$, $\int_{\overline{\mathbb{D}}^2} d\mu_n = \int_{\overline{\mathbb{D}}^2} d\mu = 1$.

We need the following version of Moser's theorem; see [**GS79**, Lemma 1].

LEMMA 3.2. *Let ω and μ be two volume forms on an oriented manifold M and let K be a connected compact set such that the support of $\omega - \mu$ is contained in the interior of K and $\int_K d\omega = \int_K d\mu$. Then there is a C^∞ diffeomorphism $\widehat{\psi} : M \to M$ such that $\widehat{\psi}|(M \setminus K) = \mathrm{id}_{(M\setminus K)}$ and $\widehat{\psi}_*\omega = \mu$.*

Note that for a fixed n, the sets $\varphi^{-1}(U'_{n,k})$ are pairwise disjoint for $k = 1, \ldots, 4^n$. Therefore, applying Lemma 3.2 to each of the 4^n connected compact sets $K_{n,k} = \varphi^{-1}(\overline{U}'_{n,k})$ and volume forms $\mu_{n+1}|\overline{U}'_{n,k}$ and $\mu_n|\overline{U}'_{n,k}$, we obtain a C^∞ diffeomorphism $\widehat{\psi}_n : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ such that $(\widehat{\psi}_n)_*\mu_{n+1} = \mu_n$ and $\widehat{\psi}_n|\varphi^{-1}(U_{n-1}) = \mathrm{id}$. Then we let

$$\psi_n = \widehat{\psi}_n \circ \cdots \circ \widehat{\psi}_1 \quad \text{and} \quad \psi = \lim_{n\to\infty} \psi_n.$$

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The construction gives $\widehat{\psi}_n(\varphi^{-1}(U'_{n,k})) = \varphi^{-1}(U'_{n,k})$. By Lemma 3.3(3), the Lipschitz constant of $\widehat{\varphi}_n^{-1}$ is less than 1 if n is large enough. Since $\operatorname{diam} U'_{n,k} \le 2\beta_n$, we obtain that $\operatorname{diam}\widehat{\varphi}_n^{-1}(U'_{n,k}) \le 2\beta_n$. This implies that $d(x, \widehat{\psi}_n(x)) \le 2\beta_n$ for any $x \in \mathbb{D}^2$, which allows us to use the same arguments as above for the sequence φ_n to show that the sequence ψ_n is uniformly Cauchy and, hence, $\psi : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ is well defined and continuous. Applying the same argument as for φ, we can also get that $\psi : \mathbb{D}^2 \to \mathbb{D}^2$ is a C^∞ diffeomorphism.

By construction, we know that $(\psi_n)_*\mu_{n+1} = \mu_1 = m_{\mathbb{D}^2}$. Note that by Lemma 3.4, $\mathbb{D}^2 = \cup_{n\ge 1}\varphi^{-1}(U_n)$. Hence, for any $x \in \mathbb{D}^2$ there are $n > 0$ and a neighborhood of x on which $\mu_{n+i} = \mu_n$ for any $i > 0$. It follows that $\psi_*\mu = (\psi_n)_*\mu_n = m_{\mathbb{D}^2}$ on the neighborhood and, hence, $\psi_*\mu = m_{\mathbb{D}^2}$ on $\mathbb{D}^2$.

Step 4. Set $h = \varphi \circ \psi$. Clearly, $h : \mathbb{D}^2 \to U$ is a C^∞ diffeomorphism and can be continuously extended to the boundary $\partial\mathbb{D}^2$. Also $h_* m_U = \psi_*(\varphi_* m_U) = \psi_*\mu = m_{\mathbb{D}^2}$. Since $h : \overline{\mathbb{D}}^2 \to \overline{U}$ is continuous, the pre-image of any open set is open and, hence, $\mathcal{N}_\varepsilon$ is a neighborhood of $\partial\mathbb{D}^2$. All the requirements of the proposition are satisfied. □

To complete the proof of Proposition 3.1, it remained to prove the two technical lemmas that were used in the above construction.

LEMMA 3.3. *There is a sequence of C^∞ diffeomorphisms $\widehat{\varphi}_n : U_n \to U_{n+1}$, $n \ge 1$, such that the following properties hold:*

(1) $\widehat{\varphi}_n | U_{n-1} = \mathrm{id}$;
(2) $\widehat{\varphi}_n$ *can be continuously extended to* ∂U_n;
(3) $d(x, \widehat{\varphi}_n(x)) \le 2\beta_n$ *for any* $x \in \overline{U}_n$;
(4) $\widehat{\varphi}_n^{-1}$ *is Lipschitz on the set* $U_{n+1} \cap \{\widehat{\varphi}_{n+1} \ne \mathrm{id}\}$ *with Lipschitz constant less than* $c\gamma_n^{-1}$, *where* $\gamma_n = \beta_{n+1}/\alpha^{n+1}$ *and* $c > 0$ *is a constant independent of* n.

Proof. By construction of $\{U_n\}$, the complement of the set $U_{n+1} \backslash U_n$ is a disjoint union of 4^n (α^{n+1}, β_n)-crosses of the form

$$U'_{n+1,k} = U_{n+1,k} \cup I_l \text{ or } U'_{n+1,k} = U_{n+1,k} \cup I_r.$$

By attaching an open square $W_{n,k}$ to the left or right edge of $\overline{U}'_{n+1,k}$, respectively, we obtain an augmented cross, denoted by $U^\sharp_{n+1,k}$, which is similar to the cross

$$\mathcal{C}_\gamma := ([0, 2+2\gamma] \times [0, 1]) \bigcup ([1+\gamma, 2+\gamma] \times [-\gamma, 1+\gamma]), \tag{3.4}$$

where $\gamma = \gamma_n = \beta_{n+1}\alpha^{-(n+1)}$. The similarity map $\eta_{n+1,k} : U^\sharp_{n+1,k} \to \mathcal{C}_\gamma$ is a composition of a translation, an enlargement given by $x \to \alpha^{-(n+1)}x$ and possibly a reflection. Note that

$$\gamma_n = \frac{\beta_{n+1}}{\alpha^{n+1}} > \frac{2^{-n-2}}{\alpha^{n+1}} > \frac{1}{(2\alpha)^n} > 10^n. \tag{3.5}$$

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Let $\sigma_{\gamma_n} : [0, 1]^2 \to \mathcal{C}_{\gamma_n}$ be the map constructed in Sublemma 3.5 below. Define a map $\widehat{\varphi}_n : U_n \to U_{n+1}$ by

$$\widehat{\varphi}_n(x) = \begin{cases} \eta_{n+1,k}^{-1} \circ \sigma_{\gamma_n} \circ \eta_{n+1,k} & \text{for } x \in W_{n,k},\ k = 1, \ldots, 2^n, \\ x & \text{elsewhere.} \end{cases} \tag{3.6}$$

Note that $\widehat{\varphi}_n : U_n \to U_{n+1}$ is a C^∞ diffeomorphism in $\mathrm{Int}(U_n)$ that can be extended to ∂U_n continuously. To see this, observe that the boundary $\partial W_{n,k}$ consists of four edges, one is on ∂U_n and the other three are in $\mathrm{Int}(U_n)$. We denote the union of the three edges in $\mathrm{Int}(U_n)$ by Γ. By Sublemma 3.5 (see Statement 2) and the construction of $\hat{\varphi}_n$ (see equation (3.6)), $\hat{\varphi}_n$ is the identity in a neighborhood of Γ, so it is C^∞ smooth.

