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FAMILIES OF INVARIANT MANIFOLDS CORRESPONDING TO NONZERO CHARACTERISTIC EXPONENTS

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Abstract. A theorem on conditional stability is proved for a family of mappings of class $C^{1+\epsilon}$, satisfying a condition more general than Ljapunov regularity. Using this theorem, families of invariant manifolds are constructed for a diffeomorphism of a smooth manifold onto a set where at least one Ljapunov characteristic exponent is nonzero. The property of absolute continuity is proved for these families.

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Introduction

0.1. In this paper we consider a cascade (that is, a dynamical system with discrete time) generated by a diffeomorphism f of a smooth closed n-dimensional manifold M and preserving a finite measure ν compatible with smoothness (that is, equivalent to the measure induced by some Riemannian metric). We fix in M an auxiliary Riemannian metric: the corresponding scalar product and norm on the tangent space T_xM will be denoted $\langle\ ,\ \rangle_x$ and $\|\cdot\|_x$ (sometimes the index x will be omitted). The required class of smoothness of f will be specified in the statements of theorems, but the smoothness of M and the Riemannian metric can, without loss of generality, be taken to be class C^∞. There is defined on the tangent bundle TM a measurable function (see [7])

$$\chi^+(x, v) = \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \| df^n v \|, \quad v \in T_xM, \tag{0.1}$$

called the Ljapunov characteristic exponent (the number $\chi^+(x, v)$ is called the characteristic exponent of the vector v at the point x).

Our basic assumption is that the measurable invariant (relative to f) set

$$\Lambda = \{x \in M : \chi^+(x, v) \neq 0 \quad \text{for some} \quad v \in T_xM\} \tag{0.2}$$

has positive measure.

If we interpret the existence of a vector $v \in T_xM$ with a nonzero exponent as a certain "partial hyperbolicity" property at x, then we may say that on Λ our cascade is "nonuniformly partially hyperbolic". "Partially" here means that the existence of vectors with zero exponent is not excluded, and "nonuniformly" means that the inequality, expressing for vec-

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tors $v \in T_xM$ with negative exponents the change of $\|df^n v\|_{f^n(x)}$ with increasing n, is not uniform in x. For comparison we note that the case when hyperbolicity is not partial but is, as we say, "complete" (no vectors with zero exponents) and is "uniform", is the well known case of U-systems [1], [2].

First we consider an individual trajectory $\{f^n(x_0)\}$, where $x_0 \in \Lambda$, and the mapping along it

$$df^n : T_{x_0}M \to T_{f^n(x_0)}M \tag{0.3}$$

(in the case of continuous time there are corresponding variational equations along the trajectory $x(t)$). Since in a linear approximation the behavior (relative to $\{f^n(x_0)\}$) of neighboring trajectories is described by these mappings, there then arises the question: does some "stability condition" of the trajectory $\{f^n(x_0)\}$ follow from our weak variant of "hyperbolicity", namely, the existence of a smooth submanifold $V(x_0) \subset M$ such that the trajectories $\{f^n(x)\}$ with initial value $x \in V(x_0)$ become arbitrarily near to $\{f^n(x_0)\}$. It is known that the answer, in general, is negative.

The similar circumstances for the problem of stability of the trivial solution $x(t) \equiv 0$ of a system of differential equations

$$\frac{dx}{dt} = A(t)x + f(t, x), \tag{0.4}$$

where x and f are vectors, A is a matrix, uniformly bounded and continuous, and $\|f\| = o(\|x\|)$ (or even $O(\|x\|^2)$) uniformly in t, are more widely known. Namely, it is known that negativity of all Ljapunov characteristic exponents of the "linear approximation" $\dot{x} = A(t)x$ is not sufficient for Ljapunov stability of the solution $x(t) = 0$ of the system (0.4) (see, for example, [4]). There exist different sufficient conditions providing an affirmative answer to the question raised.

The problem of interest to us is concerned with the case of a stability condition. In the analytic case, Ljapunov proved such a theorem for a system of differential equations satisfying a regularity condition. In this paper we prove a similar theorem for cascades of class $C^{1+\epsilon}$ and those points $x_0 \in \Lambda$ at which the linear approximation (0.3) satisfies a regularity condition or a certain general condition (see Theorems 2.1.1 and 2.1.2). In addition we are interested in the invariant manifolds $V(x_0)$ not for a single point, but for a whole set of such points having positive measure. Accordingly we obtain some information on the dependence of $V(x_0)$ on x_0.

This result is intended for use in the metric theory, playing for the smooth cascades considered here the same role as the well-known Hadamard-Perron theorem [6], [8] (more precisely, the version uniform relative to initial data [1]) plays for U-systems.

In metric theory it is appropriate to deal not with one individual trajectory but with a set of trajectories of positive measure; therefore it is necessary to consider not the separate invariant manifolds but families of such manifolds. For U-systems, Anosov showed that these families have a very important special property which one calls absolute continuity (see [1]). (In essence, it is just this property that allows the use of invariant manifolds in the metric theory.) In our case we will prove an analogous result (see §3).

The uniform hyperbolicity of U-cascades gives them rich metric properties: they are ergodic, have positive entropy and even have the K-property. In our case the hyperbolicity is weaker, but some analogue of these results is retained [9] (at least in the case when in Λ all the exponents are different from zero ; however, we will give proofs in the general case because, firstly, they do not follow from [9], and, secondly, the assertions proved are used to get a new result—a formula for the entropy of a dynamical system; see [9]). These proofs are based on the above-mentioned properties of invariant manifolds.

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0.2. In this paper we will constantly use the following notation in addition to that introduced above:

$\langle\ ,\ \rangle$ and $\|\cdot\|$ are the standard scalar product and corresponding norm in Euclidean space $\mathbf{R}^n$.

I is the identity mapping of the corresponding space (which one will be clear from the context).

$\mathbf{Z}^+$ and $\mathbf{R}^+$ are the sets of nonnegative integers or reals.

ρ and d are the distances in M and TM respectively, induced by the Riemannian metric.

$B(x, r)$ is the open ball in the manifold M with center at x and radius r.

T^*M is the cotangent bundle; $d'f = ((df)^*)^{-1}: T^*M \to T^*M$.

0.3. In conclusion we recall some information about the Ljapunov characteristic exponents (see [4] and [7]).

The upper index "plus" in the notation of exponents means that they are obtained as "time" $n \to +\infty$ (see (0.1)). Similarly we define the exponent χ^- as $n \to -\infty$. Correspondingly we should speak of "forward" and "backward" exponents, although, as a rule, working with the first we will omit the "forward".

For each $x \in M$ the restriction of χ^+ to the subspace T_xM takes not more than n values (distinct and different from $-\infty$). We denote these values in order by

$$\chi_1(x) < \chi_2(x) < \ldots < \chi_{s(x)}(x), \quad s(x) \leqslant n. \tag{0.5}$$

Put $L_i(x) = \{v \in T_xM: \chi(x, v) \leqslant \chi_i(x)\}$. The subspaces $L_i(x)$ form a filtration of T_xM; that is,

$$0 = L_0(x) \subset L_1(x) \subset \ldots \subset L_{s(x)}(x) = T_xM. \tag{0.6}$$

Put $\dim L_i(x) = k_i(x)$. The integer-valued functions $s(x), k_1(x), \ldots, k_{s(x)}(x)$ and the family of subspaces $L_i(x)$, $i = 1, \ldots, s(x)$, depend measurably on x. The characteristic exponent χ^+ is invariant; that is, for each $x \in M$

$$\chi_i(x) = \chi_i(f(x)), \quad k_i(x) = k_i(f(x)), \quad L_i(x) = L_i(f(x)).$$

If we replace df by $d'f$ on the right-hand side of (0.1), we obtain a characteristic exponent χ'^+ on the cotangent bundle, called an adjoint exponent.

Let $x \in M$, $v \in T_xM$ and $\phi \in T_x^*M$. Then

$$d'f(\varphi(dfv)) = \varphi(v) = 1. \tag{0.7}$$

From this follows the so-called adjoint condition: $\chi^+(x, v)^+ + \chi'^+(x, \phi) \geq 0$. A normalized basis $e(x) = \{e_i(x)\} \in T_xM$ is called normal if the first $k_1(x)$ vectors of the basis lie in $L_1(x)$, the following $k_2(x) - k_1(x)$ lie in $L_2(x) \setminus L_1(x)$, etc. If $e(x) = \{e_i(x)\}$ is a normal basis at the point x, then

$$e^n(x) = \left\{\frac{df^n e_i(x)}{\| df^n e_i(x) \|}\right\}$$

is a normal basis at $f^n(x)$.

Let $e'(x) = \{e_i'(x)\} \in T_x^*M$ be the dual basis. The *defect* of the pair of bases is the function

$$\gamma(x, e(x), e'(x)) = \max_i \{\chi^+(x, e_i(x)) + \chi'^+(x, e_i'(x))\}.$$

By virtue of the adjoint condition, $\gamma(x, e(x), e'(x)) \geq 0$. The irregularity coefficient of the exponent χ^+ is the function

$$\gamma(x) = \min \gamma(x, e(x), e'(x)),$$

where the minimum is taken over all possible pairs of dual bases $(e(x), e'(x))$. The exponent χ^+ (or the pair of exponents χ^+ and χ'^+) is called regular at x if $\gamma(x) = 0$, and points at which this condition is satisfied are called *forward regular*.

THEOREM 0.1 (see [4]). *Let x be a forward regular point. Then the following assertions are true*:

1) $\chi_i(x) = -\chi_i'(x)$.

2) *The filtration connected with χ'^+ consists of the subspaces $L_i^\perp(x)$ (the annihilator of the subspace $L_i(x)$).*

3) *Each basis dual to a normal basis is normal.*

4) *For each $n \in \mathbf{Z}$ the point $f^n(x)$ is forward regular.*

A point x is called *backward regular* if it is forward regular for the exponent χ^-. A point x is called *regular* if it is both forward and backward regular. We have detailed the notion of regularity, in contrast to [4] and [7] (their notion of regularity corresponds to our "forward regularity").

If ϕ is a measurable invariant function on the manifold M, then its Ljapunov characteristic exponent at x is

$$\chi^+(\varphi(x)) = \overline{\lim_{n\to+\infty}} \frac{1}{n} \ln |\varphi(f^n(x))|. \tag{0.8}$$

We similarly define the exponent $\chi^-(\phi(x))$ for $n \to -\infty$.

THEOREM 0.2 (see [7], Theorem 4). *If x is regular, then there exist subspaces $E_i(x)$, $i = 1, \ldots, s(x)$, satisfying the following conditions*:

1) $L_i(x) = \bigoplus_{j=1}^{k_i(x)} E_j(x)$, $i = 1, \ldots, s(x)$.

2) *Uniformly in $v \in E_j(x)$*

$$\lim_{n=\pm\infty} \ln \frac{1}{n} \| df^n v \| = \pm \chi_j(x).$$

3) *$\chi^-(\Gamma_j(x)) = \chi^+(\Gamma_j(x)) = (k_j(x) - k_{j-1}(x))\chi_j(x)$, where $\Gamma_j(x)$ is the volume of the parallelepiped in the space $E_j(x)$.*

4) *For each $n \in \mathbf{Z}$ the point $f^n(x)$ is regular, and $df^n E_j(x) = E_j(f^n(x))$.*

5) *There is a decomposition $T_x^*M = \bigoplus_{j=1}^{s(x)} E'_j(x)$ with similar properties relative to the exponent χ'^+; moreover, if $e(x) = \{e_i(x)\}$ is a normal basis for which $e_i(x) \in E_j(x)$, $k_{j-1}(x) < i \leqslant k_j(x)$, and if $e'(x) = \{e'_i(x)\}$ is the dual basis, then $e'_j(x) \in E'_j(x)$, $k_{j-1}(x) < i \leqslant k_j(x)$.*

THEOREM 0.3 (see [7], Theorem 1). *Let f be a dynamical system on the smooth manifold M, preserving a finite Borel measure. Then relative to this measure almost any point $x \in M$ is regular. The function $s(x)$ and the subspaces $E_1(x), \ldots, E_{s(x)}(x)$ depend measurably on x.*

§1. Invariant distribution and the Ljapunov metric

1.1. Let f be a diffeomorphism of M of class C^r, $r \geqslant 2$, preserving a measure ν, and let the measurable invariant set Λ defined by (0.2) have positive measure. We denote by $\widetilde{\Lambda}$ the set of regular points in Λ. From Theorem 0.3 it follows that $\nu(\widetilde{\Lambda}) = \nu(\Lambda)$.

Let $x \in \widetilde{\Lambda}$. We consider the filtration (0.6) and the subspaces $E_j(x)$, $j = 1, \ldots, s(x)$, at the point x (see Theorem 0.2). We denote by $k(x)$ the largest natural number such that for each $v \in L_{k(x)}(x)$

$$\chi^+(x, v) < 0.$$

According to the definition (0.2), $1 \leqslant k(x) < s(x)$ and $k(f(x)) = k(x)$.

Let $l(x)$ be an arbitrary measurable invariant function satisfying the condition $1 \leqslant l(x) \leqslant k(x)$. Put

$$E_{1x} = \bigoplus_{j=1}^{l(x)} E_j(x), \quad E_{2x} = \bigoplus_{j=l(x)+1}^{s(x)} E_j(x),$$
$$\lambda(x) = e^{\chi_{l(x)}(x)}, \quad \mu(x) = e^{\chi_{l(x)+1}(x)}. \tag{1.1.1}$$

From Theorems 0.3 and 0.2 it follows that the functions $\lambda(x)$ and $\mu(x)$ and the subspaces E_{1x} and E_{2x} depend measurably on x and satisfy the following conditions:

$$0 < \lambda(x) < \mu(x), \quad \lambda(x) < 1, \quad \lambda(f(x)) = \lambda(x), \quad \mu(f(x)) = \mu(x); \tag{1.1.2}$$

$$T_x M = E_{1x} \oplus E_{2x}, \quad df E_{ix} = E_{if(x)}, \quad i = 1, 2. \tag{1.1.3}$$

THEOREM 1.1.1. *There exist measurable functions $C(x, \epsilon)$ and $K(x, \epsilon)$, $\epsilon > 0$, $x \in \widetilde{\Lambda}$, satisfying the following conditions*:

1) *For every $m \in \mathbf{Z}$*

$$C(f^m(x), \varepsilon) \leqslant C(x, \varepsilon)\, e^{4\varepsilon|m|},$$
$$K(f^m(x), \varepsilon) \geqslant K(x, \varepsilon)\, e^{-\varepsilon|m|}. \tag{1.1.4}$$

2) *For any $n \in \mathbf{Z}^+$*

$$\begin{aligned}&\|df^n v\| \leqslant C(x,\varepsilon)\lambda^n(x)e^{\varepsilon n}\|v\|,\\&\|df^{-n}v\| \geqslant C^{-1}(x,\varepsilon)\lambda^{-n}(x)e^{-\varepsilon n}\|v\|\end{aligned}\qquad (v\in E_{1x}),$$
$$\begin{aligned}&\|df^n v\| \geqslant C^{-1}(x,\varepsilon)\mu^n(x)e^{-\varepsilon n}\|v\|,\\&\|df^{-n}v\| \leqslant C(x,\varepsilon)\mu^{-n}(x)e^{\varepsilon n}\|v\|\end{aligned}\qquad (v\in E_{2x}). \tag{1.1.5}$$

3. *Let $\gamma(x)$ be the angle between the subspaces E_{1x} and E_{2x}. Then $K(x, \epsilon) \leqslant \gamma(x)$.*

PROOF. We begin by proving

LEMMA 1.1.1. *Let $X \subset M$ be a measurable set invariant relative to f, and let $A(x, \epsilon)$ be a measurable function on X which for some $\epsilon > 0$ and alll $m \in \mathbf{Z}$ satisfies the condition*

$$M_1(x,\varepsilon)e^{-3\varepsilon|m|} \leqslant A(f^m(x),\varepsilon) \leqslant M_2(x,\varepsilon)e^{3\varepsilon|m|}, \tag{1.1.6}$$

where $M_1(x, \epsilon)$ and $M_2(x, \epsilon)$ are measurable functions. Then there are measurable functions $B_1(x, \epsilon)$ and $B_2(x, \epsilon)$ such that for any $m \in \mathbf{Z}$

$$B_1(x,\varepsilon) \leqslant A(x,\varepsilon) \leqslant B_2(x,\varepsilon), \tag{1.1.7}$$

$$\begin{aligned}&B_1(x,\varepsilon)e^{-4\varepsilon|m|} \leqslant B_1(f^m(x),\varepsilon),\\&B_2(x,\varepsilon)e^{4\varepsilon|m|} \geqslant B_2(f^m(x),\varepsilon).\end{aligned} \tag{1.1.8}$$

PROOF. (1.1.6) implies the existence of a natural number $m(x, \epsilon)$ such that for $m \in \mathbf{Z}$, $|m| \geqslant m(x, \epsilon)$,

$$-4\varepsilon \leqslant \frac{1}{|m|}\ln A(f^m(x),\varepsilon) \leqslant 4\varepsilon.$$

Put

$$B_1(x,\varepsilon) = \min_{-m(x,\varepsilon)\leqslant i\leqslant m(x,\varepsilon)}\{1, A(f^i(x),\varepsilon)e^{4\varepsilon|i|}\},$$

$$B_2(x,\varepsilon) = \max_{-m(x,\varepsilon)\leqslant i\leqslant m(x,\varepsilon)}\{1, A(f^i(x),\varepsilon)e^{-4\varepsilon|i|}\}.$$

The functions $B_j(x, \epsilon)$, $j = 1, 2$, are measurable. In addition, for each $n \in \mathbf{Z}$

$$B_1(x,\varepsilon)e^{-4\varepsilon|n|} \leqslant A(f^n(x),\varepsilon) \leqslant B_2(x,\varepsilon)e^{4\varepsilon|n|}. \tag{1.1.9}$$

If the numbers $b_1 \leqslant 1 \leqslant b_2$ have the property that for each $n \in \mathbf{Z}$

$$b_1 e^{-4\varepsilon|n|} \leqslant A(f^n(x),\varepsilon), \tag{1.1.10}$$

$$b_2 e^{4\varepsilon|n|} \geqslant A(f^n(x),\varepsilon), \tag{1.1.11}$$

then $b_1 \leqslant B_1(x, \epsilon)$ and $b_2 \geqslant B_2(x, \epsilon)$. Thus

$$\begin{aligned}&B_1(x,\varepsilon) = \sup\{b \leqslant 1:\ \text{for each } n \text{ (1.1.10) is satisfied}\},\\&B_2(x,\varepsilon) = \inf\{b \geqslant 1:\ \text{for each } n \text{ (1.1.11) is satisfied}\}.\end{aligned} \tag{1.1.12}$$

The inequality (1.1.9) implies (1.1.7), and also the inequalities

$$A(f^{n+m}(x), \varepsilon) \leqslant B_2(x, \varepsilon)\, e^{4\varepsilon|n+m|} \leqslant B_2(x, \varepsilon)\, e^{4\varepsilon|n|+4\varepsilon|m|},$$

$$A(f^{n+m}(x), \varepsilon) \geqslant B_1(x, \varepsilon)\, e^{-4\varepsilon|n+m|} \geqslant B_1(x, \varepsilon)\, e^{-4\varepsilon|n|-4\varepsilon|m|}.$$

Comparing these two inequalities with (1.1.9), written at the point $f^m(x)$, and taking account of (1.1.12), we obtain (1.1.8). The lemma is proved.