Since $\mathrm{diam} U_{n+1,k}^\sharp \le 2\beta$ and $\widehat{\varphi}_n$ is a diffeomorphism from $W_{n,k} \subset U_{n+1,k}^\sharp$ to $U_{n+1,k}^\sharp$, we must have $d(x, \widehat{\varphi}_n(x)) \le 2\beta$. So, $\widehat{\varphi}_n$ satisfies Requirements (1)–(3) of the lemma.

Note that the Lipschitz constant is preserved by a conjugacy if the latter is given by a composition of isometries, enlargements and possibly reflections. Also note that for each k, the set $U_{n+1,k}^\sharp \cap \{\widehat{\varphi}_{n+1} \neq \mathrm{id}\}$ is contained in $\eta_{n+1,k}^{-1}(\mathcal{C}_\gamma^\pm)$, where $\mathcal{C}_\gamma^\pm$ is defined in Sublemma 3.5. So, Requirement (4) of this lemma follows from Requirement (3) of the sublemma with $\gamma = \gamma_n$. □

LEMMA 3.4. *For any $x \in \mathbb{D}^2$, there exists $n(x) \ge 1$ such that $\widehat{\varphi}_j(x) = \widehat{\varphi}_{n(x)}(x)$ for any $j \ge n(x)$.*

Proof. Assuming otherwise, for any $n \ge 1$, we have that $\widehat{\varphi}_n(x) \in W_{n,k_n} \cap \{\widehat{\varphi}_n \neq \mathrm{id}\})$ for some $1 \le k_n \le 4^n$. Therefore, by (3.5) and Lemma 3.3(4),

$$\begin{aligned} d(\widehat{\varphi}_1(x), \partial U_1) &\le c\gamma_1^{-1} \cdots c\gamma_n^{-1} d(\widehat{\varphi}_{n+1}(x), \partial U_{n+1}) \\ &< \prod_{k=1}^{n} c \cdot 10^{-k} = c^n \cdot 10^{-n(n+1)/2} \to 0 \ \text{ as } n \to \infty, \end{aligned}$$

which implies that $\widehat{\varphi}_1(x) \in \partial U_1$, leading to a contradiction. □

SUBLEMMA 3.5. *Let $\mathcal{C}_\gamma$ be defined in (3.4) and let*

$$\begin{aligned} \mathcal{C}_\gamma^+ &:= [1+\gamma, 2+\gamma] \times [1 + (\gamma-1)/2,\ 1+\gamma], \\ \mathcal{C}_\gamma^- &:= [1+\gamma, 2+\gamma] \times [-\gamma,\ -(\gamma-1)/2]. \end{aligned}$$

There exists $c > 0$ such that for any $\gamma > 10$ there is a homeomorphism $\sigma_\gamma : [0, 1]^2 \to \mathcal{C}_\gamma$ which has the following properties (see Figure 2):

(1) *$\sigma_\gamma|(0, 1)^2$ is a C^∞ diffeomorphism;*
(2) *$\sigma_\gamma = \mathrm{id}$ in a neighborhood of Γ, where Γ is the boundary of the unit square $[0, 1]^2$ without its right edge;*
(3) *σ_γ^{-1} is Lipschitz and on $\mathcal{C}_\gamma^\pm$ the Lipschitz constant is less than $c\gamma^{-1}$.*

Proof. Consider the function $p(t) = \sqrt{t(1-t)}$ for $t \in [0, 1]$ and the domain

$$\Omega^\pm := \{(x_1, x_2) \in \mathbb{R}^2 :\ 0 \le x_2 \le 1,\ 0 \le x_1 \le 1 \pm p(x_2)\}.$$

It is clear that Ω^- is a neighborhood of Γ in $[0, 1]^2$.

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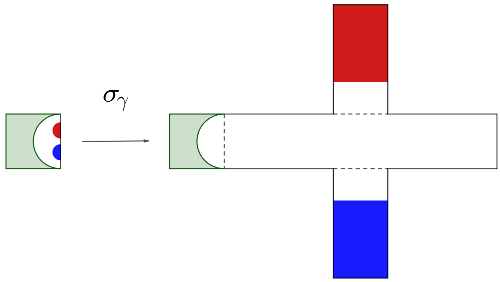

FIGURE 2. The map σ_γ.

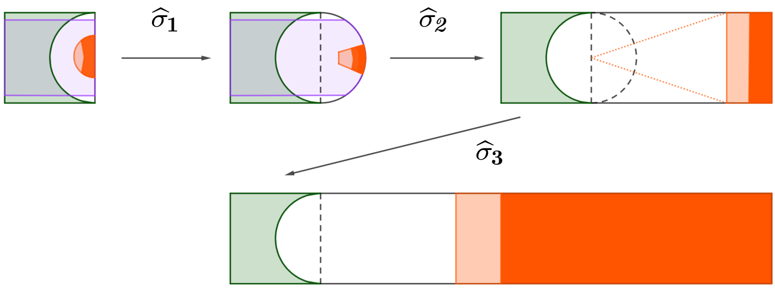

FIGURE 3. Construction of the map $\widehat{\rho}$.

We claim that there exists $c_0 > 0$ such that for any $\kappa > 3$, there exists a homeomorphism $\widehat{\rho} = \widehat{\rho}_\kappa : [0, 1]^2 \to [0, 2 + 2\kappa] \times [0, 1]$ such that:

(H1) $\widehat{\rho}|(0, 1)^2$ is a C^∞ diffeomorphism;

(H2) $\widehat{\rho} = \mathrm{id}$ on Ω^-;

(H3) $\widehat{\rho}^{-1}$ is Lipschitz on the rectangle $[1 + \kappa, 2 + 2\kappa] \times [0, 1]$ with the Lipschitz constant less than $c_0\kappa^{-1}$.

To see this, we pick a C^∞ non-decreasing function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi(t) = 0$ for all $t \leq 0$ and $\chi(t) = 1$ for all $t \geq 0.1$. Moreover, χ is sufficiently flat at $t = 0$ such that $t \mapsto t^{-k}\chi(t)$ is C^∞ for any $k > 0$. Set $c_1 := \max_{t\in\mathbb{R}} \chi'(t)$. We shall define $\widehat{\rho} = \widehat{\rho}_3 \circ \widehat{\rho}_2 \circ \widehat{\rho}_1$ as follows (see Figure 3).

First, we define a homeomorphism $\widehat{\rho}_1 : [0, 1]^2 \to \Omega^+$ by setting its inverse $\widehat{\rho}_1^{-1}(x_1, x_2) = (x_1 - S(x_1, x_2), x_2)$, where

$$S(x_1, x_2) := \begin{cases} p(x_2)\chi\left(\dfrac{x_1 - 1 + p(x_2)}{2p(x_2)}\right) & \text{if } x_2 \neq 0, 1, \\ 0 & \text{if } x_2 = 0, 1. \end{cases}$$

It is easy to check that $\widehat{\rho}_1$ is a C^∞ diffeomorphism from $(0, 1)^2$ to the interior of Ω^+ and $\widehat{\rho}_1 = \mathrm{id}$ on Ω^-. The Jacobian matrix of $d\widehat{\rho}_1^{-1}$ at $(x_1, x_2) \in \mathrm{int}(\Omega^+)$ is given by

$$d\widehat{\rho}_1^{-1}(x_1, x_2) = \begin{pmatrix} 1 - \partial_{x_1} S & -\partial_{x_2} S \\ 0 & 1 \end{pmatrix},$$

where

$$\partial_{x_1} S = \frac{1}{2}\chi'\left(\frac{x_1 - 1 + p(x_2)}{2p(x_2)}\right) \in \left[0, \frac{c_1}{2}\right].$$

Furthermore, if $x_2 \in [0.1, 0.9]$, then

$$|\partial_{x_2} S| = \left|p'(x_2)\chi\left(\frac{x_1 - 1 + p(x_2)}{2p(x_2)}\right) + \frac{(1 - x_1)p'(x_2)}{2p(x_2)}\chi'\left(\frac{x_1 - 1 + p(x_2)}{2p(x_2)}\right)\right|$$

$$\leq \max_{x_2\in[0.1,0.9]} |p'(x_2)| + \frac{\max_{x_2\in[0.1,0.9]} |p'(x_2)|}{2 \min_{x_2\in[0.1,0.9]} p(x_2)} \cdot c_1 < 2 + 4c_1.$$

Hence, $\widehat{\rho_1}^{-1}$ is Lipschitz on $\{(x_1, x_2) \in \Omega^+ : 0.1 \leq x_2 \leq 0.9\}$ with the Lipschitz constant less than $2 + 4c_1$.

Second, given any $\kappa > 3$, we define a homeomorphism $\widehat{\rho}_2 = \widehat{\rho}_{2,\kappa} : \Omega^+ \to [0, 1 + \kappa] \times [0, 1]$ by setting its inverse $(z_1, z_2) = \widehat{\rho_2}^{-1}(x_1, x_2)$ such that

$$\frac{z_1 - 1}{x_1 - 1} = \frac{z_2 - (1/2)}{x_2 - (1/2)}$$
$$= \frac{t(x_1, x_2) - 1 + \kappa}{\kappa}\chi(t(x_1, x_2))\frac{\chi(r(x_1, x_2))}{r(x_1, x_2)}\chi\left(\frac{x_1 - 1}{p(x_2)}\right),$$

where

$$t(x_1, x_2) := \frac{(x_1 - 1)^2 + 4\kappa^2(x_2 - (1/2))^2}{\kappa^2 + 4(x_1 - 1)^2(x_2 - (1/2))^2}$$

and

$$r(x_1, x_2) := \sqrt{(x_1 - 1)^2 + \left(x_2 - \tfrac{1}{2}\right)^2}.$$

Note that

$$\frac{\chi(r(1, (1/2)))}{r(1, (1/2))} := \lim_{x_1\to 1,\, x_2\to(1/2)} \frac{\chi(r(x_1, x_2))}{r(x_1, x_2)} = 0.$$

Also, for $x_2 = 0$ or 1 if $x_1 \leq 1$, then

$$\chi\left(\frac{x_1 - 1}{p(x_2)}\right) = \lim_{t\to-\infty} \chi(t) = 0$$

and, if $x_1 > 1$, then

$$\chi\left(\frac{x_1 - 1}{p(x_2)}\right) = \lim_{t\to\infty} \chi(t) = 1.$$

It is not difficult to check that $\widehat{\rho}_2$ is a C^∞ diffeomorphism from the interior of Ω^+ to $(0, 1 + \kappa) \times (0, 1)$ and $\widehat{\rho}_2 = \mathrm{id}$ on $[0, 1]^2$. Moreover, for any $(x_1, x_2) \in [1 + 0.8\kappa, 1 + \kappa] \times [0, 1]$, we have that $0.3 \leq t(x_1, x_2) \leq 2$ and $1 < 0.8\kappa \leq r(x_1, x_2) \leq 1.2\kappa$ and, hence, the inverse $(z_1, z_2) = \widehat{\rho_2}^{-1}(x_1, x_2)$ is given by

$$\frac{z_1 - 1}{x_1 - 1} = \frac{z_2 - (1/2)}{x_2 - (1/2)} = \frac{t(x_1, x_2) - 1 + \kappa}{\kappa r(x_1, x_2)}.$$

By straightforward calculations, we have

$$|\partial_{x_1} r| \leq 1, \ |\partial_{x_2} r| \leq \kappa^{-1}, \ |\partial_{x_1} t| \leq 6\kappa^{-1}, \ |\partial_{x_2} t| \leq 12,$$

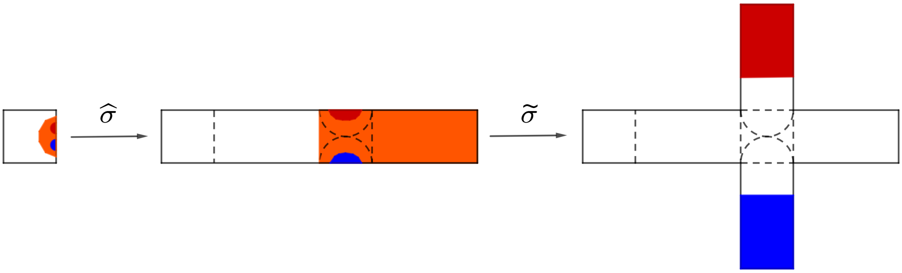

FIGURE 4. Attaching rectangular wings by $\widetilde{\rho}$.

which yields that $(\partial z_i/\partial x_j) \le 200\kappa^{-1}$ for $i = 1, 2$ and $j = 1, 2$ and thus $\widehat{\rho_2}^{-1}$ is Lipschitz on $[1 + 0.8\kappa, 1 + \kappa] \times [0, 1]$ with the Lipschitz constant less than $200\kappa^{-1}$. Also, since

$$\left|\frac{z_2 - (1/2)}{x_2 - (1/2)}\right| \le \frac{1+\kappa}{0.8\kappa^2} < 0.6,$$

we have that

$$\widehat{\rho_2}^{-1}([1 + 0.8\kappa, 1 + \kappa] \times [0, 1]) \subset \{(x_1, x_2) \in \Omega^+ : 0.1 \le x_2 \le 0.9\}.$$

Finally, we define a C^∞ diffeomorphism

$$\widehat{\rho}_3 = \widehat{\rho}_{3,\kappa} : [0, 1 + \kappa] \times [0, 1] \to [0, 2 + 2\kappa] \times [0, 1],$$

given by $\widehat{\rho}_3(x_1, x_2) = (T(x_1), x_2)$, where

$$T(x_1) = x_1 + [10(1 + \kappa^{-1})(x_1 - 1 - 0.8\kappa) - x_1]\chi\left(\frac{x_1 - 1 - 0.8\kappa}{0.2\kappa}\right).$$

Note that $\widehat{\rho}_3 = \mathrm{id}$ on $[0, 1 + 0.8\kappa] \times [0, 1]$ and $\widehat{\rho}_3$ maps $[0, 1 + 0.9\kappa] \times [0, 1]$ onto $[1 + \kappa, 2 + 2\kappa] \times [0, 1]$ with $T(x_1) = 10(1 + \kappa^{-1})(x_1 - 1 - 0.8\kappa)$ a linear map. Therefore, $\widehat{\rho_3}^{-1}$ is Lipschitz on the rectangle $[1 + \kappa, 2 + 2\kappa] \times [0, 1]$ with the Lipschitz constant no more than 1.