REMARK. If a measurable function $\psi(x)$ has exact exponents (positive and negative), equal to zero, then the function $A(x, \epsilon) = \psi(x)$ satisfies the condition of Lemma 1.1.1 for any $\epsilon > 0$.

Fix $\epsilon > 0$ and consider the angle $\gamma(x)$ between the subspaces E_{1x} and E_{2x}. By Theorem 0.2, for each $x \in \widetilde{\Lambda}$

$$\chi^+(\gamma(x)) = \chi^-(\gamma(x)) = 0; \tag{1.1.13}$$

moreover, the exponents are exact. Applying Lemma 1.1.1, we construct a function $K(x, \epsilon)$ satisfying the second inequality in (1.1.4) and assertion 3 of the theorem.

Let $x \in \widetilde{\Lambda}$. We denote $df_{jx} = df_x|_{E_j(x)}$, $j = 1, \dots, s(x)$.

LEMMA 1.1.2. *For any $\epsilon > 0$ there is a function $D(x, \epsilon)$, $x \in \widetilde{\Lambda}$, satisfying for any $m \in \mathbf{Z}$ and $1 \leqslant j \leqslant l(x)$ the condition*

$$D(f^m(x), \varepsilon) \leqslant D^2(x, \varepsilon)\, e^{2\varepsilon|m|}, \tag{1.1.14}$$

such that for each $n \in \mathbf{Z}^+$

$$\|df^n_{jx}\| \leqslant D(x, \varepsilon)\, e^{(\chi_j+\varepsilon)n},$$
$$\|df^{-n}_{jx}\| \geqslant D^{-1}(x, \varepsilon)\, e^{-(\chi_j+\varepsilon)n}, \tag{1.1.15}$$

where $\chi_j = \chi_j(x)$ (see (0.5)).

PROOF. By Theorem 0.2 there is a number $n(x, \epsilon) > 0$ such that for each $n \geqslant n(x, \epsilon)$

$$\chi_j - \varepsilon \leqslant \frac{1}{n}\ln\|df^n_{jx}\| \leqslant \chi_j + \varepsilon, \quad -\chi_j - \varepsilon \leqslant \frac{1}{n}\ln\|df^{-n}_{jx}\| \leqslant -\chi_j + \varepsilon,$$
$$-\chi_j - \varepsilon \leqslant \frac{1}{n}\ln\|d'f^n_{jx}\| \leqslant -\chi_j + \varepsilon, \quad \chi_j - \varepsilon \leqslant \frac{1}{n}\ln\|d'f^{-n}_{jx}\| \leqslant \chi_j + \varepsilon,$$

where we have put $d'f_{jx} = d'f|_{E'_j(x)}$ ($E'_j(x)$ are the subspaces of T^*_xM occurring in Theorem 0.2).

Define

$$D_1(x, \varepsilon) = \max_{\substack{-n(x,\varepsilon)\leqslant i\leqslant n(x,\varepsilon)\\ 1\leqslant j\leqslant l(x)}} \{1, \|df^i_{jx}\|\, e^{-((\operatorname{sign} i)\chi_j+\varepsilon)|i|}, \|d'f^i_{jx}\|\, e^{-(-(\operatorname{sign} i)\chi_j+\varepsilon)|i|}\},$$

$$D_2(x, \varepsilon) = \min_{\substack{-n(x,\varepsilon)\leqslant i\leqslant n(x,\varepsilon)\\ 1\leqslant j\leqslant l(x)}} \{1, \|df^i_{jx}\|\, e^{-((\operatorname{sign} i)\chi_j-\varepsilon)|i|}, \|d'f^i_{jx}\|\, e^{-(-(\operatorname{sign} i)\chi_j-\varepsilon)|i|}\},$$

$$D(x, \varepsilon) = \max\{D_1(x, \varepsilon), D_2^{-1}(x, \varepsilon)\}.$$

The function $D(x, \epsilon)$ is measurable, and for any $n \in \mathbf{Z}^+$ and $1 \leqslant j \leqslant l(x)$

$$
\begin{aligned}
D^{-1}(x,\varepsilon)e^{(\chi_j-\varepsilon)n} &\leqslant \|df^n_{jx}\| \leqslant D(x,\varepsilon)e^{(\chi_j+\varepsilon)n},\\
D^{-1}(x,\varepsilon)e^{(-\chi_j-\varepsilon)n} &\leqslant \|df^{-n}_{jx}\| \leqslant D(x,\varepsilon)e^{(-\chi_j+\varepsilon)n},\\
D^{-1}(x,\varepsilon)e^{(-\chi_j-\varepsilon)n} &\leqslant \|d'f^n_{jx}\| \leqslant D(x,\varepsilon)e^{(-\chi_j+\varepsilon)n},\\
D^{-1}(x,\varepsilon)e^{(\chi_j-\varepsilon)n} &\leqslant \|d'f^{-n}_{jx}\| \leqslant D(x,\varepsilon)e^{(\chi_j+\varepsilon)n}.
\end{aligned}
\tag{1.1.16}
$$

If a number $d \geqslant 1$ has the property that for any $n \in \mathbf{Z}^+$ (1.1.16) are satisfied on replacing $D(x, \epsilon)$ by d, then $d \geqslant D(x, \epsilon)$, Thus

$$
D(x,\epsilon) = \inf\{d \geqslant 1:\ \text{for each } n > 0 \text{ (1.1.16) is satisfied on replacing } D(x,\epsilon) \text{ by } d\}. \tag{1.1.17}
$$

We will compare the values of $D(x, \epsilon)$ at the points x and $f^m x$, $m \in \mathbf{Z}^+$. We make some preliminary remarks. We define a mapping $\tau_x\colon T^*_xM \to T_xM$, putting $\tau_x(\phi) = v_\phi$ for $\phi \in T^*_xM$, where the vector v_ϕ satisfies

$$
\langle v_\varphi, v\rangle = \varphi(v)
$$

for each $v \in T_xM$. Let $\{e^n_i\}$ be an orthonormal basis in $E_j(f^n(x))$, $\{e'^n_i\}$ the dual basis in $E'_j(f^n(x))$. Then $v_{e'^n_i} = e^n_i$. Let the mappings $df^n_{jf^m(x)}$ and $\tau_{f^{n+m}(x)} d'f^n_{jf^m(x)} \tau^{-1}_{f^m(x)}$ relative to the basis $\{e^n_i\}$ be given by matrices $A^j_{n,m}$ and $B^j_{n,m}$ respectively. Condition (0.7) written in our notation gives

$$
A^j_{0,m}(B^j_{0,m})^\top = I,
$$

where $^\top$ denotes the transposed matrix. Therefore for each $n > 0$ the matrix of $df^n_{f^m(x)}$ has the form

$$
A^j_{0,n+m}(A^j_{0,m})^{-1} = A^j_{0,n+m}(B^j_{0,m})^\top.
$$

We will use the inequalities (1.1.16) in order to estimate the norm of the operator $df^n_{jf^m(x)}$ for different n. We consider the following cases:

1. $n > 0$.

$$
\|df^n_{jf^m(x)}\| \leqslant D^2(x,\varepsilon)e^{(\chi_j+\varepsilon)(n+m)+(-\chi_j+\varepsilon)m} = D^2(x,\varepsilon)e^{2\varepsilon m}e^{(\chi_j+\varepsilon)n},
$$

$$
\|df^n_{jf^m(x)}\| \geqslant D^{-2}(x,\varepsilon)e^{(\chi_j-\varepsilon)(n+m)+(-\chi_j-\varepsilon)m} = D^{-2}(x,\varepsilon)e^{-2\varepsilon m}e^{(\chi_j-\varepsilon)n}.
$$

2. $n > 0$, $m-n > 0$.

$$
\|df^{-n}_{jf^m(x)}\| \leqslant D^2(x,\varepsilon)e^{(\chi_j+\varepsilon)(m-n)+(-\chi_j+\varepsilon)m} \leqslant D^2(x,\varepsilon)e^{2\varepsilon m}e^{(-\chi_j+\varepsilon)n},
$$

$$
\|df^{-n}_{jf^m(x)}\| \geqslant D^{-2}(x,\varepsilon)e^{(\chi_j-\varepsilon)(m-n)+(-\chi_j-\varepsilon)m} \geqslant D^{-2}(x,\varepsilon)e^{-2\varepsilon m}e^{(-\chi_j-\varepsilon)n}.
$$

3. $n > 0$, $n-m > 0$.

$$
\|df^{-n}_{jf^m(x)}\| \leqslant D^2(x,\varepsilon)e^{(-\chi_j+\varepsilon)(n-m)+(-\chi_j+\varepsilon)m} = D^2(x,\varepsilon)e^{2\varepsilon m}e^{(-\chi_j+\varepsilon)n},
$$

$$
\|df^{-n}_{jf^m(x)}\| \geqslant D^{-2}(x,\varepsilon)e^{(-\chi_j-\varepsilon)(n-m)+(-\chi_j-\varepsilon)m} = D^{-2}(x,\varepsilon)e^{-2\varepsilon m}e^{(-\chi_j-\varepsilon)n}.
$$

Similar inequalities hold for $d'f^n_{jf^m(x)}$, $n, m \in \mathbf{Z}^+$. Comparing the relations obtained and (1.1.16), written out for $f^m(x)$, and also taking account of (1.1.17), we get that (1.1.14) holds for any $m \in \mathbf{Z}$. Arguing similarly, it is not difficult to show that this inequality also holds for each integer $m \leqslant 0$. The lemma is proved.

If we replace m by $-m$ and x by $f^m(x)$ in (1.1.14), we obtain

$$D(f^m(x), \varepsilon) \geqslant \sqrt{D(x, \varepsilon)}\, e^{-2\varepsilon|m|}. \tag{1.1.18}$$

Let $x \in \widetilde{\Lambda}$. We consider two disjoint subsets σ_1 and σ_2 of the set of natural numbers from 1 to $l(x)$. Put

$$L(x) = \bigoplus_{i \in \sigma_1} E_i(x), \qquad M(x) = \bigoplus_{i \in \sigma_1} E_i(x)$$

and consider the angle $\gamma_{\sigma_1\sigma_2}(x)$ between the subspaces $L(x)$ and $M(x)$. By Theorem 0.1, (1.1.13) is satisfied for each $x \in \widetilde{\Lambda}$. Therefore, by Lemma 1.1.1 the function $\gamma_{\sigma_1\sigma_2}(x)$ satisfies the estimate

$$\gamma_{\sigma_1\sigma_2}(x) \geqslant K_{\sigma_1\sigma_2}(x, \varepsilon)$$

with some function $K_{\sigma_1\sigma_2}(x, \epsilon)$ satisfying the second inequality in (1.1.4). Put

$$T(x, \varepsilon) = \min K_{\sigma_1\sigma_2}(x, \varepsilon),$$

where the minimum is taken over all possible pairs of disjoint subsets σ_1 and σ_2 of the set of natural numbers from 1 to $l(x)$. The function $T(x, \epsilon)$ also satisfies the second inequality in (1.1.4).

Let $v \in E_{1x}$, $v = \Sigma_1^{l(x)} v_j$, where $v_j \in E_j(x)$. From the above follows the existence of a constant $L > 1$ such that

$$\sum_{j=1}^{l(x)} \|v_j\| \leqslant L T^{-1}(x, \varepsilon) \|v\|. \tag{1.1.19}$$

Put

$$C_1'(x, \varepsilon) = L D(x, \varepsilon) T^{-1}(x, \varepsilon).$$

From (1.1.4) and (1.1.18) it follows that $C_1'(x, \epsilon)$ satisfies (1.1.6) of Lemma 1.1.1, where

$$M_1(x, \varepsilon) = \frac{2}{\pi} L \sqrt{D(x, \varepsilon)}, \qquad M_2(x, \varepsilon) = L D^2(x, \varepsilon) T^{-1}(x, \varepsilon).$$

Therefore there exists a function $C_1(x, \epsilon) \geqslant C_1'(x, \epsilon)$ satisfying the first inequality in (1.1.4) for all $m \in \mathbf{Z}$. In addition, by virtue of (1.1.15) and (1.1.19), the first two inequalities in (1.1.5) are satisfied with the function $C_1(x, \epsilon)$ for each $v \in E_{1x}$ and $n \in \mathbf{Z}^+$.

Our construction is symmetric relative to the passage to the inverse mapping. Here the subspaces E_{1x} and E_{2x} exchange roles. Thus, repeating the preceding argument with the mapping f^{-1} and the subspace E_{2x}, we construct a measurable function $C_2(x, \epsilon)$, satisfying the first inequality in (1.1.4). In addition, for any $v \in E_{2x}$ and $n \in \mathbf{Z}^+$ the first two inequalities in (1.1.5) are satisfied with the function $C_2(x, \epsilon)$.

Put

$$C(x,\varepsilon)=\max\{C_1(x,\varepsilon),\ C_2(x,\varepsilon)\}.$$

It is not difficult to verify that the measurable function $C(x,\epsilon)$ satisfies the first inequality in (1.1.4), and that the inequalities (1.1.5) are satisfied with $C(x,\epsilon)$. The theorem is proved.

1.2. It follows from Theorem 1.1.1 that it is possible to introduce a nonuniform partially hyperbolic structure in $\widetilde{\Lambda}$. We will briefly describe the corresponding construction.

As above, let $k(x)$ denote the largest natural number such that $\chi^+(x,v)<0$ for each $v\in L_{k(x)}(x)$. Put

$$E_{1x}=\bigoplus_{j=1}^{k(x)} E_j(x),\quad E_{2x}=\bigoplus_{j=k(x)+2}^{s(x)} E_j(x),$$
$$E_{0x}=E_{k(x)+1}(x),\quad \lambda(x)=e^{\chi_{k(x)}(x)},\quad \mu(x)=e^{\chi_{k(x)+2}(x)}. \tag{1.2.1}$$

It is not difficult to see that

$$T_xM=E_{1x}\oplus E_{0x}\oplus E_{2x},\quad dfE_{ix}=E_{if(x)},\quad i=1,2,0,$$
$$0<\lambda(x)<1<\mu(x),\quad \lambda(f(x))=\lambda(x),\quad \mu(f(x))=\mu(x). \tag{1.2.2}$$

In addition, $\chi_{k(x)+1}(x)=0$ and $\chi^+(x,v)>0$ for each $v\in E_{2x}$.

The following is proved in the same way as Theorem 1.1.1.