Finally, we set $\widehat{\rho} = \widehat{\rho}_3 \circ \widehat{\rho}_2 \circ \widehat{\rho}_1$. It is easy to see that the function $\widehat{\rho}$ has all the desired properties (H1)–(H3) with the constant $c_0 = 200(2 + 2c_1)$.

We now proceed with the proof of Sublemma 3.5. Set $c = c_0^2$. For any $\gamma > 10$, the above claim yields a homeomorphism $\widehat{\rho} = \widehat{\rho}_\gamma$ from $[0, 1]^2$ onto $[0, 2 + 2\gamma] \times [0, 1]$ having Properties (H1)–(H3). We then attach two rectangular wings to the rectangle $[0, 2 + 2\gamma] \times [0, 1]$ as follows; see Figure 4.

Take another homeomorphism $\widehat{\rho}' = \widehat{\rho}_{\gamma-1/2}$ from $[0, 1]^2$ onto $[0, 1 + \gamma] \times [0, 1]$ having Properties (H1)–(H3). Note that there is a unique planar isometry $\eta_\pm : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $[\gamma, 1 + \gamma] \times [0, 1]$ onto $[0, 1]^2$ such that $\eta_+(\gamma, 0) = (0, 1)$ and $\eta_+(\gamma, 1) = (1, 1)$, while $\eta_-(\gamma, 0) = (1, 0)$ and $\eta_-(\gamma, 1) = (0, 0)$. We then define two homeomorphisms

$$\widetilde{\rho}_+ : [\gamma, 1 + \gamma] \times [0, 1] \to [\gamma, 1 + \gamma] \times [0, 1 + \gamma]$$

and

$$\widetilde{\rho}_- : [\gamma, 1 + \gamma] \times [0, 1] \to [\gamma, 1 + \gamma] \times [-\gamma, 1],$$

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which are given by $\widetilde{\rho}_{\pm} = \eta_{\pm}^{-1} \circ \widehat{\rho}' \circ \eta_{\pm}$. Further, we define a homeomorphism $\widetilde{\rho} : [0, 2 + 2\gamma] \times [0, 1] \to \mathcal{C}_\gamma$ by

$$\widetilde{\rho}(x) = \begin{cases} \widetilde{\rho}_+ \circ \widetilde{\rho}_-(x), & x \in [\gamma, 1+\gamma] \times [0, 1], \\ x & \text{elsewhere.} \end{cases}$$

Finally, we take $\rho_\gamma = \widetilde{\rho} \circ \widehat{\rho}$, which obviously satisfies Statements (1) and (2) of Sublemma 3.5. It remains to show that Statement (3) holds for ρ_γ. Note that

$$\widetilde{\rho}^{-1} | \mathcal{C}_\gamma^{\pm} = \widetilde{\rho}_{\pm}^{-1} | \mathcal{C}_\gamma^{\pm} = \eta_{\pm}^{-1} \circ (\widehat{\rho}')^{-1} | [(1+\gamma)/2, 1+\gamma] \times [0, 1] \circ \eta_{\pm}$$

is Lipschitz with the Lipschitz constant less than $c_0((\gamma - 1)/2)^{-1} < c_0$. Moreover, $\widetilde{\rho}^{-1}(\mathcal{C}_\gamma^{\pm})$ is a subset of $[\gamma, 1+\gamma] \times [0, 1]$, on which $\widehat{\rho}^{-1}$ is Lipschitz with the Lipschitz constant less than $c_0\gamma^{-1}$. Therefore, $\rho_\gamma^{-1} = \widehat{\rho}^{-1} \circ \widetilde{\rho}^{-1}$ is Lipschitz on $\mathcal{C}_\gamma^{\pm}$ with the Lipschitz constant less than $c_0^2\gamma^{-1} = c\gamma^{-1}$. The proof of Sublemma 3.5 is complete. $\square$

4. *Proofs of Theorems C and B*

An important ingredient of our proof of Theorem C is the map $g : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ constructed by Katok in [**Kat79**]. We summarize its properties in the following statement.

PROPOSITION 4.1. *There is a C^∞ area-preserving diffeomorphism $g : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ which has the following properties:*

(1) *g is ergodic and, in fact, is isomorphic to a Bernoulli map;*
(2) *g has non-zero Lyapunov exponents almost everywhere;*
(3) *there are a neighborhood $\mathcal{N}$ of $\partial\mathbb{D}^2$ and a smooth vector field Z on $\mathcal{N}$ such that $g|\mathcal{N}$ is the time-1 map of the flow generated by Z;*
(4) *the map g can be constructed to be arbitrarily flat near the boundary of the disk; more precisely, given any sequence of positive numbers $\rho_n \to 0$ and any sequence of decreasing neighborhoods $\mathcal{N}_n$ of $\partial\mathbb{D}^2$ satisfying*

$$\mathcal{N}_n \subset \overline{\mathcal{N}}_n \subset \mathcal{N}_{n+1} \text{ and } \bigcap_{n \geq 1} \mathcal{N}_n = \partial\mathbb{D}^2, \tag{4.1}$$

one can construct a C^∞ area-preserving diffeomorphism g of $\overline{\mathbb{D}}^2$ which has Properties (1)–(3) of the proposition and such that

$$\mathcal{N}_{n-1} \subset g(\mathcal{N}_n) \subset \mathcal{N}_{n+1} \text{ and } \|g - \mathrm{id}\|_{C^n(\mathcal{N}_n)} \leq \rho_n. \tag{4.2}$$

In particular, $g|\partial\mathbb{D}^2 = \mathrm{id}$.

Proof of Theorem C. Let U be the open dense set and $h : \mathbb{D}^2 \to U \subset \mathbb{T}^2$ the map both constructed in Proposition 3.1.

Set $V_n = \{x \in U : \mathrm{dist}(x, \partial U) < (1/n)\}$ and $\mathcal{N}_n = h^{-1}(V_n)$. By Proposition 3.1, $\mathcal{N}_n$ is a neighborhood of $\partial\mathbb{D}^2$.

Given a C^∞ map $g : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ and a number $\lambda \in (0, 1)$, choose a sequence $\rho_n \to 0$ such that

$$\rho_n \leq \lambda^n / (\|h\|_{C^n(V_{n+2} \setminus V_{n-1})} \|h^{-1}\|_{C^n(V_{n+1} \setminus V_n)}).$$

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Let g be the C^∞ map constructed in Proposition 4.1 with respect to the sequence ρ_n and the sequence of decreasing neighborhoods $\mathcal{N}_n$ of $\partial\mathbb{D}^2$. Note that

$$\begin{aligned}\|h\circ(g-\mathrm{id})\circ h^{-1}\|_{C^n(V_{n+1}\setminus V_n)} &\\ \leq \|h\|_{C^n(V_{n+2}\setminus V_{n-1})}\|g-\mathrm{id}\|_{C^n(\mathcal{N}_n)}\|h^{-1}\|_{C^n(V_{n+1}\setminus V_n)} &\\ \leq \|h\|_{C^n(V_{n+2}\setminus V_{n-1})}\rho_n\|h^{-1}\|_{C^n(V_{n+1}\setminus V_n)} \leq \lambda^n. &\end{aligned}$$

It follows that the map $f:\mathbb{T}^2\to\mathbb{T}^2$ given by

$$f(x)=\begin{cases}(h\circ g\circ h^{-1})(x), & x\in U,\\ \mathrm{id} & \text{elsewhere}\end{cases}$$

is well defined and that $\|f-\mathrm{id}\|_{C^n(V_n)}\leq\lambda^n$. This implies that f is C^∞-tangent to id near ∂U. It is obvious that f satisfies all other requirements of Theorem C. □

To prove Theorem B, we also need a result from [**HPT04**] (see Proposition 4) showing that there is a smooth isotopy connecting the identity map and the Katok map.