THEOREM 1.2.1. *There exist measurable functions $C(x,\epsilon)$ and $K(x,\epsilon)$, $\epsilon>0$, $x\in\widetilde{\Lambda}$, satisfying* (1.1.4) *and the following conditions*:

1. *For each $v\in E_{1x}$ (or $v\in E_{2x}$) the inequalities* (1.1.5) *hold.*
2. *For any $n\in\mathbf{Z}$ and $v\in E_{0x}$*

$$C^{-1}(x,\varepsilon)e^{-\varepsilon|n|}\|v\|\leqslant\|df^nv\|\leqslant C(x,\varepsilon)e^{\varepsilon|n|}\|v\|.$$

3. *Let $\gamma_{ij}(x)$ be the angle between the subspaces E_{ix} and E_{jx}, $j=0,1,2$, $i\neq j$. Then $K(x,\epsilon)\leqslant\gamma(x)$.*

1.3. For integers $s>r\geqslant 1$ consider the set

$$\widetilde{\Lambda}_{s,r}=\{x\in\widetilde{\Lambda}:\ \frac{r-1}{s}<\lambda(x)\leqslant\frac{r}{s}<\frac{r+2}{s}\leqslant\mu(x),\ \text{where } s \text{ is the smallest number satisfying these inequalities for some } r\}.$$

It is obvious that $\widetilde{\Lambda}_{s,r}$ is measurable and invariant relative to f. Moreover, $\bigcup_{s>r\geqslant 1}\widetilde{\Lambda}_{s,r}=\widetilde{\Lambda}$, and if $s_1\neq s_2$ or $r_1\neq r_2$, then $\widetilde{\Lambda}_{s_1,r_1}\cap\widetilde{\Lambda}_{s_2,r_2}=\varnothing$.

Consider the measurable invariant function defined on $\widetilde{\Lambda}$ by the equality

$$\varepsilon(x)=\varepsilon_s=\frac{1}{100}\ln\left(1+\frac{2}{s}\right),\quad x\in\widetilde{\Lambda}_{s,r}. \tag{1.3.1}$$

It is easy to verify that for each $x\in\widetilde{\Lambda}$

$$\lambda(x)e^{50\varepsilon(x)}<1,\quad \frac{r}{s}e^{50\varepsilon(x)}<\frac{r+2}{s}e^{-50\varepsilon(x)}. \tag{1.3.2}$$

In addition, the functions $C(x) = C(x, \epsilon(x))$ and $K(x) = K(x, \epsilon(x))$ are measurable on $\widetilde{\Lambda}$. For each integer $l \geqslant 1$, consider the set

$$\widetilde{\Lambda}^l_{s,r} = \{x \in \widetilde{\Lambda}_{s,r} : C(x) \leqslant l,\ K^{-1}(x) \leqslant l\}.$$

It is obvious that $\widetilde{\Lambda}^l_{s,r}$ is measurable, $\bigcup \widetilde{\Lambda}^l_{s,r} = \widetilde{\Lambda}_{s,r}$ and $\widetilde{\Lambda}^l_{s,r} \subset \widetilde{\Lambda}^{l+1}_{s,r}$. On the basis of (1.1.4) and (1.1.5), for each $x \in \widetilde{\Lambda}^l_{s,r}$, any $m \in \mathbf{Z}$ and any $n \in \mathbf{Z}^+$ we obtain

$$\begin{aligned} &\| df^n_{f^m(x)} v \| \leqslant l \left(\frac{r}{s}\right)^n e^{\varepsilon_s n + 4\varepsilon_s |m|} \| v \| \\ & \qquad\qquad\qquad\qquad (v \in E_{1 f^m(x)}), \\ &\| df^{-n}_{f^m(x)} v \| \geqslant l^{-1} \left(\frac{r}{s}\right)^{-n} e^{-\varepsilon_s n - 4\varepsilon_s |m|} \| v \| \\ &\| df^n_{f^m(x)} v \| \geqslant l^{-1} \left(\frac{r+2}{s}\right)^n e^{-\varepsilon_s n - 4\varepsilon_s |m|} \| v \|, \\ & \qquad\qquad\qquad\qquad (v \in E_{2 f^m(x)}), \\ &\| df^{-n}_{f^m(x)} v \| \leqslant l \left(\frac{r+2}{s}\right)^{-n} e^{\varepsilon_s n + 4\varepsilon_s |m|} \| v \| \end{aligned} \tag{1.3.3}$$

$$l^{-1} e^{-\varepsilon_s |m|} \leqslant \gamma(f^m(x)). \tag{1.3.4}$$

We denote by $\Lambda^l_{s,r}$ the set of points $x \in M$ satisfying the following conditions:

(1.3.5) *There exist subspaces E_{1x} and E_{2x} for which $T_x M = E_{1x} \oplus E_{2x}$.*

(1.3.6) *For vectors v lying in $df^m(E_{ix})$, $i = 1, 2$, the estimates* (1.3.3) *hold.*

(1.3.7) *The angle $\gamma(f^m(x))$ between the subspaces $df^m(E_{1x})$ and $df^m(E_{2x})$ satisfies* (1.3.4).

THEOREM 1.3.1. 1. *$\widetilde{\Lambda}^l_{s,r} \subset \Lambda^l_{s,r} \subset \Lambda$ and $\Lambda^l_{s,r} \subset \Lambda^{l+1}_{s,r}$.*
2. *The set $\Lambda^l_{s,r}$ is closed.*
3. *The subspaces E_{1x} and E_{2x} depend continuously on x in the set $\Lambda^l_{s,r}$.*
4. *For each integer q and $l \geqslant 1$ there is an $\alpha = \alpha(q, l, s) \in \mathbf{Z}^+$ such that*

$$f^q(\Lambda^l_{s,r}) \subset \Lambda^\alpha_{s,r}.$$

5. *The set $\Lambda_{s,r} = \bigcup_{l \geqslant 1} \Lambda^l_{s,r}$ is invariant relative to f.*

PROOF. 1. The inclusion $\widetilde{\Lambda}^l_{s,r} \subset \Lambda^l_{s,r}$ is obvious. If $x \in \Lambda^l_{s,r}$, then for any $v \in E_{1x}$

$$\chi^+(x, v) < \ln \frac{r}{s} + \varepsilon_s < 0.$$

Thus $x \in \Lambda$. Since (1.3.3) and (1.3.4) remain valid on replacing l by $l + 1$, it follows that $\Lambda^l_{s,r} \subset \Lambda^{l+1}_{s,r}$; moreover, the subspaces E_{1x} and E_{2x} do not depend on which of the sets $\Lambda^l_{s,r}$ contain x.

2 *and* 3. It is easy to see that a decomposition satisfying (1.3.5)–(1.3.7) is unique. Let $x \in M$, and let $x_i \in \Lambda^l_{s,r}$ be a sequence of points converging to x. Passing to a subsequence, we may suppose that for each i

$$\dim E_{1x_i} = k, \quad \dim E_{2x_i} = n - k$$

and the subspaces E_{1x_i} and E_{2x_i} converge, as $i \to \infty$, to subspaces E_{1x}, $E_{2x} \subset T_xM$. We will prove that these subspaces satisfy (1.3.5)–(1.3.7). Let $v \in E_{1x}$ and $v_i \in E_{1x_i}$, $v_i \to v$. Since the inequalities (1.3.3) hold for v_i, on putting $m = 0$ in these and passing to the limit as $i \to \infty$ we obtain that (1.3.3), with $m = 0$, is valid for v. A similar assertion holds for vectors $v \in E_{2x}$. Hence it follows that $E_{1x} \cap E_{2x} = 0$, so that $T_xM = E_{1x} \oplus E_{2x}$. Fix an integer m. Since $x_i \to x$, it follows that $df^m_{x_i} \to df^m_x$, and consequently the subspaces $df^m(E_{jx_i})$ converge to the subspace $df^m(E_{jk})$, $j = 1, 2$. Let the sequence of vectors $v_i \in df^m(E_{jx_i})$ converge to a vector $v \in df^m(E_{jx})$, $j = 1, 2$. Since (1.3.3), with a given m, is valid for the v_i, the same will be true for v. Arguing similarly, we obtain that (1.3.4) holds for each $x \in \Lambda^l_{s,r}$ and any m.

4. Let q and $l > 0$ be integers. Choose an integer $\alpha = \alpha(q, l, s)$ such that

$$le^{4\varepsilon_s|q|} \leqslant \alpha. \tag{1.3.8}$$

If $x \in f^q(\Lambda^l_{s,r})$, we put $E_{jx} = df^q(E_{jf^{-q}(x)})$, $j = 1, 2$. For $v \in E_{1x}$, by (1.3.8) we have

$$\|df^n_{f^m(x)}v\| = \|df^n_{f^{m+q}(f^{-q}(x))}v\| \leqslant l\left(\frac{r}{s}\right)^n e^{\varepsilon_s n + 4\varepsilon_s|m+q|}\|v\|$$

$$\leqslant le^{4\varepsilon_s|q|}\left(\frac{r}{s}\right)^n e^{\varepsilon_s n+4\varepsilon_s|m|}\|v\| \leqslant \alpha\left(\frac{r}{s}\right)^n e^{\varepsilon_s n+4\varepsilon_s|m|}.$$

The remaining estimates are deduced similarly. Thus $x \in \Lambda^\alpha_{s,r}$.

Assertion 5 follows immediately from 4. The theorem is proved.

Put

$$\hat\Lambda = \bigcup_{s,r} \Lambda_{s,r}, \quad A_k = \{x \in \hat\Lambda : \dim E_{1x} = k\},$$

$$\Lambda_{k,s,r} = A_k \cap \Lambda_{s,r}, \quad \Lambda^l_{k,s,r} = A_k \cap \Lambda^l_{s,r},$$

$$\tilde\Lambda_{k,s,r} = A_k \cap \tilde\Lambda_{s,r}, \quad \tilde\Lambda^l_{k,s,r} = A_k \cap \tilde\Lambda^l_{s,r}.$$

It is obvious that the sets $\tilde\Lambda^l_{k,s,r}$ and $\Lambda^l_{k,s,r}$ are measurable and the sets $\tilde\Lambda_{k,s,r}$ and $\Lambda_{k,s,r}$ are measurable and invariant. We also put $l(x) = \dim E_{1x}$.

1.4. The starting point for further constructions lies in the nonuniform (partial) hyperbolicity condition on the set $\tilde\Lambda$ given by Theorem 1.1.1 (see also Theorem 1.2.1), where in (1.1.4) and (1.1.5) we must put $\epsilon = \epsilon(x)$ (see (1.3.1)). In fact this condition holds on a set $\hat\Lambda \supset \tilde\Lambda$. Therefore the following construction can be carried out on $\hat\Lambda$. We note however, that, by Theorem 0.3, $\Lambda = \hat\Lambda = \tilde\Lambda$ (mod 0). We stress that the hyperbolicity conditions on the sets $\Lambda^l_{k,s,r}$ are uniform, although the sets themselves are not invariant.

1.5. Now we construct a special (in general measurable) Riemannian metric, a systematic use of which will significantly simplify the arguments.

Consider the restriction $T\hat\Lambda$ of the tangent bundle TM to the set $\hat\Lambda$. Since $\hat\Lambda$ is measurable (and also Borel), $T\hat\Lambda$ can be regarded as a measurable linear bundle in the sense of [7]. A trivialization of the measurable bundle $T\hat\Lambda$ is a measurable family of isomorphisms τ_x:

$T_xM \to \mathbf{R}^n$. A measurable Riemannian metric on $T\hat{\Lambda}$ is a measurable family of positive bilinear forms (scalar products) in the spaces T_xM. Let ψ be a measurable Riemannian metric. A trivialization $\tau = \{\tau_x\}$ is called a Riemannian trivialization (relative to ψ) if for any $x \in \hat{\Lambda}$ and $v_1, v_2 \in T_xM$

$$\langle \tau_x v_1, \tau_x v_2 \rangle = \psi_x(v_1, v_2). \tag{1.5.1}$$

It is not difficult to show that a Riemannian trivialization exists for any measurable Riemannian metric.

THEOREM 1.5.1. *On $\hat{\Lambda}$ there exist a measurable Riemannian metric $\langle\ ,\ \rangle'_x$, measurable functions $\lambda'(x)$ and $\mu'(x)$ invariant relative to f, and a measurable function $A(x)$ such that for any $x \in \hat{\Lambda}$ and $m \in \mathbf{Z}$*

$$0 < \lambda(x) < \lambda'(x) < \mu'(x) < \mu(x), \quad \lambda'(x) < 1; \tag{1.5.2}$$

$$A(f^m(x)) \leqslant A(x)e^{5\varepsilon(x)|m|}; \tag{1.5.3}$$

$$\sup_{x \in \Lambda^l_{s,r}} A(x) = A^l_{s,r} < \infty; \tag{1.5.4}$$

$$\|df|_{E_{1x}}\|' < \lambda'(x), \quad (\|df^{-1}|_{E_{2x}}\|')^{-1} \geqslant \mu'(x), \quad \gamma'(x) = \frac{\pi}{2}, \tag{1.5.5}$$

where $\gamma'(x)$ denotes the angle between the subspaces E_{1x} and E_{2x} in the metric $\langle\ ,\ \rangle'_x$;

$$\frac{\sqrt{2}}{2}\|\cdot\|_x \leqslant \|\cdot\|'_x \leqslant A(x)\|\cdot\|_x. \tag{1.5.6}$$

PROOF. For $x \in \hat{\Lambda}$ put

$$\lambda'(x) = \lambda(x)e^{3\varepsilon(x)}, \quad \mu'(x) = \mu(x)e^{-3\varepsilon(x)}. \tag{1.5.7}$$

The inequality (1.5.2) follows from (1.3.2). Further, put

$$\begin{aligned} \langle v_1, v_2\rangle'_x &= \sum_{k=0}^{\infty} (\lambda'(x))^{-2k} \langle df^k v_1, df^k v_2\rangle_{f^k(x)}, \quad v_1, v_2 \in E_{1x}, \\ \langle v_1, v_2\rangle'_x &= \sum_{k=0}^{\infty} (\mu'(x))^{2k} \langle df^{-k} v_1, df^{-k} v_2\rangle_{f^{-k}(x)}, \quad v_1, v_2 \in E_{2x}, \end{aligned} \tag{1.5.8}$$

and if $v_i = v_i^1 + v_i^2$, where $v_i^j \in E_{jx}$, $i, j = 1, 2$, then

$$\langle v_1, v_2\rangle'_x = \langle v_1^1, v_2^1\rangle'_x + \langle v_1^2, v_2^2\rangle'_x. \tag{1.5.9}$$

We note that the series on the right-hand side of (1.5.8) converges. In addition, by (1.1.5), (1.5.2) and the Cauchy-Schwarz inequality,

$$|\langle v_1, v_2\rangle_x'| \leqslant C^2(x)\left[1-\left(\frac{\lambda(x)}{\lambda'(x)}e^{\varepsilon(x)}\right)^2\right]^{-1}\|v_1\|_x\|v_2\|_x, \quad v_1, v_2 \in E_{1x},$$

$$|\langle v_1, v_2\rangle_x'| \leqslant C^2(x)\left[1-\left(\frac{\mu'(x)}{\mu(x)}e^{\varepsilon(x)}\right)^2\right]^{-1}\|v_1\|_x\|v_2\|_x. \quad v_1, v_2 \in E_{2x} \tag{1.5.10}$$

Using (1.5.8), we obtain that for any $x \in \hat{\Lambda}$ and $v \in E_{1x}$

$$\|dfv\|' = \left[\sum_{k=0}^{\infty}(\lambda'(x))^{-2k}\|df^k dfv\|^2_{f^{k+1}(x)}\right]^{1/2}$$
$$= [(\lambda'(x)\|v\|_x')^2 - (\lambda'(x)\|v\|_x)^2]^{1/2} \leqslant \lambda'(x)\|v\|_x'.$$

Arguing similarly, it can be shown that for $v \in E_{2x}$

$$\|dfv\|' \geqslant \mu'(x)\|v\|_x'.$$

Finally, it follows from (1.5.9) that $\gamma(x) = \pi/2$.

We denote by τ_x any Riemannian trivialization corresponding to the Riemannian metric $\langle\ ,\ \rangle_x$. Let $\kappa^i(x)$ be orthogonal (in the metric $\langle\ ,\ \rangle_x$) mappings:

$$\varkappa^1(x): E_{1x} \to \tau_x^{-1}(\mathbf{R}^{l(x)}), \qquad \varkappa^2(x): E_{2x} \to \tau_x^{-1}(\mathbf{R}^{n-l(x)}),$$

and $\kappa(x)$: $T_xM \to T_xM$ the mapping defined in the following way:

$$\varkappa(x)(v_1+v_2) = \varkappa^1(x)v_1 + \varkappa^2(x)v_2,$$

where $v_i \in E_{ix}$, $i = 1, 2$. It is easily seen that there is a constant $M > 0$ such that for any $x \in \hat{\Lambda}$

$$M\gamma(x) \leqslant \|\varkappa(x)\|_x \leqslant \sqrt{2}.$$

From this and (1.5.10) it follows that for each $x \in \hat{\Lambda}$ and any $v \in T_xM$ the inequality (1.5.6) is satisfied, where

$$A(x) = C(x)M^{-1}K^{-1}(x)\max\left\{\left[1-\left(\frac{\lambda(x)}{\lambda'(x)}e^{\varepsilon(x)}\right)^2\right]^{-\frac{1}{2}}, \left[1-\left(\frac{\mu'(x)}{\mu(x)}e^{\varepsilon(x)}\right)^2\right]^{-\frac{1}{2}}\right\}. \tag{1.5.11}$$

The inequality (1.5.3) follows from (1.1.4), and (1.5.4) follows from the definition of $\Lambda^l_{s,r}$. The theorem is proved.

The measurable Riemannian metric $\langle\ ,\ \rangle_x'$ will be called a *measurable Ljapunov metric.*

1.6. In this subsection we will introduce special coordinates in a neighborhood of each point $x \in \hat{\Lambda}$ with the help of a certain Riemannian trivialization, corresponding to the measurable Ljapunov metric, and also give the representation of f in these coordinates. This representation will be used in §2 for the construction of families of local stable manifolds and in §3 for the proof of their absolute continuity.

For each $x \in \hat{\Lambda}$ consider the mapping

$$\widetilde{f}_x = \exp^{-1}_{f(x)} \circ f \circ \exp_x, \tag{1.6.1}$$

defined in some neighborhood $U_x \subset T_xM$. The neighborhoods U_x can be chosen so that for each $x \in \hat{\Lambda}$

$$\tau_x(U_x) = U. \tag{1.6.2}$$

Here U is some fixed neighborhood of zero in $\mathbf{R}^n$ and τ_x is a Riemannian trivialization corresponding to the measurable Ljapunov metric. This trivialization can be chosen so that for each $x \in \hat{\Lambda}$

$$\tau_x'(E_{1x}) = \mathbf{R}^{l(x)}, \quad \tau_x'(E_{2x}) = \mathbf{R}^{n-l(x)}. \tag{1.6.3}$$

Consider the mapping

$$f_x' = \tau_{f(x)}' \circ f_x \circ (\tau_x')^{-1} : U \to \mathbf{R}^n, \tag{1.6.4}$$

which, on the basis of (1.6.3), can be written in the form

$$f_x'(u, v) = (A_x u + g_x(u, v), B_x v + h_x(u, v)). \tag{1.6.5}$$

Here $u \in \mathbf{R}^{l(x)}$, $v \in \mathbf{R}^{n-l(x)}$, $A_x\colon \mathbf{R}^{l(x)} \to \mathbf{R}^{l(x)}$, $B_x\colon \mathbf{R}^{n-l(x)} \to \mathbf{R}^{n-l(x)}$, $g_x\colon U \to \mathbf{R}^{l(x)}$, $g_x \in C^r$, $h_x\colon U \to \mathbf{R}^{n-l(x)}$ and $h_x \in C^r$.

THEOREM 1.6.1.

$$\|A_x\| \leqslant \lambda'(x), \quad \|B_x^{-1}\|^{-1} \geqslant \mu'(x), \tag{1.6.6}$$

$$g_x(0) = 0, \quad h_x(0) = 0, \quad dg_x(0) = 0, \quad dh_x(0) = 0. \tag{1.6.7}$$

There exists $M > 0$ such that for any $x \in \hat{\Lambda}$, $z_1, z_2 \in U$ and $l = 1, 2, \ldots, r-1$

$$\|d^l t_x(z_1) - d^l t_x(z_2)\| \leqslant MA(x)\|z_1 - z_2\|, \tag{1.6.8}$$

where $t_x\colon T_xM \to T_{f(x)}M$, $t_x = (g_x, h_x)$.

PROOF. Since τ_x' is an isometry (see (1.5.1)), on the basis of (1.5.5), for any $u \in \mathbf{R}^{l(x)}$ we obtain

$$\|A_x u\| = \|\tau_{f(x)}' d f_x' (\tau_x')^{-1} u\| = \|d f_x' (\tau_x')^{-1} u\|_{f(x)}' \leqslant \lambda'(x) \|(\tau_x')^{-1} u\|_x' = \lambda'(x)\|u\|,$$

and if $v \in \mathbf{R}^{n-l(x)}$ similar inequalities give

$$\|B_x^{-1} v\|^{-1} \geqslant \mu'(x)\|v\|.$$

Since $f \in C^r$, $r \geqslant 2$, from the definition of f_x (see (1.6.1)) and the compactness of M follows the existence of a constant $C > 0$ such that for any $x \in M$, $w_1, w_2 \in T_xM$ and $l = 1, 2, \ldots, r-1$

$$\|d^l f_x(w_1) - d^l f_x(w_2)\|_{f(x)} \leqslant C\|w_1 - w_2\|_x.$$

From this and (1.5.6) it follows that for any $z_1, z_2 \in U$ and $l = 1, 2, \ldots, r-1$ ($w_i = (\tau_x')^{-1} z_i$, $i = 1, 2$)

$$\| d^l t_x(z_1) - d^l t_x(z_2) \| = \| d^l (f'_x(z_1) - df'_x(0) z_1) - d^l (f'_x(z_2) - df'_x(0) z_2) \|$$
$$= \| \tau'_{f(x)} (d^l f_x(w_1) - d^l f_x(w_2) \| = \| d^l f_x(w_1) - d^l f_x(w_2) \|'_{f(x)}$$
$$\leqslant A(x) \| d^l f_x(w_1) - d^l f_x(w_2) \|_{f(x)} \leqslant A(x) C \| w_1 - w_2 \|_x$$
$$\leqslant A(x) C \sqrt{2} \| w_1 - w_2 \|'_x = A(x) \sqrt{2} C \| z_1 - z_2 \|.$$

The theorem is proved.

§2. The construction of local stable manifolds.

2.1. Let F be Euclidean space with norm $\|\cdot\|$. Suppose that F can be decomposed as a direct sum of subspaces E_1 and E_2, $F = E_1 \oplus E_2$. Choose open convex neighborhoods of zero U_1 and U_2 in E_1 and E_2 respectively. Let F_m be a countable family of Euclidean spaces with norms $\|\cdot\|_m$, $m \in \mathbf{Z}^+$, isomorphic to F by isometries (a trivialization) $\tau_m: F_m \to F$. Put $E_{im} = \tau_m^{-1} E_i$ and $U_{im} = \tau_m^{-1}(U_i)$, $i = 1, 2$, and consider the family of mappings f_m,

$$\tau_{m+1} \circ f_m \circ \tau_m^{-1}(u, v) = (A_m u + g_{1m}(u, v), B_m v + g_{2m}(u, v)), \tag{2.1.1}$$

where $u \in U_1$ and $v \in U_2$.

For brevity we denote the composition $f_m \circ f_{m-1} \circ \cdots \circ f_0 = \Pi_0^m f_i$.

THEOREM 2.1.1. *Suppose that the following conditions are satisfied*:

1. *There exist numbers* λ *and* μ *such that*

$$0 < \lambda < 1, \quad \lambda < \mu, \tag{2.1.2}$$

and for any $m \in \mathbf{Z}^+$

$$\| A_m \| \leqslant \lambda, \quad \| B_m^{-1} \|^{-1} \geqslant \mu. \tag{2.1.3}$$

2. *The function* $g_m = (g_{1m}, g_{2m}) \in C^1$, *and for any* $m \in \mathbf{Z}^+$

$$g_m(0) = 0, \quad dg_m(0) = 0. \tag{2.1.4}$$

3. *There exist constants* K, α *and* ν *such that*

$$\lambda^\alpha < \nu < 1, \quad 0 < \alpha \leqslant 1, \quad K > 0, \tag{2.1.5}$$

and for any $z_1, z_2 \in U_1 \times U_2$ *and* $m \in \mathbf{Z}^+$

$$\| dg_m(z_1) - dg_m(z_2) \| \leqslant K \nu^{-m} \| z_1 - z_2 \|^\alpha. \tag{2.1.6}$$

Let κ *be any number satisfying*

$$\lambda < \varkappa < \min(\mu, \nu^{\frac{1}{\alpha}}). \tag{2.1.7}$$

Then there exist positive constants C *and* r_0 *and a mapping* $\phi: S \to E_{20}$, *where* S *is the ball in* E_{10} *with center at zero and radius* r_0, *satisfying the following conditions*:

1) $\phi \in C^1$,

$$\varphi(0)=0,\quad d\varphi(0)=0, \tag{2.1.8}$$

and for any $u_1, u_2 \in S$

$$\|d\varphi(u_1)-d\varphi(u_2)\|_0 \leq C\|u_1-u_2\|_0^{\alpha}. \tag{2.1.9}$$

2) *For each* $m \in \mathbf{Z}^+$ *and any* $u \in S$

$$\left(\prod_{i=0}^{m} f_i\right)(u, \varphi(u)) \in U_{1m}\times U_{2m},$$
$$\left\|\left(\prod_{i=0}^{m} f_i\right)(u, \varphi(u))\right\|_m \leq 200\varkappa^m\|(u, \varphi(u))\|_0. \tag{2.1.10}$$

3) *Let* $(u, v) \in F_0$, $u \in F$, *and let there be a number* $C > 0$ *such that for any* $m \in \mathbf{Z}^+$

$$\left(\prod_{i=0}^{m} f_i\right)(u, v) \in U_{1m}\times U_{2m},\qquad \left\|\left(\prod_{i=0}^{m} f_i\right)(u, v)\right\|_m \leq C\varkappa^m.$$

Then $v = \phi(u)$.

4) *There exist continuous functions* $\psi_i = \psi_i(\lambda, \mu, \nu, \kappa, \alpha)$, $i = 1, 2$, *defined on the open set in* $\mathbf{R}^4$ *given by the inequalities* (2.1.2), (2.1.5) *and* (2.1.7), *such that*

$$r_0=K^{-\frac{1}{\alpha}}\psi_1(\lambda, \mu, \nu, \varkappa, \alpha),\qquad C=K^{-\frac{1}{\alpha}}\psi_2(\lambda, \mu, \nu, \varkappa, \alpha). \tag{2.1.11}$$

PROOF. Consider the linear space Γ_κ of sequences of vectors $\{z(m)\}$, $z(m) \in F_m$, satisfying

$$\|z\|_\varkappa=\sup_{m\in\mathbf{Z}^+}\varkappa^{-m}\|z(m)\|_m<\infty.$$

The norm $\|\cdot\|_\kappa$ makes Γ_κ into a Banach space. Consider the open set

$$W=\{z\in\Gamma_\varkappa : z(m)\in U_{1m}\times U_{2m}\}$$

and the mapping $\Phi_\kappa\colon U_1 \times W \to \Gamma_\kappa$:

$$\tau_0\Phi_\varkappa(y, z)(0)=\left(y, -\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k} B_s^{-1}\right)g_{2k}(\tau_k z(k))\right),$$

$$\tau_m\Phi_\varkappa(y, z)(m)=\left(\left(\prod_{s=0}^{m-1} A_s\right)y+\sum_{k=0}^{m-1}\left(\prod_{s=k+1}^{m-1} A_s\right)g_{1k}(\tau_k z(k)),\right. \tag{2.1.12}$$
$$\left.-\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k} B_{s+m}^{-1}\right)g_{2k+m}(\tau_{k+m}z(k+m))\right)-z(m).$$

Here for uniformity of notation we regard $\Pi_{s=m}^{m-1} A_s = I$. First we will show that the mapping Φ_κ is well defined. For this we note that by the mean value theorem, (2.1.4) and (2.1.6), for any $z \in U_{1m} \times U_{2m}$, $m \in \mathbf{Z}^+$,

$$\|g_m(\tau_m z)\| = \|g_m(\tau_m z) - g_m(0)\| \leqslant \|dg_m(\xi)\| \|\tau_m z\| = \|dg_m(\xi) - dg_m(0)\| \times \|\tau_m z\| \leqslant K\nu^{-m} \|\xi\|^{\alpha} \|\tau_m z\| \leqslant K\nu^{-m} \|\tau_m z\|^{1+\alpha}. \tag{2.1.13}$$

Here $\xi = \xi(m)$ lies on the segment joining 0 and $\tau_m z$. Using (2.1.3), (2.1.13) and the definition (2.1.12) of Φ_κ, we get

$$\|\Phi_\varkappa(y, z)\|_\varkappa = \sup_{m \geqslant 0} \varkappa^{-m} \|\tau_m \Phi_\varkappa(y, z)(m)\|$$

$$\leqslant \sup_{m\geqslant 0} \left\{ \varkappa^{-m} \left[\prod_{s=0}^{m-1} \|A_s\| \|y\| + \sum_{k=0}^{m-1} \left(\prod_{s=k+1}^{m-1} \|A_k\| \right) K\nu^{-k} \|z(k)\|_k^{1+\alpha} \right.\right.$$

$$\left.\left. + \sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} \|B_{s+m}^{-1}\| \right) K\nu^{-(m+k)} \|z(m+k)\|_{m+k}^{1+\alpha} \right] \right\} + \sup_{m\geqslant 0} \varkappa^{-m} \|z(m)\|_m$$

$$\leqslant \sup_{m\geqslant 0} \varkappa^{-m}\lambda^m \|y\| + \sup_{m\geqslant 0} \varkappa^{-m} K \|z\|_\varkappa \left[\sum_{k=0}^{m-1} \lambda^{m-k-1} \nu^{-k} \varkappa^{(1+\alpha)k} \right.$$

$$\left. + \sum_{k=0}^{\infty} \mu^{-(k+1)} \nu^{-(m+k)} \varkappa^{(1+\alpha)(m+k)} \right] + \|z\|_\varkappa.$$

(2.1.7) implies the estimates

$$\sup_{m\geqslant 0} \varkappa^{-m}\lambda^m = 1, \tag{2.1.14}$$

$$\sup_{m\geqslant 0} \varkappa^{-m}\lambda^{m-1} \sum_{k=0}^{m-1} (\lambda^{-1}\nu^{-1}\varkappa^{1+\alpha})^k \leqslant -\lambda^{-1}e^{-1}\left(\ln \max\left\{\frac{\varkappa^\alpha}{\nu}, \frac{\lambda}{\varkappa}\right\}\right)^{-1}, \tag{2.1.15}$$

$$\sup_{m\geqslant 0} \varkappa^{-m}\nu^{-m}\varkappa^{(1+\alpha)m}\mu^{-1} \sum_{k=0}^{\infty} (\mu^{-1}\nu^{-1}\varkappa^{1+\alpha})^k = \frac{\mu^{-1}}{1-\mu^{-1}\nu^{-1}\varkappa^{1+\alpha}}. \tag{2.1.16}$$

Put

$$M = \max\left\{ -\frac{1}{\lambda e \ln \max\left\{\frac{\varkappa^\alpha}{\nu}, \frac{\lambda}{\mu}\right\}}, \frac{\mu^{-1}}{1-\mu^{-1}\nu^{-1}\varkappa^{1+\alpha}} \right\}. \tag{2.1.17}$$

From (2.1.14)–(2.1.17) it follows that

$$\|\Phi_\varkappa(y, z)\|_\varkappa \leqslant \|y\| + 2KM\|z\|_\varkappa^{1+\alpha} + \|z\|_\varkappa. \tag{2.1.18}$$

We have thus proved that Φ_κ is well defined. From (2.1.4) it follows that

$$\Phi_\varkappa(0, 0) = (0, 0). \tag{2.1.19}$$

We will show that $\Phi_\kappa \in C^1$. For this it is sufficient to prove the existence and continuity of the partial derivatives of Φ_κ with respect to y and z. Let $y \in U_1$, $h \in E_1$ and $y + h \in U_1$. It follows from (2.1.12) that for any $z \in W$ and $m \in \mathbf{Z}^+$

$$\tau_m\,[\Phi_\varkappa(y+h,\,z)(m)-\Phi_\varkappa(y,\,z)(m)]=\left(\left(\prod_{s=0}^{m-1}A_s\right)h,\,0\right).$$

Therefore

$$\tau_m d_y\Phi_\varkappa(y,\,z)(m)=\left(\prod_{s=0}^{m-1}A_s,\,0\right). \tag{2.1.20}$$

We will prove the continuous differentiability of the mapping Φ_κ relative to z.