PROPOSITION 4.2. *Let $g:\overline{\mathbb{D}}^2\to\overline{\mathbb{D}}^2$ be a map given in Proposition 4.1. Then there is a C^∞ map $G:\overline{\mathbb{D}}^2\times[0,1]\to\overline{\mathbb{D}}^2$ such that:*

(1) *for any $t\in[0,1]$, the map $g_t=G(\cdot,t):\overline{\mathbb{D}}^2\to\overline{\mathbb{D}}^2$ is an area-preserving diffeomorphism;*
(2) *$g_0=\mathrm{id}$ and $g_1=g$;*
(3) *$D^nG(x,1)=D^nG(g(x),0)$ for any $n\geq 0$;*
(4) *in a neighborhood $\mathcal{N}$ of $\partial\mathbb{D}^2$, $g_t|\mathcal{N}$ is the flow generated by Z;*
(5) *the maps g_t are arbitrarily flat near $\partial\mathbb{D}^2$; more precisely, given any sequence of positive numbers $\rho_n\to 0$ and any sequence of decreasing neighborhoods $\mathcal{N}_n$ of $\partial\mathbb{D}^2$ satisfying (4.1), one has that for any $t\in[0,1]$ the map $g=g_t$ satisfies (4.2).*

To prove Theorem B, we start with the smooth isotopy $G(x,t)$ from the above proposition and use the conjugacy map h from Proposition 3.1 to get a smooth isotopy $F(x,t)$ on $\mathbb{T}^2$ that connects the identity map with the map f_1 constructed in Theorem C. We then use this isotopy to define a flow on $\mathbb{T}^3$ that exhibits essential coexistence. Finally, we make a time change in this flow to obtain a new flow which is ergodic and, in fact, is a Bernoulli flow.

Proof of Theorem B. Let $G:\overline{\mathbb{D}}^2\times[0,1]\to\overline{\mathbb{D}}^2$ be the smooth isotopy constructed in Proposition 4.2, $g_t=G(\cdot,t)$ and $h:\mathbb{D}^2\to U\subset\mathbb{T}^2$ be the diffeomorphism constructed in Proposition 3.1.

Choose V_n, $\mathcal{N}_n$ and ρ_n in the same way as in the proof of Theorem C, and choose g_t such that for all $n\geq 1$ and $t\in[0,1]$ the map $g=g_t$ satisfies (4.2).

Then we define $F:\mathbb{T}^3\to\mathbb{T}^2$ by

$$F(x,t)=\begin{cases}h\circ G(h^{-1}(x),t), & x\in U,\\ \mathrm{id} & \text{elsewhere}\end{cases}$$

and denote

$$f_t = F(\cdot, t) = h \circ g_t \circ h^{-1} : \mathbb{T}^2 \to \mathbb{T}^2. \tag{4.3}$$

Similarly to the above, we have

$$\sup_{t\in[0,1]} \|F(\cdot, t) - \mathrm{id}\|_{C^n(V_n)} \le \lambda^n,$$

which implies that f_t is C^∞-tangent to id near ∂U for each $t \in [0, 1]$. Also, by Proposition 4.2(3), we have $d^n F(x, 1) = d^n F(f_1(x), 0)$ for any $n \ge 0$. In particular,

$$F(x, 1) = F(f_1(x), 0). \tag{4.4}$$

Next, we use F to define a map on $\mathbb{T}^3$ and then define a vector field $\widehat{X}$.

Let $\mathcal{K}$ be the suspension manifold over f_1, that is,

$$\mathcal{K} = \{(x, \theta) \in \mathbb{T}^2 \times [0, 1] : (x, 1) \sim (f_1(x), 0)\}.$$

The suspension flow $\widehat{f^t} : \mathcal{K} \to \mathcal{K}$ is generated by the vertical vector field $\partial/\partial\theta$. Note that the restriction of $\widehat{f^t}$ on the set $\{(x, 0) \in U \times [0, 1] : (x, 1) \sim (f_1(x), 0)\}$ has non-zero Lyapunov exponents almost everywhere (except for the Lyapunov exponent along the flow direction, which is zero), while $\widehat{f^t}|\partial U \times \mathbb{T} = \mathrm{id}$ and has all zero Lyapunov exponents.

We view the 3-torus as

$$\mathbb{T}^3 = \{(x, \theta) \in \mathbb{T}^2 \times [0, 1] : (x, 1) \sim (x, 0)\}.$$

The map $\widehat{F} : \mathcal{K} \to \mathbb{T}^3$, given by $\widehat{F}(x, \theta) = (F(x, \theta), \theta)$, is well defined, since, by (4.4),

$$\begin{aligned} \widehat{F}(x, 1) &= (F(x, 1), 1) = (F(f_1(x), 0), 1) \\ &= (F(f_1(x), 0), 0) = \widehat{F}(f_1(x), 0). \end{aligned}$$

Furthermore, $\widehat{F}$ is a C^∞ diffeomorphism and $\widehat{F}|\partial U \times \mathbb{T} = \mathrm{id}$. We now define a C^∞ vector field on $\mathbb{T}^3$ by $\widehat{X} = \widehat{F}_*(\partial/\partial\theta)$ and we have that

$$\widehat{X}(x, \theta) =: (X_1(x_1, x_2, \theta), X_2(x_1, x_2, \theta), 1).$$

It is clear that $\widehat{X}|\partial U \times \mathbb{T} = \partial/\partial\theta$. Since for each $t \in [0, 1]$ the map (4.3) is area preserving, it is easy to see that $\widehat{X}$ is divergence free along the $\mathbb{T}^2$-direction, that is, for any fixed $\theta \in \mathbb{T}$,

$$\frac{\partial}{\partial x_1} X_1(x_1, x_2, \theta) + \frac{\partial}{\partial x_2} X_2(x_1, x_2, \theta) = 0. \tag{4.5}$$

Note that the flow generated by the vector field $\hat{X}$ has Properties (a) and (b) of Theorem A. It is also ergodic on the open set $U \times \mathbb{T}$ but fails to be mixing let alone Bernoulli. Indeed, from the above construction of $\hat{X}$, it is easy to see that this flow is C^∞ conjugate to the suspension flow on the suspension manifold $\mathcal{K}$ with constant roof function. Therefore, it is not mixing and is not Bernoulli. To this end, following the approach in [**HPT04**], we will slightly modify $\hat{X}$ to gain the Bernoulli property, by making a small perturbation of the third component of $\hat{X}$.