Consider the difference $\Phi_\kappa(y,\,z+h)-\Phi_\kappa(y,\,z)$, where $z\in W$, $h\in\Gamma_\kappa$ and $z+h\in W$. Using (2.1.12) and condition 2 of the theorem, we obtain

$$\Phi_\varkappa(y,\,z+h)-\Phi_\varkappa(y,\,z)=(\gamma_\varkappa(z)-I)\,h+o(y,\,z,\,h),$$

where I is the identity mapping of Γ_κ and

$$\gamma_\varkappa(z)\,h\,(m)=\tau_m^{-1}\left(\sum_{k=0}^{m-1}\left(\prod_{s=k+1}^{m-1}A_s\right)dg_{1k}(\tau_k z(k))\,\tau_k h\,(k),\right.$$

$$\left.-\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k}B_{s+m}^{-1}\right)dg_{2m+k}(\tau_{m+k}z(m+k))\,\tau_{m+k}h\,(m+k)\right),$$

$$o(y,\,z,\,h)(m)=\tau_m^{-1}\left(\sum_{k=0}^{m-1}\left(\prod_{s=k+1}^{m-1}A_s\right)o_1(z,\,h)(k),\right.$$

$$\left.-\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k}B_{s+m}^{-1}\right)o_2(z,\,h)(m+k)\right).$$

Here $o_i(z,\,h)\,(m)$, $i=1,2$, is defined by

$$o_i(z,\,h)(m)=g_{im}(\tau_m\,(z+h)(m))-g_{im}(\tau_m z(m))-dg_{im}(\tau_m z(m))\,\tau_m h\,(m). \tag{2.1.21}$$

Inequality (2.1.6) and the mean value theorem imply

$$\begin{gathered}\|o_i(z,\,h)(m)\|\leqslant\|dg_{im}(\xi(m))\,\tau_m h\,(m)\\ -dg_{im}(\tau_m z(m))\,\tau_m h\,(m)\|\leqslant K\nu^{-m}\|h\,(m)\|_m^{1+\alpha},\end{gathered} \tag{2.1.22}$$

where $\xi(m)$ lies on the segment joining $\tau_m z(m)$ and $\tau_m(z+h)(m)$.

It follows from (2.1.22) that for any $z_1,\ z_2\in W$ and $h\in\Gamma_\kappa$

$$\|(\gamma_\varkappa(z_1)-\gamma_\varkappa(z_2))\,h\|_\varkappa$$

$$\leqslant\sup_{m\geqslant 0}\left\{\varkappa^{-m}\left[\sum_{k=0}^{m-1}\left(\prod_{s=k+1}^{m-1}\|A_s\|\right)K\nu^{-k}\|z_1(k)-z_2(k)\|_k^\alpha\|h_1(k)\|_k\right.\right.$$

$$\left.\left.+\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k}\|B_{s+m}^{-1}\|\right)K\nu^{-(m+k)}\|z_1(m+k)-z_2(m+k)\|_{m+k}^\alpha\|h\,(m+k)\|_{m+k}\right]\right\}.$$

The relations (2.1.14)–(2.1.17) imply

$$\|(\gamma_\varkappa(z_1)-\gamma_\varkappa(z_2))h\|_\varkappa \leqslant 2KM\|z_1-z_2\|_\varkappa^\alpha \|h\|_\varkappa. \tag{2.1.23}$$

In a similar way it is proved that

$$\|o(y, z, h)\|_\varkappa \leqslant 2KM\|h\|_\varkappa^{1+\alpha}. \tag{2.1.24}$$

It follows from (2.1.4) that $\gamma_\kappa(0) = 0$. Therefore by (2.1.23)

$$\|\gamma_\varkappa(z)\|_\varkappa \leqslant 2KM\|z\|_\varkappa^\alpha, \quad z\in W. \tag{2.1.25}$$

This inequality together with (2.1.24) shows that Φ_κ is continuously differentiable with respect to z and

$$d_z\Phi_\varkappa(y, z)=\gamma_\varkappa(z)-I. \tag{2.1.26}$$

In addition, for any $y \in U_1$

$$d_z\Phi_\varkappa(y, 0)=-I. \tag{2.1.27}$$

From the above and the equalities (2.1.19) and (2.1.27) it follows that Φ_κ satisfies all the conditions of the implicit function theorem. In what follows we will need a version of this theorem, which we quote below.

LEMMA 2.1.1. *Let E, F and G be three Banach spaces and f a continuously differentiable mapping of the product $A = A_1 \times A_2$ into G, where $A_1 \subset E$, $A_2 \subset F$, and A_i is a ball about zero with radius r_i, $i = 1, 2$. Suppose that df satisfies a Hölder condition in A with constant a and exponent α, $f(0, 0) = 0$ and the partial derivative $T_0 = D_2 f(0, 0)$ is a linear isomorphism of F onto G. Let S be the ball in E with center at zero and radius r_0,*

$$r_0=\min\left\{r_1, r_2, \frac{r_2}{2cb}, \frac{1}{(1+2cb)(2ca)^{1/\alpha}}\right\}, \tag{2.1.28}$$

where $b = \max_{(x,0)\in A}\|d_x f(x, 0)\|$ *and* $C = \|T_0^{-1}\|$. *Then there exists a unique mapping $u: S \to A_2$ of class C^1 satisfying the following conditions:*

$$f(x, u(x))=0, \quad u(0)=0,$$

$$\left\|\frac{du}{dx}(x_1)-\frac{du}{dx}(x_2)\right\| \leqslant 8ac(1+2bc)^2\|x_1-x_2\|^\alpha, \tag{2.1.29}$$

$$\left\|\frac{du}{dx}(x)\right\| \leqslant 1+2bc.$$

PROOF. We use the method of proof of Theorem 10.21 in [5]. Denote

$$g(x, y)=y-T_0^{-1}f(x, y).$$

Let $(x, y_1), (x, y_2) \in A$. Then

$$g(x, y_1)-g(x, y_2)=T_0^{-1}(T_0(y_1-y_2)-(f(x, y_1)-f(x, y_2))).$$

Put $r_3 = 2cbr_0$. By (2.1.28), $r_0 \leqslant r_1$, $r_3 \leqslant r_2$. Let $S \subset E$ and $Q \subset F$ be balls about zero with radii r_0 and r_3 respectively. If $x \in S$ and $y_1, y_2 \in Q$, then by the choice of r_0 and r_3 and the mean value theorem we have

$$\|f(x, y_1)-f(x, y_2)-T_0(y_1-y_2)\| \leqslant a(r_0+r_3)^\alpha \|y_1-y_2\| \leqslant \frac{1}{2c}\|y_1-y_2\|.$$

Therefore for any $x \in S$ and $y_1, y_2 \in Q$

$$\|g(x, y_1)-g(x, y_2)\| \leqslant \frac{1}{2}\|y_1-y_2\|.$$

On the other hand, for any $x \in S$

$$\|g(x, 0)\| = \|-T_0^{-1}f(x, 0)\| \leqslant cbr_0 = \frac{1}{2}r_3.$$

Thus the mapping g: $S \times Q \to Q$ satisfies the conditions of a fixed point theorem of [5] (Theorem 10.14), from which follows the existence and uniqueness of a continuous mapping u: $S \to Q$ such that $u(0) = 0$ and $f(x, u(x)) = 0$ for each $x \in S$. The proof that u is continuously differentiable in S is a verbatim repeat of the corresponding argument in [5] (see p. 267). From the inequality

$$\|df(x, y)-df(0, 0)\| \leqslant a(r_0+r_3)^\alpha \leqslant \frac{1}{2c},$$

valid for all $(x, y) \in S \times Q$, it follows that

$$\|d_xf(x, y)\| \leqslant b+\frac{1}{2c}; \tag{2.1.30}$$

in addition,

$$\|d_yf(x, y)\| \geqslant \frac{1}{c}-\frac{1}{2c}=\frac{1}{2c}.$$

Therefore the mapping $d_yf(x, y)$ is invertible in $S \times Q$, and it is not difficult to show that

$$\|d_y^{-1}f(x, y)\| \leqslant 2c. \tag{2.1.31}$$

Differentiating the equality $f(x, u(x)) = 0$ with respect to x, we get

$$d_xf(x, u(x))+d_yf(x, u(x))\frac{du(x)}{dx}=0.$$

Inequalities (2.1.30) and (2.1.31) imply that for $x \in S$

$$\left\|\frac{du(x)}{dx}\right\| \leqslant 1+2bc. \tag{2.1.32}$$

In addition, for any $x_1, x_2 \in S$

$$d_x f(x_1, u(x_1)) - d_x f(x_2, u(x_2)) + (d_y f(x_1, u(x_1)) - d_y f(x_2, u(x_2))) \frac{du(x_1)}{dx}$$

$$+ d_y f(x_2, u(x_2)) \left(\frac{du(x_1)}{dx} - \frac{du(x_2)}{dx}\right) = 0.$$

Inequalities (2.1.30), (2.1.31) and (2.1.32) imply

$$\left\| \frac{du(x_1)}{dx} - \frac{du(x_2)}{dx} \right\| \leqslant 4ac(1+bc)^2 \|x_1 - x_2\|^\alpha$$

$$+ 4ac(1+2bc)^2 \|x_1 - x_2\|^\alpha \leqslant 8ac(1+2bc)^2 \|x_1 - x_2\|^\alpha.$$

The lemma is proved.

It follows from (2.1.23), (2.1.20) and (2.1.27) that Φ_κ satisfies the conditions of the lemma with

$$c = 1, \quad b = 1, \quad a = 2KM. \tag{2.1.33}$$

According to the lemma there exists a ball S in E_{10} of radius r_0, $0 \in S \subset U$, and a mapping $\psi\colon S \to W$, $\psi \in C^1$, such that for any $y \in S$

$$\psi(0) = 0, \quad \Phi_\varkappa(y, \psi(y)) = 0. \tag{2.1.34}$$

By virtue of (2.1.17), (2.1.28) and (2.1.33) the number r_0 has the form (2.1.11).

Differentiating the second equality in (2.1.34) with respect to y, we get

$$d\psi(y) = -[d_z\Phi_\varkappa(y, \psi(y))]^{-1} d_y\Phi_\varkappa(y, \psi(y)). \tag{2.1.35}$$

Putting $y = 0$ in this equality and taking account of (2.1.20), (2.1.27) and (2.1.34), we find that

$$d\psi(0)(m) = \left(\prod_{s=0}^{m-1} A_s, 0\right). \tag{2.1.36}$$

We represent $\psi(y)(m)$ in the form $\psi(y)(m) = (\psi_1(y)(m), \psi_2(y)(m))$, where $\psi_i(y)(m) \in E_{im}$, $i = 1, 2$. It follows from (2.1.12) and (2.1.34) that

$$\tau_0\psi_1(y)(0) = y,$$

$$\tau_m\psi_1(y)(m) = \left(\prod_{s=0}^{m-1} A_s\right) y + \sum_{k=0}^{m-1} \left(\prod_{s=k+1}^{m-1} A_s\right) g_{1k}(\tau_k\psi(y)(k)), \quad m \geqslant 1, \tag{2.1.37}$$

$$\tau_m\psi_2(y)(m) = -\sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} B_{s+m}^{-1}\right) g_{2m+k}(\tau_{m+k}\psi(y)(m+k)), \quad m \geqslant 0. \tag{2.1.38}$$

It is easy to see that these equalities imply the relations

$$\tau_{m+1}\psi_1(y)(m+1) = A_m\tau_m\psi_1(y)(m) + g_{1m}(\tau_m\psi(y)(m)),$$

$$\tau_{m+1}\psi_2(y)(m+1) = B_m\tau_m\psi_2(y)(m) + g_{2m}(\tau_m\psi(y)(m)),$$

which mean that for all $m \in \mathbf{Z}^+$ and $u \in S$

$$f_m(\psi(y)(m)) = \psi(y)(m+1). \tag{2.1.39}$$

Define $\phi\colon S \to E_{20}$ from the conditions

$$\varphi(u) = \psi_2(y)(0), \quad u = \psi_1(y)(0), \quad y \in S. \tag{2.1.40}$$

We will show that $\phi(u)$ satisfies assertions 1–3 of the theorem. The differentiability of ϕ follows from the differentiability of ψ, and (2.1.8) follows from (2.1.34) and (2.1.36). Using (2.1.29), (2.1.33), (2.1.37) and (2.1.40), we obtain that for any u_1, $u_2 \in S$

$$\begin{gathered}\| d\varphi(u_1) - d\varphi(u_2) \|_0 \leqslant \| (d\psi(y_1) - d\psi(y_2))(0) \|_0 \\ \leqslant \| d\psi(y_1) - d\psi(y_2) \|_\varkappa \leqslant 200KM \| y_1 - y_2 \|^\alpha = 200KM \| u_1 - u_2 \|^\alpha.\end{gathered}$$

This proves (2.1.9). Assertion 1 is proved. In addition, $C = 200KM$ has the form (2.1.11).

Since $\psi(y) \in W$ for any $y \in S$, using (2.1.29), (2.1.33), (2.1.37) and (2.1.40), we find that for any $u \in S$ and $m \in \mathbf{Z}^+$

$$\begin{gathered}\left\| \left(\prod_{i=1}^{m} f_i \right)(u, \varphi(u)) \right\|_m = \left\| \left(\prod_{i=1}^{m} f_i \right)(\psi(y)(0)) \right\|_m = \| \psi(y)(m) \|_m \leqslant \varkappa^m \| \psi(y) \|_\varkappa \\ \leqslant \varkappa^m \| d\psi(y) \| \| y \| \leqslant 200\varkappa^m \| y \| = 200\varkappa^m \| u \|_0 \leqslant 200\varkappa^m \| (u, \varphi(u)) \|_0.\end{gathered}$$

In addition,

$$\left(\prod_{i=1}^{m} f_i \right)(u, \varphi(u)) = \psi(y)(m) \in U_{1m} \times U_{2m}.$$

Assertion 2 is proved.

Let the point (u, v) and the number $C > 0$ be chosen in accordance with the conditions of assertion 3. Consider the sequence $\psi(l) = (\Pi_{i=1}^{l} f_i)(u, v)$. Since

$$\sup_{l \geqslant 0} \varkappa^{-l} \left\| \left(\prod_{i=1}^{l} f_i \right)(u, v) \right\|_l \leqslant C \sup_{l \geqslant 0} \varkappa^{-l} \varkappa^l = C,$$

we see that $\psi \in \Gamma_\kappa$. From the definition of ψ it follows that $f_l \psi(l) = \psi(l+1)$. Therefore

$$\begin{gathered}\tau_{l+1}\psi_1(l+1) = A_l \tau_l \psi_1(l) + g_{1l}(\tau_l \psi(l)), \\ \tau_{l+1}\psi_2(l+1) = B_l \tau_l \psi_2(l) + g_{2l}(\tau_l \psi(l)).\end{gathered}$$

Hence it follows that for any $n \geqslant l$

$$\tau_l \psi_1(l) = \left(\prod_{s=0}^{l-1} A_s \right) y + \sum_{k=0}^{l-1} \left(\prod_{s=k+1}^{l-1} A_s \right) g_{1k}(\tau_k \psi(k)),$$

$$\tau_l \psi_2(l) = \left(\prod_{s=0}^{n-l-1} B_{s+l}^{-1} \right) \tau_n \psi_2(n) - \sum_{k=0}^{n-l-1} \left(\prod_{s=0}^{k} B_{s+l}^{-1} \right) g_{2k+l}(\tau_{k+l} \psi(k+l)). \tag{2.1.41}$$

Since

$$\left\|\left(\prod_{s=0}^{n-l-1} B_{s+l}^{-1}\right)\tau_n\psi_2(n)\right\| \leqslant \frac{1}{\mu^{n-l}}\left\|\left(\prod_{i=0}^{n-l-1} f_{i+l}\right)\psi(l)\right\|_n \leqslant C\left(\frac{\varkappa}{\mu}\right)^{n-l},$$

on passing to the limit as $n \to \infty$ in the second equality of (2.1.41) we obtain

$$\tau_l\psi_2(l) = -\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k} B_{s+l}^{-1}\right) g_{2k+l}(\tau_{k+l}\psi(k+l)).$$

Hence from (2.1.38) and (2.1.41) it follows that $\Phi_\kappa(y, \psi) = 0$. Since $y \in S$, by the uniqueness statement in Lemma 2.1.1 we have $\psi = \psi(y)$. Therefore

$$\psi_1(y) = \psi_1(0) = u, \quad \varphi(u) = \psi_2(y)(0) = \psi(0) = v.$$

Assertion 3, and together with it the theorem, is proved.

Under additional smoothness assumptions on f_m it can be proved that the mapping ϕ constructed in Theorem 2.1.1 also has a higher degree of smoothness.