More precisely, define a C^∞ vector field X on $\mathbb{T}^3$ by modifying the last component of $\widehat{X}$ as follows. Given a sufficiently small $\varepsilon > 0$, we let

$$X(x_1, x_2, \theta) := (X_1(x_1, x_2, \theta), X_2(x_1, x_2, \theta), \tau(x_1, x_2)), \tag{4.6}$$

where $\tau : \mathbb{T}^2 \to [1, 1+\varepsilon]$ is a C^∞ function such that:

(A1) $\|\tau - 1\|_{C^1} \leq \varepsilon$;

(A2) τ is a constant greater that 1 on the closure of some open subset of $\mathbb{T}^2$;

(A3) $\tau = 1$ on a neighborhood of ∂U.

By (4.5), the vector field X is divergence free along the $\mathbb{T}^2$-direction. It is also easy to see that $X = \partial/\partial\theta$ on $\partial U \times \mathbb{T}$.

We denote by $f^t : \mathbb{T}^3 \to \mathbb{T}^3$ the flow generated by X. It follows from what was said above that f^t is volume preserving and, following the arguments in [**HPT04**], it is not hard to show that $f^t|U \times \mathbb{T}$ is a Bernoulli flow with non-zero Lyapunov exponents almost everywhere (except for the Lyapunov exponent along the flow direction, which is zero), while $f^t|\partial U \times \mathbb{T} = \mathrm{id}$ and has all zero Lyapunov exponents. In particular, f^t exhibits essential coexistence. □

5. *The Hamiltonian flow: proof of Theorem A*

We shall prove that the vector field $X = (X_1, X_2, \tau)$ given by (4.6) can be embedded as a Hamiltonian vector field in the four-dimensional manifold $\mathcal{M} = \mathbb{T}^3 \times \mathbb{R}$.

Let f^t be the flow on $\mathbb{T}^3$ which is generated by the vector field X. Denote by $\Theta = \Theta(x_1, x_2, \theta)$ the backward hitting time to the zero level for the flow f^t initiated at $(x_1, x_2, \theta) \in \mathbb{T}^3$, that is, there are a unique point $(\widehat{x}_1, \widehat{x}_2)$ and a unique value $\Theta \in [0, 1) \cong \mathbb{T}$ such that $f^\Theta(\widehat{x}_1, \widehat{x}_2, 0) = (x_1, x_2, \theta)$. It is easy to see that the function Θ is C^∞ smooth on $\mathbb{T}^3$ and satisfies

$$X_1 \frac{\partial\Theta}{\partial x_1} + X_2 \frac{\partial\Theta}{\partial x_2} + \tau \frac{\partial\Theta}{\partial\theta} = 1. \tag{5.1}$$

Indeed, if $\gamma(t) = f^t(\widehat{x}_1, \widehat{x}_2, 0)$ for $t \in [0, \Theta]$, then $\Theta(\gamma(t)) = t$. Taking the derivative d/dt of both sides and then letting $t = \Theta$, we obtain (5.1).

We now consider the four-dimensional manifold $\mathcal{M} = \mathbb{T}^3 \times \mathbb{R}$, endowed with the standard symplectic form $\omega = dx_1 \wedge dx_2 + d\theta \wedge dI$. The diffeomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ given by

$$\Phi(x_1, x_2, \theta, I) = (x_1, x_2, \Theta, I) \tag{5.2}$$

pulls ω back to a closed 2-form

$$\begin{aligned}\widehat{\omega} = \Phi^*\omega &= dx_1 \wedge dx_2 + d\Theta \wedge dI \\ &= dx_1 \wedge dx_2 + \frac{\partial\Theta}{\partial x_1} dx_1 \wedge dI + \frac{\partial\Theta}{\partial x_2} dx_2 \wedge dI + \frac{\partial\Theta}{\partial\theta} d\theta \wedge dI.\end{aligned}$$

We may further assume that $\widehat{\omega}$ is non-degenerate, since $\|\widehat{\omega} - \omega\|_{C^0} \leq \|\Theta - \theta\|_{C^1}$ could be made sufficiently small if the function τ in (4.6) is chosen such that $\|\tau - 1\|_{C^1}$ is sufficiently small. Therefore, $\widehat{\omega}$ is also a symplectic form on $\mathcal{M}$.

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Let $\widetilde{H} = \widetilde{H}(x_1, x_2, \theta) : \mathbb{T}^3 \to \mathbb{R}$ be a function which for each $\theta \in \mathbb{T}$ is a solution of the following system of equations:

$$\begin{cases} \dfrac{\partial \widetilde{H}}{\partial x_2} = X_1, \; -\dfrac{\partial \widetilde{H}}{\partial x_1} = X_2 & \text{on } U, \\ \\ \widetilde{H} = 0 & \text{on } \partial U, \end{cases} \tag{5.3}$$

whose existence is provided by Lemma 5.1 below. Define a function $\widehat{H} : \mathcal{M} \to \mathbb{R}$ by

$$\widehat{H}(x_1, x_2, \theta, I) = \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta)) + I$$

and a vector field on $\mathcal{M}$ by

$$\begin{aligned} X_{\widehat{H}} &= X_{\widehat{H}}(x_1, x_2, \theta, I) \\ &= (X_1(x_1, x_2, \theta), X_2(x_1, x_2, \theta), \tau(x_1, x_2), v(x_1, x_2, \theta, I)), \end{aligned} \tag{5.4}$$

where

$$v(x_1, x_2, \theta, I) := -\frac{\partial \widetilde{H}}{\partial \theta}(x_1, x_2, \Theta(x_1, x_2, \theta)) \tag{5.5}$$

is independent of I. By Lemma 5.2 below, $X_{\widehat{H}}$ is a Hamiltonian vector field on $\mathcal{M}$ for the Hamiltonian function $\widehat{H}$ with respect to the non-standard symplectic form $\widehat{\omega}$, that is,

$$\widehat{\omega}(X_{\widehat{H}}, \cdot) = d\widehat{H}. \tag{5.6}$$

Note that given any $e \in \mathbb{R}$, the energy surface $\{\widehat{H} = e\}$ is non-degenerate (indeed, since $\partial \widehat{H}/\partial I = 1$, the differential $d\widehat{H}$ is non-degenerate) and, hence,

$$\widehat{\mathcal{M}}_e := \{\widehat{H} = e\} = \{(x_1, x_2, \theta, I) \in \mathcal{M} : \; I = e - \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta))\}$$

is a compact smooth submanifold in $\mathcal{M}$.

Finally, using the diffeomorphism Φ given by (5.2), we set

$$H := \Phi_* \widehat{H} = \widehat{H} \circ \Phi^{-1} \text{ and } X_H := \Phi_* X_{\widehat{H}} = d\Phi(X_{\widehat{H}}(\Phi^{-1})). \tag{5.7}$$

We claim that X_H is a Hamiltonian vector field on $\mathcal{M}$ for the Hamiltonian function H with respect to the standard symplectic form ω, that is,

$$\omega(X_H, \cdot) = dH. \tag{5.8}$$

To see this, note that given a 2-form η, vector fields X_1, X_2 and a diffeomorphism Φ, we have that

$$\Phi_* \eta(X_1, X_2) = [\eta(\Phi^* X_1, \Phi^* X_2)] \circ \Phi^{-1} = \Phi_*[\eta(\Phi^* X_1, \Phi^* X_2)]. \tag{5.9}$$

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Finally, we find that

$$\begin{aligned}\omega(X_H, Y) &= \Phi_* \widehat{\omega}(\Phi_* X_{\widehat{H}}, Y) && (\text{using } \widehat{\omega} = \Phi^*\omega \text{ and } (5.9))\\ &= \Phi_*[\widehat{\omega}(\Phi^*\Phi_* X_{\widehat{H}}, \Phi^* Y)] && (\text{using } \Phi^*\Phi_* = \mathrm{Id})\\ &= \Phi_*[\widehat{\omega}(X_{\widehat{H}}, \Phi^* Y)] && (\text{using } (5.6))\\ &= \Phi_*[d\widehat{H}(\Phi^* Y)] && (\text{using } (5.7))\\ &= (\Phi_* d\widehat{H})(Y) && (\text{using } (5.9))\\ &= d\Phi_*\widehat{H}(Y) = dH(Y), && (\text{using } (5.7))\end{aligned}$$

implying (5.8).

Since $H = \widehat{H} \circ \Phi^{-1}$ or, equivalently, $\widehat{H} = H \circ \Phi$ and since Φ is a diffeomorphism, for any $(x_1, x_2, \theta, I) \in \mathcal{M}$, we have that

$$H(\Phi(x_1, x_2, \theta, I)) = (H \circ \Phi)(x_1, x_2, \theta, I) = \widehat{H}(x_1, x_2, \theta, I).$$

This implies that for any $e \in \mathbb{R}$, the corresponding energy surface is given by

$$\mathcal{M}_e = \{H = e\} = \Phi\{\widehat{H} = e\} = \Phi\widehat{\mathcal{M}}_e$$

and is a compact smooth submanifold in $\mathcal{M}$.

Let f_H^t be the Hamiltonian flow restricted to $\mathcal{M}_e$ (with respect to the standard symplectic form ω). We shall show that this flow exhibits essential coexistence. To this end, define a C^∞ diffeomorphism $\widehat{\Psi}_e : \mathbb{T}^3 \to \widehat{\mathcal{M}}_e$ by

$$\widehat{\Psi}_e(x_1, x_2, \theta) = (x_1, x_2, \theta, e - \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta))).$$

It follows from (4.6), (5.1), (5.4) and (5.5) that

$$\left(\widehat{\Psi}_e\right)_* X := d\widehat{\Psi}_e(X(\widehat{\Psi}_e^{-1})) = X_{\widehat{H}}. \tag{5.10}$$

To see this, for every point $A = (x_1, x_2, \theta, I) \in \widehat{\mathcal{M}}_e$ where $I = e - \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta))$, consider the vector

$$X(\widehat{\Psi}_e^{-1})(A)) = (X_1(\widehat{\Psi}_e^{-1})(A)), X_2(\widehat{\Psi}_e^{-1})(A)), \tau(\widehat{\Psi}_e^{-1})(A)).$$

This gives a vector field X on $\mathbb{T}^3$. Note that

$$(\widehat{\Psi}_e)_* X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2} & \frac{\partial J}{\partial \theta} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \tau \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ \tau \\ X_1 \frac{\partial J}{\partial x_1} + X_2 \frac{\partial J}{\partial x_2} + \tau \frac{\partial J}{\partial \theta} \end{pmatrix},$$

where J is the last component of the map $\widehat{\Psi}_e(x_1, x_2, \theta)$, that is,

$$J = J(x_1, x_2, \theta) = e - \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta)).$$

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By (5.1) and (5.3), the fourth component of $(\widehat{\Psi}_e)_* X$ is equal to (we omit the variables $(x_1, x_2, \Theta(x_1, x_2, \theta))$)

$$
\begin{aligned}
X_1\bigg(-\frac{\partial \widetilde{H}}{\partial x_1}-\frac{\partial \widetilde{H}}{\partial \theta}\frac{\partial \Theta}{\partial x_1}\bigg)&+X_2\bigg(-\frac{\partial \widetilde{H}}{\partial x_2}-\frac{\partial \widetilde{H}}{\partial \theta}\frac{\partial \Theta}{\partial x_2}\bigg)+\tau\bigg(-\frac{\partial \widetilde{H}}{\partial \theta}\frac{\partial \Theta}{\partial \theta}\bigg)\\
&=-\bigg(X_1\frac{\partial \widetilde{H}}{\partial x_1}+X_2\frac{\partial \widetilde{H}}{\partial x_2}\bigg)-\frac{\partial \widetilde{H}}{\partial \theta}\bigg(X_1\frac{\partial \Theta}{\partial x_1}+X_2\frac{\partial \Theta}{\partial x_2}+\tau\frac{\partial \Theta}{\partial \theta}\bigg)\\
&=-\bigg(-X_1X_2+X_2X_1\bigg)-\frac{\partial \widetilde{H}}{\partial \theta}\cdot 1=-\frac{\partial \widetilde{H}}{\partial \theta}=v.
\end{aligned}
$$

It follows that the relation (5.10) holds and, hence, the Hamiltonian flow restricted to $\widehat{\mathcal{M}}_e$ (under the non-standard symplectic form $\widehat{\omega}$) is $\widehat{\Psi}_e \circ f^t \circ \widehat{\Psi}_e^{-1}$.

Furthermore, consider the diffeomorphism $\Psi_e = \Phi \circ \widehat{\Psi}_e : \mathbb{T}^3 \to \mathcal{M}_e$. It is clear that $(\Psi_e)_* X = X_H$ and, hence, for the Hamiltonian flow f_H^t restricted to $\mathcal{M}_e$ we have that $f_H^t = \Psi_e \circ f^t \circ \Psi_e^{-1}$ (where we recall that f^t is the flow on $\mathbb{T}^3$ generated by X). It follows that f_H^t is a volume-preserving Bernoulli flow which exhibits essential coexistence. This completes the proof of Theorem A subject to the two technical results mentioned above, whose proofs we now present.

LEMMA 5.1. *The system* (5.3) *has a solution* $\widetilde{H} : \mathbb{T}^3 \to \mathbb{R}$.

Proof. Recall that

$$X(x_1, x_2, \theta) = (X_1(x_1, x_2, \theta), X_2(x_1, x_2, \theta), \tau(x_1, x_2))$$

is a C^∞ vector field on $\mathbb{T}^3$ such that $X|(\partial U \times \mathbb{T}) = \partial/\partial\theta$, which means that

$$X_1(x_1, x_2, \theta) = X_2(x_1, x_2, \theta) = 0 \quad \text{for any } (x_1, x_2) \in \partial U \text{ and } \theta \in \mathbb{T}.$$

Moreover, X is divergence free along the $\mathbb{T}^2$-direction, that is, (4.5) holds for any $\theta \in \mathbb{T}$.