THEOREM 2.1.2. *Assume that, in the conditions of Theorem* 2.1.1, *for each* $m \in \mathbf{Z}^+$ *the function* $g_m(u, v) \in C^r$, $(u, v) \in U_1 \times U_2$, $r \geqslant 2$, *and moreover there exist positive constants* K *and* K_l, $l = 1, \ldots, r$, *such that*

$$\sup_{z \in U_1 \times U_2} \|d^l g_m(z)\| \leqslant K_l \nu^{-m}, \tag{2.1.42}$$

$$\|d^r g_m(z_1) - d^r g_m(z_2)\| \leqslant K\nu^{-m}\|z_1 - z_2\|^\alpha, \tag{2.1.43}$$

where $z_1, z_2 \in U_1 \times U_2$. *Let* $\phi(u)$ *be the mapping constructed in Theorem* 2.1.1. *Then there exist positive numbers* r_0, N *and* N_l, $l = 1, \ldots, r$, *depending only on* K, K_l, λ, μ, ν, κ *and* α, *such that for any* $u, u_1, u_2 \in S$ *and* $\phi(u) \in C^r$

$$\sup_{u \in S}\|d^l\varphi(u)\| \leqslant N_l, \tag{2.1.44}$$

$$\|d^r\varphi(u_1) - d^r\varphi(u_2)\| \leqslant N\|u_1 - u_2\|^\alpha. \tag{2.1.45}$$

PROOF. We use the notation and constructions of Theorem 2.1.1 and show that $\Phi_\kappa \in C^r$. It follows from (2.1.20) that for any $y \in U_1$ and $z \in W$

$$d_y^l\Phi_\varkappa(y, z) = (0, 0), \quad 2 \leqslant l \leqslant r.$$

We will show that Φ_κ is r times continuously differentiable with respect to z. Formally differentiating (2.1.12) l times with respect to z gives

$$\tau_m d_z^l\Phi_\varkappa(y, z)(m) = \left(\sum_{k=0}^{m-1}\left(\prod_{s=k+1}^{m-1} A_s\right) d^l g_{1k}(\tau_k z(k)),\right.$$

$$\left. -\sum_{k=0}^{\infty}\left(\prod_{s=0}^{k} B_{s+m}^{-1}\right) d^l g_{2m+k}(\tau_{m+k} z(m+k))\right).$$

The inequality (2.1.42) means that the multilinear form $d^l g_m(z)$ is uniformly bounded; that is, for any $z \in U_1 \times U_2$ and $h_1, \ldots, h_l \in F$, $l \geqslant 2$, we have

$$\| d^l g_m(z)(h_1, \ldots, h_l) \| \leqslant K_l \nu^{-m} \prod_{i=1}^{l} \| h_i \|.$$

Let $h_i \in \Gamma_\kappa$. From the preceding inequality and (2.1.14)–(2.1.17) it follows that

$$\| d_z^l \Phi_\varkappa (y, z)(h_1, \ldots, h_l) \|_\varkappa \leqslant \sup_{m \geqslant 0} \varkappa^{-m} \left[\sum_{k=0}^{m-1} \lambda^{m-k-1} K_l \nu^{-k} \varkappa^{lk} \prod_{i=1}^{l} \| h_i \|_\varkappa \right.$$
$$\left. + \sum_{k=0}^{\infty} \mu^{-(k+1)} K_l \nu^{-(m+k)} \varkappa^{(m+k)l} \prod_{i=1}^{l} \| h_i \|_\varkappa \right] \leqslant 2 K_l M \prod_{i=1}^{l} \| h_i \|_\varkappa .$$

Thus the multilinear form $d_z^l \Phi_\kappa(y, z)$ is bounded. Obviously it is continuous relative to y and z. Now we show by induction that $d_z^l \Phi_\kappa(y, z)$ is in fact the lth derivative with respect to z of $\Phi_\kappa(y, z)$. For $l = 1$ this was proved in Theorem 2.1.1. If $h \in \Gamma_\kappa$ is a small increment, then

$$d_z^{l-1} \Phi_\varkappa (y, z+h) - d_z^{l-1} \Phi_\varkappa (y, z) = d_z^l \Phi_\varkappa (y, z) h + o_l(y, z, h),$$

where $o_l(y, z, h)$ is defined by

$$\tau_m o_l (y, z, h)(m) = \left(\sum_{k=0}^{m-1} \left(\prod_{s=k+1}^{m-1} A_s \right) o_{1l} (y, z, h)(k), \right.$$
$$\left. - \sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} B_{s+m}^{-1} \right) o_{2l} (y, z, h)(m+k) \right);$$

moreover $o_{il}(y, z, h)$ is defined by (2.1.22), in which g_i is replaced by its l-derivative. Using (2.1.42) and (2.1.43), we obtain that

$$\| o_{il} (y, z, h)(m) \| = \| d^l g_{im} (\xi(m)) \tau_m h(m)$$
$$- d^l g_{im} (\tau_m z(m)) \tau_m h(m) \| \leqslant K_{l+1} \nu^{-m} \| h(m) \|_m^2 \quad \text{for } 2 \leqslant l < r,$$

and

$$\| o_{ir} (y, z, h)(m) \| \leqslant K \nu^{-m} \| h(m) \|_m^{1+\alpha} \quad \text{for } l = r.$$

Hence, arguing as above and using (2.1.14)–(2.1.17), we obtain

$$\| o_l (y, z, h) \|_\varkappa \leqslant 2 K_{l+1} M \| h \|_\varkappa^2 \quad \text{for } 2 \leqslant l < r,$$

and

$$\| o_r (y, z, h) \|_\varkappa \leqslant 2 K M \| h \|_\varkappa^{1+\alpha} \quad \text{for } l = r.$$

This proves that $\Phi_\kappa \in C^r$. From the implicit function theorem we conclude that $\psi \in C^r$. Furthermore, there exist constants $N_1 > 0$, depending only on λ, μ, ν, κ and α, such that

$$\sup_{y \in S} \| d^l \psi(y) \|_\varkappa \leqslant N_l, \quad 2 \leqslant l \leqslant r.$$

We will use a version of the implicit function theorem whose proof is similar to that of Lemma 2.1.1.

LEMMA 2.1.2. *In the conditions of Lemma* 2.1.1 *assume that $f \in C^r$ and $d^r f$ satisfies a Hölder condition in A with constant a and exponent α. Then $u \in C^r$, and there is a constant N such that for any $x_1, x_2 \in S$*

$$\left\| \frac{d^r u(x_1)}{dx^r} - \frac{d^r u(x_2)}{dx^r} \right\| \leqslant N \| x_1 - x_2 \|^\alpha. \tag{2.1.46}$$

From the definition of $\phi(u)$ (see (2.1.40)) and Lemma 2.1.2 (see (2.1.46)) it follows that for any $u_1, u_2 \in S$

$$\| d^r \varphi(u_1) - d^r \varphi(u_2) \|_0 \leqslant \| d^r \psi(y_1) - d^r \psi(y_2) \|_\varkappa \leqslant N \| u_1 - u_2 \|_0^\alpha.$$

The theorem is proved.

2.2. Let f be a diffeomorphism of M of class C^r, $r \geqslant 2$, and Λ the invariant set of positive measure defined by (0.2). We use the notation of 1.5 and consider the bundle $T\Lambda$ with the Riemannian trivialization τ', corresponding to the Ljapunov measurable metric $\langle\, , \rangle'_x$, constructed in Theorem 1.5.1.

PROPOSITION 2.2.1. *Let $x \in \Lambda_{s,r}$. The family of trivializations*

$$\tau_m = \tau'_{f(m)} : T_{f^m(x)} \to \mathbf{R}^n$$

and mappings $f_m = f_{f^m(x)}$ (see (1.6.6)*), defined in the neighborhoods $U_m = U_{f^m(x)}$ (see* (1.6.2)*), satisfies the conditions of Theorems* 2.1.1 *and* 2.1.2, *where*

$$\lambda = \lambda_{s,r} = \frac{r}{s} e^{3\varepsilon_s}, \qquad \mu = \mu_{s,r} = \frac{r+2}{s} e^{-3\varepsilon_s},$$
$$\nu = \nu_{s,r} = e^{-5\varepsilon_s}, \qquad \varkappa = \varkappa_{s,r} = \lambda_{s,r} e^{\varepsilon_s}, \tag{2.2.1}$$
$$\alpha = 1, \qquad \varepsilon_s = \frac{1}{100} \ln\left(1 + \frac{2}{s}\right), \qquad K = MA(x).$$

Here the function $A(x)$ constructed in Theorem 1.5.1 *and the constant M are the same as in* (1.6.8).

PROOF. The representation (2.1.1) follows from (1.6.3)–(1.6.5). Inequalities (2.1.2), (2.1.5) and (2.1.7) follow from (1.3.2); inequality (2.1.3), from (1.6.6) and (1.5.7). Conditions (2.1.4) are corollaries of (1.6.7). The inequalities (1.6.8) and (1.5.3) imply that for any $m \in \mathbf{Z}^+$, $z_1, z_2 \in U$ (see (1.6.2)) and $l = 1, \ldots, r-1$

$$\| d^l t_m(z_1) - d^l t_m(z_2) \| \leqslant MA(x) e^{6\varepsilon_s m} \| z_1 - z_2 \|.$$

Finally, (2.1.43) is an immediate corollary of (1.6.8). The proposition is proved.

Put

$$\varkappa(x)=\varkappa_{s,r}, \quad x\in\Lambda_{s,r}. \tag{2.2.2}$$

If $\delta(x)$ is a positive measurable function on $\hat{\Lambda}$ we denote

$$\begin{gathered} B^i(\delta(x))=\{u\in E_{ix}: \|u\|_x\leqslant\delta(x)\}, \quad i=1,2, \\ B(\delta(x))=B^1(\delta(x))\times B^2(\delta(x)), \\ U(x,\delta(x))=\exp_x B(\delta(x)), \quad U'_x=\exp_x(\tau'_x)^{-1}Q. \end{gathered} \tag{2.2.3}$$

THEOREM 2.2.1. *There exist a measurable function $\delta(x)$, $x\in\hat{\Lambda}$, a family of mappings $\phi(x)$: $B^1(\delta(x))\to B^2(\delta(x))$ of class C^{r-1}, depending measurably on $x\in\hat{\Lambda}$, and a constant $N>0$ satisfying the following conditions*:

1. *The set $V(x)=\{\exp_x(u,\phi(x)u): u\in B^1(\delta(x))\}$ is a submanifold in M of class C^{r-1}.*
2. $x\in V(x)$.
3. $T_xV(x)=E_{1x}$.
4. *For $y\in V(x)$ and $n\in\mathbf{Z}^+$ we have $f^n(y)\in U'_{f^n(x)}$ and*

$$\rho(f^n(x),f^n(y))\leqslant NA(x)\varkappa^n(x)\rho(x,y). \tag{2.2.4}$$

5. *If there are a point $y\in U(x,\delta(x))$ and a constant $C>0$ such that for each $n\in\mathbf{Z}^+$ we have $f^n(y)\in U'_{f^n(x)}$ and*

$$\rho(f^n(x),f^n(y))\leqslant C\varkappa^n(x), \tag{2.2.5}$$

then $y\in V(x)$.

6. *$U(x,\delta(x))\in U'_x$, where U'_x is a neighborhood of x (see (2.2.3)) and for any $m\in\mathbf{Z}^+$*

$$\begin{gathered} \delta(f^m(x))\geqslant\delta(x)e^{-10\varepsilon(x)m}, \\ \delta^l_{s,r}=\inf_{x\in\Lambda^l_{s,r}}\delta(x)>0, \quad \delta(x)<1. \end{gathered} \tag{2.2.6}$$

7. $f(V(x))\cap U(f(x),\delta(f(x)))\subseteq V(f(x))$.
8. *There exists a measurable function $G(x)$, $x\in\hat{\Lambda}$, such that*

$$G(f(x))=G(x), \quad G_{s,r}=\sup_{x\in\Lambda_{s,r}}G(x)<\infty,$$

and if $y\in V(x)$, then

$$d(T_yV(x),T_xV(x))\leqslant G(x)A^2(x)\rho(x,y). \tag{2.2.7}$$

PROOF. Let $x\in\hat{\Lambda}$. Choose a number r_0 and a mapping ϕ in accordance with Theorem 2.1.1 and Proposition 2.2.1. By Theorem 2.1.2, $\phi\in C^{r-1}$. The number r_0 and the mapping

ϕ depend on the choice of x, so $r_0 = r_0(x)$ and $\phi = \phi(x)$. Put $\delta(x) = r_0(x)A^{-1}(x)$. The inequalities (2.2.6) are a corollary of (2.1.11), (2.2.1), (1.5.4) and (1.5.3). Assertion 1 is obvious, and 2 and 3 follow from (2.1.8). We will prove 4. It is known (see [10]) that there exist δ, C_1 and C_2 such that for any $x, y \in M$ for which $\rho(x, y) \leqslant \delta$ we have

$$C_1 \| \exp_x^{-1} y \|_x \leqslant \rho(x, y) \leqslant C_2 \| \exp_x^{-1} y \|_x. \tag{2.2.8}$$

From (1.5.6) it follows that

$$A^{-1}(x) C_1 \| \exp_x^{-1} y \|'_x \leqslant \rho(x, y) \leqslant \sqrt{2} C_2 \| \exp_x^{-1} y \|'_x. \tag{2.2.9}$$

It follows from the above and (2.1.10) that

$$\rho(f^n(x), f^n(y)) \leqslant \sqrt{2} C_2 \left\| \left(\prod_{i=0}^{n} f_i \right) (u, \varphi_i(u)) \right\|_{f^n(x)}$$

$$\leqslant 200 \sqrt{2} C_2 \varkappa^n(x) \| (u, \varphi(u)) \|'_x \leqslant 200 \sqrt{2} C_2 C_1^{-1} A(x) \varkappa^n(x) \rho(x, y).$$

Now 4 follows from 2 of Theorem 2.1.1. Assertion 5 is deduced from 3 of Theorem 2.1.1 in a similar way. Let $y \in f(V(x)) \cap U(f(x), \delta(f(x)))$; then $f^{-1}(y) \in V(x)$, and, by assertion 4, $f^n(y) = f^{n+1}(f^{-1}(y)) \in U'_{f^{n+1}(x)}$ and

$$\rho(f^n(x), f^n(f^{-1}(y)) = \rho(f^{n-1}(f(x)), f^{n-1}(y))$$

$$\leqslant NA(x) \varkappa^n(x) \rho(x, f^{-1}(y)) = NA(x) \varkappa(x) \varkappa^{n-1}(x) \rho(x, f^{-1}(y)).$$

Since $y \in U(f(x), \delta(f(x)))$, by 5, in which we must put $C = NA(x)\kappa(x)\rho(x, f^{-1}(y))$, we obtain $y \in V(f(x))$. Thus 7 is proved. From (2.1.9) and (2.1.8) it follows that for any $v \in S$

$$\| d\varphi v \|'_0 \leqslant C(x) \| v \|'_0. \tag{2.2.10}$$

Here $C(x)$ is the measurable function on $\hat{\Lambda}$ given by (2.1.11). Therefore $C(x) = A(x)L(x)$, where $L(x)$ is a measurable function constant on the trajectories of the diffeomorphism f. From (2.1.11), (2.1.9), (2.2.10), (2.2.1) and (1.5.6) it follows that for any $y \in V(x)$

$$d(T_y V(x), T_x V(x)) \leqslant \sqrt{2} C_2 \max_{\|v\|_0 \leqslant \|u\|_0} \| d\varphi(v) \|'_0$$

$$\leqslant \sqrt{2} C_2 C(x) \max_{\|v\|_0 \leqslant \|u\|_0} \| v \|'_0 \leqslant \sqrt{2} C_2 C(x) A(x) \max_{\|v\|_0 \leqslant \|u\|_0} \| v \|_0$$

$$\leqslant \sqrt{2} C_2 C(x) A(x) \| \exp_x^{-1} y \|_x \leqslant \sqrt{2} C_2 C_1^{-1} L(x) A^2(x) \rho(x, y).$$

The theorem is proved.

DEFINITION 2.2.1. The submanifold $V(x)$ is called the *local stable manifold passing through x.*

2.3. Some additional properties of local stable manifolds are described in the following result.

THEOREM 2.3.1. 1. *If* $x \in \Lambda^l_{s,r}$, $y \in \Lambda^{l_1}_{s,r}$, $l_1 \geqslant l$, $y \in U(x, \frac{1}{4}\delta^l_{s,r})$ *and* $y \notin V(x)$, *then*

$$V(x)\cap V(y)\cap U\left(y,\frac{1}{4}\delta_{s,r}^{l_1}\right)=\varnothing.$$

2. *If* $x\in\Lambda_{s,r}^{l}$, $y\in V(x)\cap\Lambda_{s,r}^{l_1}$ *and* $l_1\geqslant l$, *then*

$$V(y)\cap U(x,\delta_{s,r}^{l})\subseteq V(x).$$

3. *If* $x\in\Lambda_{s,r}^{l}$, $x_i\in\Lambda_{s,r}^{l}$, $i=1,2,\ldots$, *and* $x_i\to x$, *then* $V(x_i)\to V(x)$ *in the* C^1-*topology in a neighborhood* $U(x,q)$, *where* $0<q<\delta_{s,r}^{l}$.

PROOF. 1. Let $z\in V(x)\cap V(y)\cap U(y,\frac{1}{4}\delta_{s,r}^{l_1})$. Then $z\in U(x,\delta_{s,r}^{l})$, and for every $\epsilon>0$ there are points $z_1,z_2\in U(x,\delta_{s,r}^{l})$ such that $\rho(z_1,z_2)\leqslant\epsilon$, $z_1\in V(x)\cap V(y)$, $z_2\in V(y)$, $z_2\notin V(x)$. On the basis of Theorem 2.2.1 (see 4), for every $n\in\mathbf{Z}^+$ we have

$$\rho(f^n(x),f^n(z_1))\leqslant NA(x)\varkappa^n(x)\,\rho(x,z_1),$$

$$\rho(f^n(z_1),f^n(z_2))\leqslant NA(y)\varkappa^n(y)(\rho(z_1,y)+\rho(z_2,y)).$$

Put $C=NA_{s,r}^{l_1}(\rho(z_1,y)+\rho(z_2,y)+\rho(z_1,x))$. From the latter two inequalities, the triangle inequality and (1.5.4), it follows that for any $n\in\mathbf{Z}^+$

$$\rho(f^n(x),f^n(z_2))\leqslant C(\varkappa_{s,r})^n. \tag{2.3.1}$$

Choose an integer n_0 so that for all $n\geqslant n_0$

$$C(\varkappa_{s,r})^n\leqslant\delta_{s,r}^{l}e^{-10\varepsilon_s n}. \tag{2.3.2}$$

This can be done because $\kappa_{s,r}<e^{-10\epsilon_s n}<1$. Choosing ϵ sufficiently small and using (2.3.1) and (2.3.2), we conclude that $f^n(z_2)\in U'_{f^n(x)}$ for all $n\in\mathbf{Z}^+$. Therefore from Theorem 2.2.1 (see 5) it follows that $z_2\in V(x)$, which contradicts the choice of z_2. Assertion 2 is proved similarly.

3. By Theorem 1.3.1 (see 3), $E_{1x_i}\to E_{1x}$. Therefore from Theorem 2.2.1 it follows that for any q, $0<q<\delta_{s,r}^{l}$, for sufficiently large i, $V(x_i)$ has the form

$$V(x_i)=\{\exp_x(u,\chi_i(u)):\|u\|_x\leqslant q\},$$

where $\chi_i(u)$ is a mapping of class C^{r-1} of the ball of radius q in E_{1x} into the space E_{2x}. From (2.2.9) and 8 of Theorem 2.2.1 follows the existence of a constant $C_{s,r}^{l}$ such that for sufficiently large i and any $u_1,u_2\in E_{1x}$, $\|u_1\|_x\leqslant q$, $\|u_2\|_x\leqslant q$,

$$\sup_{\|u\|_x\leqslant q}\|d\chi_i(u)\|_x\leqslant C_{s,r}^{l},$$

$$\|d\chi_i(u_1)-d\chi_i(u_2)\|_x\leqslant C_{s,r}^{l}\|u_1-u_2\|_x.$$

Therefore the family of functions χ_i is compact in the C^1-topology. Let ψ be a limit point of the sequence χ_i and $\chi_{i_p}\to\psi$. Since $\chi_{i_p}(0)\to 0$ and $d\chi_{i_p}(0)\to 0$, we have $\psi(0)=0$ and $d\psi(0)=0$. Fix $m\in\mathbf{Z}^+$ and $u\in E_{1x}$, $\|u\|_x\leqslant q$. Put

$$z_{i_p}=\exp_x(u,\chi_{i_p}(u)),\qquad z=\exp_x(u,\psi(u)).$$

Since $z_{i_p} \to z$, for every $\epsilon > 0$ for sufficiently large i_p we have

$$\rho(f^m(z_{i_p}), f^m(z)) \leqslant \varepsilon, \quad \rho(f^m(x_{i_p}), f^m(x)) \leqslant \varepsilon,$$
$$\rho(z_{i_p}, z) \leqslant \varepsilon, \quad \rho(x_{i_p}, x) \leqslant \varepsilon.$$

Since $z_{i_p} \in V(x_{i_p})$, (2.2.4) and (1.5.4) imply

$$\rho(f^m(z_{i_p}), f^m(x_{i_p})) \leqslant N(\varkappa_{s,r})^m A^l_{s,r} \rho(z_{i_p}, x_{i_p}).$$

From the above and the triangle inequality it follows that

$$\rho(f^m(z), f^m(x)) \leqslant N A^l_{s,r} (\varkappa_{s,r})^m \rho(x, z).$$

Since m is arbitrary, from the latter inequality and Theorem 2.2.1 (see assertion 5), we obtain

$$\exp_x\{(u, \psi(u))\} = V(x), \quad \|u\|_x \leqslant q.$$

Thus $\chi_i \to \psi$ in the C^1-topology. The theorem is proved.

REMARK 2.3.1. *Without loss of generality, on the strength of Theorems* 2.2.1 *and* 2.3.1 *we may suppose that the number* $\delta^l_{s,r}$ *is so small that for any* $x, y \in \Lambda^l_{k,s,r}$, $y \in U(x, \delta^l_{s,r}/8)$,

$$V(y) \cap U(x, \delta^l_{s,r}) \supset \left\{\exp_x(u, \varphi_y(u)) : u \in B\left(\frac{1}{4}\delta^l_{s,r}\right)\right\},$$

where $\phi_y: B(\frac{1}{4}\delta^l_{s,r}) \to E_{2x}$ *is a mapping of class* C^{r-1}, *with*

$$\max_{y \in \Lambda^l_{s,r} \cap U\left(x, \frac{1}{8}\delta^l_{s,r}\right)} \max_{u \in B\left(\frac{1}{4}\delta^l_{s,r}\right)} [\|\varphi_y(u)\| + \|d\varphi_y(u)\|] \leqslant 1.$$

DEFINITION 2.3.1. Let $x \in \Lambda^l_{k,s,r}$. The collection of local stable manifolds passing through points $y \in \Lambda^l_{k,s,r} \cap U(x, \delta^l_{s,r}/8)$ is called the *family of local stable manifolds* $S^l_{k,s,r}(x)$.

In conclusion we give one further result on local stable manifolds. Let $x \in \hat{\Lambda}$ and $y \in V(x)$. Since the trajectories of the points x and y, under the action of f^n, $n = 0, 1, \dots$, come together with exponential speed, the variational equation along the trajectory of y can be considered as obtained from the variational equation along the trajectory of x by a rapidly decaying (with increasing time) perturbation of the first order. From this and Theorems 15.2.1 and 17.1.1 of [4] follows

PROPOSITION 2.3.1. 1. *Let* $x \in \tilde{\Lambda}$ *and* $y \in V(x)$. *Then* y *is forward regular, and*

$$s(x) = s(y), \quad \chi_i(x) = \chi_i(y), \quad i = 1, \dots, s(x),$$

where $\chi_i(x)$ *is the value of the multiplier* χ^+ *at* x *(see* §0.3).

2. *Let* $x \in \tilde{\Lambda}^l_{s,r}$ *and* $y \in \tilde{\Lambda} \cap V(x)$. *There exists* $K = K(l, s, r)$ *such that for any* $n \in \mathbf{Z}^+$

$$\| df^n v \|_{f^n(y)} \leqslant K\left(\frac{r}{s}\right)^n e^{5\varepsilon_s n} \| v \|_y, \quad v \in E_{1y} = T_y V(x), \tag{2.3.3}$$

$$\| df^n v \|_{f^n(y)} \geqslant K^{-1}\left(\frac{r+2}{s}\right)^n e^{-5\varepsilon_s n} \| v \|_y, \quad v \in E_{2y}. \tag{2.3.4}$$

3. *Let $x \in \hat{\Lambda}$ and $y \in V(x)$. Then for every $x \in T_y V(x)$*

$$\chi^+(y, v) < 0.$$

§3. Absolute continuity of families of local stable manifolds

3.1. Let Λ be the set of positive measure, invariant relative to f, satisfying (0.2). We consider a set $\Lambda^l_{k,s,r}$ (see §1.3) having positive measure. For $y \in \Lambda^l_{k,s,r}$ we denote by $V(y)$ the local stable manifold in the neighborhood $U(y, \delta^l_{s,r})$ of y (see Theorem 2.2.1). Let x be a point of density of $\Lambda^l_{k,s,r}$. We consider the family $S^l_{k,s,r}(x)$ of local stable manifolds (see (2.3.1)). Choose q, $0 < q \leqslant \delta^l_{s,r}/8$, and put

$$\hat{\Lambda}^l_{k,s,r}(x) = \bigcup_{y \in \Delta^l_{k,s,r} \cap \bar{U}(x,q)} \bar{V}(y) \cap \bar{U}(x, q). \tag{3.1.1}$$

It follows from Theorem 1.3.1 that $\hat{\Lambda}^l_{k,s,r}$ is closed in $\bar{U}(x, q)$. Consider a local open smooth submanifold $W \subset U(x, q)$ such that the set $\exp_x^{-1} W$ is the graph of a smooth mapping $\psi\colon U \to E_{1x}$ defined in some neighborhood $U \subset E_{2x}$ by the relation

$$t(\psi(u)) = u, \quad u \in U, \tag{3.1.2}$$

where t is the projection onto E_{2x} parallel to E_{1x}. Put

$$|W| = \max_{u \in U} \| \psi(u) \|_x + \max_{u \in U} \| d\psi(u) \|_x. \tag{3.1.3}$$

It is easy to see that there is a constant $\epsilon^l_{k,s,r} > 0$ such that if

$$|W| \leqslant \varepsilon^l_{k,s,r}, \tag{3.1.4}$$

then the submanifold W intersects each local stable manifold $V(y)$, $y \in \Lambda^l_{k,s,r} \cap U(x, q)$, in not more than one point; moreover, the intersection is transversal. A submanifold W satisfying the above conditions (see (3.1.2) and (3.1.4)) will be called a transversal to the family $S^l_{k,s,r}(x)$.

Let W^1 and W^2 be two smooth submanifolds, transversals to the family $S^l_{k,s,r}(x)$. There exist open subsets $\widetilde{W}^1 \subset W^1$ and $\widetilde{W}^2 \subset W^2$ for which the successor mapping

$$p\colon \hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^1 \to \hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^2$$

is defined. Namely if $y \in \widetilde{W}^1 \cap V(w)$ and $w \in U(x, q) \cap \Lambda^l_{k,s,r}$, then

$$p(y) = \widetilde{W}^2 \cap V(w). \tag{3.1.5}$$

Theorem 2.3.1 implies that p is a homeomorphism.

DEFINITION 3.1.1. The family $S^l_{k,s,r}(x)$ is called *absolutely continuous* if any successor mapping, constructed as above, is absolutely continuous.

3.2. THEOREM 3.2.1. *There exist constants $q^l_{k,s,r}$ and $J^l_{k,s,r}$ satisfying the following conditions*:

1. *The family $S^l_{k,s,r}(x)$ in the neighborhood $U(x, q^l_{k,s,r})$ is absolutely continuous.*
2. *If y is a point of density of $\hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^1$, then the Jacobian $J(p)(y)$ satisfies*

$$|J(p)(y)-1| \leqslant J^l_{k,s,r}(|W^1|+|W^2|). \tag{3.2.1}$$

PROOF. Consider a point $w \in \Lambda^l_{k,s,r} \cap U(x, q)$, $0 < q \leqslant \delta^l_{s,r}/8$, a family of trivializations $\tau_m = \tau'_{f^m(w)}: T_{f^m(w)}M \to \mathbf{R}^n$, and the mappings $f_m = f_{f^{m-1}(w)}$ defined in the neighborhoods $U_m = U_{f^m(w)} \subset T_{f^m(w)}M$ (see §1.6). We denote by $\|\cdot\|'_m$ the measurable Ljapunov norm in $T_{f^m(w)}M$, by $\|\cdot\|_m$ the Riemannian norm in $T_{f^m(w)}M$, and by $\|\cdot\|$ the norm in $\mathbf{R}^n$. We define the numbers λ, μ, ν, κ, α and ϵ_s by (2.2.1). Let

$$\varphi(f^m(w)): B^1(\delta(f^m(w))) \to B^2(\delta(f^m(w)))$$

be the smooth mapping constructed in Theorem 2.2.1. We put $\phi_m = \phi(f^m(w))$ and

$$B^i_m = \{v \in E_{if^m(w)}: \|v\|_m \leqslant q_m\}, \quad i=1, 2, \quad B_m = B^1_m \times B^2_m,$$

where $q_m = q_0 e^{-10\epsilon_s m}$, $q_0 \leqslant \delta(w)$. By Theorem 2.2.1, $B^i_m \subset B^i(\delta(f^m(w)))$, $i = 1, 2$. Consider a point $P = (u_0, v_0) \in B_0$ on the graph of the function ϕ_0, $v_0 = \phi_0(u_0)$, and put, for $m \in \mathbf{Z}^+$,

$$P_m = (u_m, v_m) = f_m(P), \quad u_m \in B^1_m, \quad v_m \in B^2_m.$$

Theorem 2.2.1 (see assertion 7) implies that $v_m = \phi_m(u_m)$.

Put

$$\xi = \frac{r}{s} e^{6\varepsilon_s}, \quad \eta = e^{-6\varepsilon_s}.$$

It is not difficult to verify, using (1.3.1) and (2.2.1), that

$$\varkappa < \xi < \mu, \quad \frac{\lambda}{\mu} < \eta, \quad \xi < \eta < \nu, \quad \xi\nu^{-1} < \eta. \tag{3.2.2}$$

Fix a $\delta_0 > 0$, and write $\delta_m = \delta_0 \xi^m$ and

$$B(v_m, \delta_m) = \{v \in E_{2f^m(m)}: \|v - v_m\|_m \leqslant \delta_m\}.$$

LEMMA 3.2.1. *If $\|v_0\|_0 \leqslant q_0(200\sqrt{2}A^l_{s,r})^{-1}$ and $\delta_0 \leqslant q_0/2$, then $B(v_m, q_m) \subset B^2_m$.*

PROOF. If $v \in B(v_m, \delta_m)$, then from (1 4), (1.5.6), (2.1.10), the conditions of the lemma and (3.2.2) it follows that

$$\|v\|_m \leqslant \|v_m\|_m + \delta_m \leqslant \sqrt{2}\|v_m\|'_m + \delta_m$$
$$\leqslant 200\sqrt{2}\varkappa^m \|v_0\|'_0 + \delta_m \leqslant 200\sqrt{2}A^l_{s,r}\varkappa^m \|v_0\|_0 + \delta_0\xi^m \leqslant q_0\xi^m.$$

The lemma is proved.

LEMMA 3.2.2. *Let $\psi_0: B(v_0, \delta_0) \to E_{10}$ be a smooth mapping, where*

$$\max_{v \in B(v_0, \delta_0)} \|\psi_0(v)\|_0' = C', \qquad \max_{v \in B(v_0, \delta_0)} \|d\psi_0(v)\|_0' = C. \tag{3.2.3}$$

Then there exists a $\hat{q} = \hat{q}(l, s, r, C) \leqslant \delta_{s,r}^l/8$ such that if $q_0 \leqslant \hat{q}$ and $\delta_0 \leqslant q_0/2$, then there is a sequence of smooth mappings $\psi_m: B(v_m, \delta_m) \to E_{1f^m(w)}$, $m > 0$, for which

$$\{(\psi_m(v), v) : v \in B(v_m, \delta_m)\} \subset \{f_m(\psi_{m-1}(v), v) : v \in B(v_{m-1}, \delta_{m-1})\}, \tag{3.2.4}$$

$$a_m = \max_{v \in B(v_m, \delta_m)} \|\psi_m(v)\|_m' \leqslant (200 + CA_{s,r}^l) q_0 \xi^m, \tag{3.2.5}$$

$$b_m = \max_{v \in B(v_m, \delta_m)} \|d\psi_m(v)\|_m' \leqslant C\eta^m. \tag{3.2.6}$$

PROOF. Denote

$$\begin{gathered} C_2 = C, \quad C_1 = 200 + CA_{s,r}^l, \\ \hat{q} = (C+1) MA_{s,r}^l (C_1 + A_{s,r}^l) \min\{(\mu\eta - \lambda), (\mu - \xi)\}. \end{gathered} \tag{3.2.7}$$

Here M is the same as in (1.6.8). Lemma 3.2.1 implies that $\hat{q} > 0$. It is easy to see that if $q_0 \leqslant \hat{q}$, then

$$\mu > MA_{s,r}^l (C_1 + A_{s,r}^l)(C+1) q_0 + \xi. \tag{3.2.8}$$

For the proof of the lemma we proceed by induction. We suppose that we have constructed functions ψ_k, $k = 0, 1, \ldots, m$, satisfying the assertions of the lemma. Put

$$\tilde{v} = \tau_m v, \quad \tilde{B}_m = \tau_m B(v_m, \delta_m), \quad \tilde{\psi}_m(\tilde{v}) = \tau_m \psi_m(v).$$

It follows from (1.6.5) that for $v \in B(v_m, \delta_m)$

$$\begin{gathered} \tau_{m+1} f_{m+1}(\psi_m(v), v) = (\hat{u}, \hat{v}) = (A_{m+1}\tilde{\psi}_m(\tilde{v}) + g_{m+1}(\tilde{\psi}_m(\tilde{v}), \tilde{v}), \\ B_{m+1}\tilde{v} + h_{m+1}(\tilde{\psi}_m(\tilde{v}), \tilde{v})), \end{gathered} \tag{3.2.9}$$

where

$$A_{m+1} = A_{f^m(w)}, \quad B_{m+1} = B_{f^m(w)}, \quad g_{m+1} = g_{f^m(w)}, \quad h_{m+1} = h_{f^m(w)}.$$