Now we extend the domains of X_1 and X_2 onto $\mathbb{R}^3$ by periodicity, that is, we set

$$X_i(\widetilde{x}_1, \widetilde{x}_2, \widetilde{\theta}) = X_i(x_1, x_2, \theta), \quad i = 1, 2$$

for any $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{\theta}) \in \mathbb{R}^3$ such that $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{\theta}) \equiv (x_1, x_2, \theta) \pmod{\mathbb{Z}^3}$, where $(x_1, x_2, \theta) \in \mathbb{T}^3$. Further, we consider a family of smooth 1-forms on $\mathbb{R}^2$ given by

$$\omega_{\widetilde{\theta}}(\widetilde{x}_1, \widetilde{x}_2) = X_2(\widetilde{x}_1, \widetilde{x}_2, \widetilde{\theta})\, d\widetilde{x}_1 - X_1(\widetilde{x}_1, \widetilde{x}_2, \widetilde{\theta})\, d\widetilde{x}_2,$$

in which we view $\widetilde{\theta}$ as a parameter. It is clear that $\omega_{\widetilde{\theta}}(\widetilde{x}_1, \widetilde{x}_2)$ are periodic in both $\widetilde{x}_1$ and $\widetilde{x}_2$ as well as in the parameter $\widetilde{\theta}$. Moreover, $\omega_{\widetilde{\theta}}(\widetilde{x}_1, \widetilde{x}_2) = 0$ if either $\widetilde{x}_1 \in \mathbb{Z}$ or $\widetilde{x}_2 \in \mathbb{Z}$.

For each fixed $\widetilde{\theta} \in \mathbb{R}$, equation (4.5) implies that $d\omega_{\widetilde{\theta}} = 0$, that is, the 1-form $\omega_{\widetilde{\theta}}$ is closed. By the Poincaré lemma, the form $\omega_{\widetilde{\theta}}$ is exact since $\mathbb{R}^2$ is contractible (see [**Mun91**, Ch. 8]). More precisely, $\omega_{\widetilde{\theta}} = du_{\widetilde{\theta}}$, where we choose a particular potential function $u_{\widetilde{\theta}}$ by

$$u_{\widetilde{\theta}}(\widetilde{x}_1, \widetilde{x}_2) = \int_\Gamma \omega_{\widetilde{\theta}},$$

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for any smooth path Γ from $(0,0)$ to (x_1,x_2) in $\mathbb{R}^2$ (note that the path integral is independent of the choice of Γ). Moreover, we claim that $u_{\widetilde{\theta}}$ is periodic in $(\widetilde{x}_1,\widetilde{x}_2)$, that is,

$$u_{\widetilde{\theta}}(\widetilde{x}'_1,\widetilde{x}'_2)=u_{\widetilde{\theta}}(\widetilde{x}_1,\widetilde{x}_2)\quad \text{if } (\widetilde{x}'_1,\widetilde{x}'_2)\equiv(\widetilde{x}_1,\widetilde{x}_2) \ (\mathrm{mod}\ \mathbb{Z}^2).$$

Indeed, by periodicity of $\omega_{\widetilde{\theta}}$, it suffices to show that $\int_\Gamma \omega_{\widetilde{\theta}}=0$ for two special types of paths:

$$\Gamma=\Gamma_a^1:=\{(\widetilde{x}_1,a):0\le\widetilde{x}_1\le 1\},\qquad \Gamma=\Gamma_a^2:=\{(a,\widetilde{x}_2):0\le\widetilde{x}_2\le 1\}$$

for any $a\in\mathbb{R}$. When $\Gamma=\Gamma_a^1$, we denote the bounded region

$$\Omega_a^1:=\{(\widetilde{x}_1,\widetilde{x}_2):0\le\widetilde{x}_1\le 1,\ 0\le\widetilde{x}_2\le a\}$$

and note that $\omega_{\widetilde{\theta}}$ vanishes on $\partial\Omega_a^1\backslash\Gamma_a^1$. By Stokes' theorem and the fact that $\omega_{\widetilde{\theta}}$ is a closed form, we have

$$\int_{\Gamma_a^1}\omega_{\widetilde{\theta}}=\int_{\partial\Omega_a^1}\omega_{\widetilde{\theta}}=\int_{\Omega_a^1}d\omega_{\widetilde{\theta}}=0.$$

In a similar fashion, we can show that $\int_{\Gamma_a^2}\omega_{\widetilde{\theta}}=0$ as well. Thus, $u_{\widetilde{\theta}}$ is periodic in $(\widetilde{x}_1,\widetilde{x}_2)$.

We further notice that if $\widetilde{\theta}'\equiv\widetilde{\theta}$ (mod $\mathbb{Z}$), then $\omega_{\widetilde{\theta}'}=\omega_{\widetilde{\theta}}$ and, hence, $u_{\widetilde{\theta}'}=u_{\widetilde{\theta}}$. To summarize, $u_{\widetilde{\theta}}(\widetilde{x}_1,\widetilde{x}_2)$ is periodic in all arguments, that is,

$$u_{\widetilde{\theta}'}(\widetilde{x}'_1,\widetilde{x}'_2)=u_{\widetilde{\theta}}(\widetilde{x}_1,\widetilde{x}_2)\quad \text{if } (\widetilde{x}'_1,\widetilde{x}'_2,\widetilde{\theta}')\equiv(\widetilde{x}_1,\widetilde{x}_2,\widetilde{\theta}) \ (\mathrm{mod}\ \mathbb{Z}^3).$$

Therefore, the function $\widetilde{H}:\mathbb{T}^3\to\mathbb{R}$ given by

$$\widetilde{H}(x_1,x_2,\theta)=u_{\widetilde{\theta}}(\widetilde{x}_1,\widetilde{x}_2),$$

where $(x_1,x_2,\theta)\equiv(\widetilde{x}_1,\widetilde{x}_2,\widetilde{\theta})$ (mod $\mathbb{Z}^3$), is well defined. It follows from the definition of $\omega_{\widetilde{\theta}}$ and $u_{\widetilde{\theta}}$ that $\widetilde{H}$ is a solution of (5.3). □

LEMMA 5.2. *The Hamiltonian vector field in $\mathcal{M}$ corresponding to the non-standard symplectic form $\widehat{\omega}$ and the Hamiltonian function*

$$\widehat{H}(x_1,x_2,\theta,I)=\widetilde{H}(x_1,x_2,\Theta(x_1,x_2,\theta))+I$$

is the vector field $X_{\hat{H}}$ given by (5.4) and (5.5), that is, $\hat{\omega}(X_{\hat{H}},\cdot)=d\hat{H}$.

Proof. Using (5.1), it is straightforward to show that

$$\begin{aligned}\widehat{\omega}(X_{\widehat{H}},\cdot)&=\bigg(-X_2-v\frac{\partial\Theta}{\partial x_1}\bigg)\,dx_1+\bigg(X_1-v\frac{\partial\Theta}{\partial x_2}\bigg)\,dx_2-v\frac{\partial\Theta}{\partial\theta}\,d\theta\\&\quad+\bigg(X_1\frac{\partial\Theta}{\partial x_1}+X_2\frac{\partial\Theta}{\partial x_2}+\tau\frac{\partial\Theta}{\partial\theta}\bigg)\,dI\\&=\frac{\partial\widehat{H}}{\partial x_1}\,dx_1+\frac{\partial\widehat{H}}{\partial x_2}\,dx_2+\frac{\partial\widehat{H}}{\partial\theta}\,d\theta+dI=d\widehat{H}.\end{aligned}$$

Thus, $X_{\widehat{H}}$ is the Hamiltonian vector field of $\widehat{H}$ under the symplectic form $\widehat{\omega}$. □

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