Denote by $t_m: \tilde{B}_m \to \mathbf{R}^{n-k}$ the mapping associating the vector $\tilde{v}$ to the vector $\hat{v}$ (see (3.2.9)). We will show that

$$\|t_{m+1}\tilde{v}_1 - t_{m+1}\tilde{v}_2\| \geqslant \xi \|\tilde{v}_1 - \tilde{v}_2\|. \tag{3.2.10}$$

From the mean value theorem it follows that for $v_1, v_2 \in B(v_m, \delta_m)$

$$\|h_{m+1}(\tilde{\psi}_m(\tilde{v}_1), \tilde{v}_1) - h_{m+1}(\tilde{\psi}_m(\tilde{v}_2), \tilde{v})\| \leqslant \sup \|dh_{m+1}(z)\| (\|\tilde{\psi}_m(\tilde{v}_1) - \tilde{\psi}_m(\tilde{v}_2)\| + \|\tilde{v}_1 - \tilde{v}_2\|). \tag{3.2.11}$$

Here the supremum is taken over all points z lying on the segment joining the points $(\tilde{\psi}_m(\tilde{v}_1), \tilde{v}_1)$ and $(\tilde{\psi}_m(\tilde{v}_2), \tilde{v}_2)$. By (1.5.3), (1.5.4) and (1.6.8) we have

$$\| dh_{m+1}(z)\| = \| dh_{m+1}(z) - dh_{m+1}(0)\| \leqslant MA^l_{s,r}\nu^{-m}\|z\|. \tag{3.2.12}$$

Since ψ_m, by the induction hypothesis, satisfies (3.2.5) and (3.2.6), it follows that (3.2.7) and the inequalities (1.5.3), (1.5.4) and (1.5.6) imply the estimates

$$\begin{gathered}\|\widetilde{\psi}_m(\widetilde{v}_1) - \widetilde{\psi}_m(\widetilde{v}_2)\| \leqslant C\eta^m \|\widetilde{v}_1 - \widetilde{v}_2\|, \\ \|z\| \leqslant \max_{i=1,2}\{\|\widetilde{\psi}_m(\widetilde{v}_i)\| + \|\widetilde{v}_i\|\} \leqslant C_1 q_0 \xi^m + A^l_{s,r}\nu^{-m}\delta_0\xi^m.\end{gathered} \tag{3.2.13}$$

Therefore (3.2.11)–(3.2.13) imply that

$$h_{m+1}(\widetilde{\psi}_m(\widetilde{v}_1), \widetilde{v}_1) - h_{m+1}(\widetilde{\psi}_m(\widetilde{v}_2), \widetilde{v}_2)\| \leqslant MA^l_{s,r}(C_1 + A^l_{s,r})(C+1)q_0(\eta\xi\nu^{-2})^m. \tag{3.2.14}$$

On the other hand, by (1.6.6)

$$\|B_{m+1}(\widetilde{v}_1 - \widetilde{v}_2)\| \geqslant \mu\|\widetilde{v}_1 - \widetilde{v}_2\|. \tag{3.2.15}$$

Therefore (3.2.2) and (3.2.8) imply (3.2.10). Since $\delta_{m+1} = \xi\delta_m$, we have

$$f_m(B_m(v_m, \delta_m)) \supset B(v_{m+1}, \delta_{m+1}). \tag{3.2.16}$$

Hence, in turn, it follows that the mapping $\psi_{m+1}(v) = \tau^{-1}_{m+1}t_{m+1}\tau_m(v)$ satisfies (3.2.4).

On the basis of (2.1.10), we obtain

$$a_{m+1} \leqslant \|\widetilde{\psi}_{m+1}(\widetilde{v}_{m+1})\| + \max_{v\in\widetilde{B}_m}\|d\widetilde{\psi}_{m+1}(v)\|\|\widetilde{v}\| \leqslant 200q_0\varkappa^{m+1} + b_{m+1}q_0\xi^{m+1}A^l_{s,r}\nu^{-(m+1)}. \tag{3.2.17}$$

Choose $\hat{\tau}$, near to zero, such that $\hat{v} + \hat{\tau} \in \widetilde{B}_{m+1}$. There exists a unique $\widetilde{\tau} \in \mathbf{R}^{n-k}$ such that

$$\hat{v} + \hat{\tau} = B_{m+1}(\widetilde{v} + \widetilde{\tau}) + h_{m+1}(\widetilde{\psi}_m(\widetilde{v} + \widetilde{\tau}), \widetilde{v} + \widetilde{\tau}).$$

Using (3.2.9), we obtain that

$$\hat{\tau} = B_{m+1}\widetilde{\tau} + h_{m+1}(\widetilde{\psi}_m(\widetilde{v} + \widetilde{\tau}), \widetilde{v} + \widetilde{\tau}) - h_{m+1}(\widetilde{\psi}_m(\widetilde{v}), \widetilde{v}).$$

From (3.2.11)–(3.2.15), in which we must put $\widetilde{v}_1 = \widetilde{v} + \widetilde{\tau}$ and $\widetilde{v}_2 = \widetilde{v}$, it follows that

$$\|\hat{\tau}\| \geqslant (\mu - MA^l_{s,r}(C_1 + A^l_{s,r})(C+1)q_0(\eta\xi\nu^{-2})^m)\|\widetilde{\tau}\|.$$

Again using (3.2.9), we obtain that

$$\begin{gathered}\|\widetilde{\psi}_{m+1}(\widetilde{v} + \widetilde{\tau}) - \widetilde{\psi}_{m+1}(\widetilde{v})\| \leqslant \|A_{m+1}\|\|\widetilde{\psi}_m(\widetilde{v} + \widetilde{\tau}) - \widetilde{\psi}_m(\widetilde{v})\| \\ + \|g_{m+1}(\widetilde{\psi}_m(\widetilde{v} + \widetilde{\tau}), \widetilde{v} + \widetilde{\tau}) - g_{m+1}(\widetilde{\psi}_m(\widetilde{v}), \widetilde{v})\|.\end{gathered} \tag{3.2.18}$$

Since inequalities similar to (3.2.11) and (3.2.12) hold for g_{m+1}, on dividing (3.2.18) by $\|\hat{\tau}\|$, passing to the limit as $\hat{\tau} \to 0$ and taking the least upper bound over all $\widetilde{v} \in \widetilde{B}_m$, we obtain

$$b_{m+1} \leqslant \frac{\lambda b_m + MA^l_{s,r}(C_1 + A^l_{s,r})(C+1)q_0(\eta\xi\nu^{-2})^m}{\mu - MA^l_{s,r}(C_1 + A^l_{s,r})(C+1)q_0(\eta\xi\nu^{-2})^m}.$$

Since, according to the induction hypothesis, $b_m \leqslant C_2\eta^m = C\eta^m$, (3.2.7) implies that $b_{m+1} \leqslant C\eta^{m+1}$. From this (3.2.17) and (3.2.2) it follows that

$$a_{m+1} \leqslant (200q_0 + CA^l_{s,r}q_0)(\eta\xi\nu^{-1})^{m+1} \leqslant C_1 q_0 \xi^{m+1}.$$

The lemma is proved.

From Lemma 3.2.2 and (1.5.3), (1.5.4) and (1.5.6) follows

$$\max_{v\in B(v_m,\delta_m)} \|\psi_m(v)\| \leqslant \sqrt{2}C_1 q_0 \xi^m, \qquad \max_{v\in B(v_m,\delta_m)} \|d\psi_m(v)\| \leqslant \sqrt{2}C_2 A^l_{s,r}(\eta\nu^{-1})^m. \tag{3.2.19}$$

Taking $C = 1$ in (3.2.7), we define a number $\hat{q}$ depending only on the numbers l, k, s and r. There exists a $q^l_{k,s,r}$ such that

$$q^l_{k,s,r} \leqslant (200\sqrt{2}A^l_{s,r})^{-1}\hat{q} \tag{3.2.20}$$

and for any point $w \in U(x, q^l_{k,s,r})$

$$\exp_w B_*((200\sqrt{2}A^l_{s,r})^{-1}\hat{q}) \supset U(x, q^l_{k,s,r}). \tag{3.2.21}$$

We consider in the neighborhood $U(x, q^l_{k,s,r})$ a smooth submanifold W, transversal to the family $S^l_{k,s,r}(x)$. Let $w \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$. We choose $q^l_{k,s,r}$ so small that

the set $\exp_w^{-1} W$ is the graph of a smooth mapping $\psi: U \to E_{1w}$ (see (3.1.2)), given in some neighborhood $U \subset E_{2w}$. (3.2.22)

Thus we will suppose that $q^l_{k,s,r}$ is chosen in accordance with the conditions (3.2.20)–(3.2.22). Without loss of generality we can assume that (see (2.3.3))

$$\max_{v\in U} \|\psi(v)\|'_0 \leqslant 1, \qquad \max_{v\in U} \|d\psi(v)\|''_0 \leqslant 1.$$

Let $y = V(w) \cap W$. Put

$$\exp_w^{-1} y = P = (u_0, v_0), \quad f_m(P) = P_m = (u_m, v_m), \quad m \in \mathbf{Z}^+,$$

$$q(w, W) = 200\sqrt{2}A^l_{s,r}\max_{v\in U}\|\psi(u)\|.$$

Then

$$q(w, W) \leqslant \frac{1}{8}\delta^l_{s,r}, \quad q(w, W) \leqslant \hat{q}, \qquad \|v_0\|_0 \leqslant (200\sqrt{2}A^l_{s,r})^{-1}\, q(w, W). \tag{3.2.23}$$

Further, by (3.1.3), (3.2.3), (3.2.7), (1.5.6) and (3.2.21) there is a constant $C_3 = C_3(l, s, r)$ such that for any $w \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$

$$C_1q(w, W)+C_2\leqslant C_3|W|. \tag{3.2.24}$$

Taking $\delta_0 = \frac{1}{2}q(w, W)$ and applying Lemmas 3.2.1 and 3.2.2, we construct a mapping $\psi_m: B(v_m, \delta_m) \to E_{1f^m(w)}$. We denote

$$W_m=W_m(w, y, \delta_0)=\{\exp_{f^m(w)}(\psi_m(v), v): v\in B(v_m, \delta_m)\}. \tag{3.2.25}$$

It is obvious that W_m is a smooth submanifold in M. By (3.2.4) and (3.2.16)

$$f^{-1}(W_m)\subset W_{m-1}. \tag{3.2.26}$$

We consider in the neighborhood $U(x, q^l_{k,s,r})$ the set $\hat\Lambda^l_{k,s,r}$ (see (3.1.1)), and two smooth submanifolds W^1 and W^2, transversal to the family $S^l_{k,s,r}(x)$ and intersecting $V(w_1)$, $w_1 \in \Lambda^l_{k,s,r} \cap U(q^l_{k,s,r})$, in the points y^1 and y^2 respectively.

Let $W^i_m = W_m(w_1, y^i, \delta_0)$ be the submanifolds defined by (3.2.25), $i = 1, 2$. By (3.2.26)

$$z^1=f^{-m}(W^1_m\cap f^m(\hat\Lambda^l_{k,s,r}))\in W^1_0\cap \hat\Lambda^l_{k,s,r}.$$

Therefore there exists a point $w_2 \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$ such that $z^1 \in V(w_1)$. Put

$$P^i_0=\exp^{-1}_{w_1}y^i=(u^i_0, v^i_0),\quad P^i_k=f_k(P^i)=(u^i_k, v^i_k),$$

$$z^2=V(w_2)\cap W^2,\quad \exp^{-1}_{w_1}z^i=(u_0, v_0),$$
$$\exp^{-1}_{f^k(w_1)}f^k(z^i)=(\bar u^i_k, \bar v^i_k)=f_k(\bar u^i_0, \bar v^i_0),$$
$$i=1, 2,\quad k=1, \ldots, m.$$

LEMMA 3.2.3. *For every $\alpha > 0$ there exists $m_1 = m_1(l, s, r, \alpha)$ such that for any $m \geqslant m_1$*

$$\bar v^2_m\in B(v^2_m, \bar\delta_m), \tag{3.2.27}$$

where $\bar\delta_m = (\delta_0 + \alpha)\xi^m$.

PROOF. We will estimate $\|\bar v^2_m - v^2_m\|_m$. We have

$$\|\bar v^2_m-v^2_m\|_m\leqslant\|\bar v^2_m-\bar v^1_m\|_m+\|\bar v^1_m-v^2_m\|_m. \tag{3.2.28}$$

Since $f^m(z^1) \in W^1_m$, (2.1.10), (1.5.4) and (1.5.6) imply

$$\begin{aligned}\|\bar v^1_m-v^2_m\|_m&\leqslant\|\bar v^1_m-v^1_m\|_m+\|v^1_m-v^2_m\|_m\\ \leqslant\delta_m+\sqrt2(\|v^1_m\|'_m+\|v^2_m\|'_m)&\leqslant\delta_m+200\sqrt2A^l_{s,r}(\varkappa_{s,r})^m(|W^1|+|W^2|).\end{aligned} \tag{3.2.29}$$

Since $z^1, z^2 \in V(w_2)$, (2.2.8), (2.2.4) and (1.5.3) imply

$$\begin{aligned}\|\bar v^2_m-\bar v^1_m\|_m&\leqslant C_1^{-1}\rho(f^m(z^1), f^m(z^2))\\ \leqslant C_1^{-1}N(\varkappa_{s,r})^mA^l_{s,r}\rho(z^1, z^2)&\leqslant C_1^{-1}NA^l_{s,r}(\varkappa_{s,r})^m(|W^1|+|W^2|).\end{aligned} \tag{3.2.30}$$

Thus it follows from (3.2.28)–(3.2.30) and (3.2.2) that $\|\bar{v}_m^2 - v_m^2\|_m \leqslant \delta_m$ if m is so large that

$$A_{s,r}^l (200\sqrt{2} + C_1^{-1}N)(\varkappa_{s,r})^m (|W^1| + |W^2|) \leqslant \alpha\xi^m.$$

The lemma is proved.

LEMMA 3.2.4. *For any $\alpha > 0$, $\delta_0 \leqslant \frac{1}{2}(|W^1| + |W^2|)$ and $m \geqslant m_1(\alpha)$ (see Lemma 3.2.3)*

$$p(f^{-m}(W_m(w, y^1, \delta_0)) \cap \hat{\Lambda}_{k,s,r}^l) \subset f^{-m}(W_m(w, y^2, \delta_0 + \alpha)),$$

where p is the successor mapping connected with the manifolds W^1 and W^2.

The proof follows from Lemmas 3.2.1, 3.2.3, the inclusion (3.2.26) and the inequality (3.2.23).

The last assertion shows that to compare the measures of the sets $A \subset \hat{\Lambda}_{k,s,r}^l \cap W^1$ and $p(A)$ we must learn to compare the parts of these sets lying in the inverse images of the submanifolds $W_m(w, y^1, \delta_0)$ and $W_m(w, y^2, \delta_0 + \alpha)$ respectively.

LEMMA 3.2.5 (see [3], Lemma 6). *Let $z^1 = f^m(z) \in W_m^1$. There exist constants $C_5 = C_5(l, s, r)$ and $C_6 = C_6(l, s, r)$ such that*

$$\begin{aligned} &\left| \, |df_{f^m(z)}^{-m}|_{T_{f^m(z)}W_m^1}| \cdot |df_{f^m(y^1)}^{-m}|_{T_{f^m(y^1)}W_m^1}|^{-1} - 1 \right| \leqslant C_5(|W^1| + |W^2|), \\ &\left| \, |df_{f^m(y^2)}^{-m}|_{T_{f^m(y^2)}W_m^2}| \cdot |df_{f^m(y^1)}^{-m}|_{T_{f^m(y^1)}W_m^1}|^{-1} - 1 \right| \leqslant C_6(|W^1| + |W^2|), \end{aligned} \tag{3.2.31}$$

where $|B|$ denotes the coefficient of volume expansion for the mapping B.

PROOF. Since f is of class C^2, there exists a $C_7 > 0$ such that

$$\|df_{z_1}^{-1} - df_{z_2}^{-1}\| \leqslant C_7\rho(z_1, z_2),$$

where z_1 and z_2 are arbitrary points in M. Therefore there exists C_8 such that for any subspace A

$$\left| \, |df_{z_1}^{-1}|_A| - |df_{z_2}^{-1}|_A| \, \right| \leqslant C_8\rho(z_1, z_2). \tag{3.2.32}$$

In addition, there exists a $C_9 > 0$ such that for any $z \in M$ and two subspaces $A_1, A_2 \subset T_zM$ of the same dimension

$$\left| \, |df_z^{-1}|_{A_1}| - |df_z^{-1}|_{A_2}| \, \right| \leqslant C_9\, d(A_1, A_2). \tag{3.2.33}$$

Denote

$$E_z^k = T_{f^{-k}(z)}W_k^1, \quad J(k, z) = |df_{f^{-k}(z)}^{-1}|_{E_z^k}|.$$

It follows from (3.2.19), (3.2.24), (3.2.32) and (3.2.33) that

$$|J(k, z) - J(k, y_1)| \leqslant C_8 \rho(f^{-k}(z), f^{-k}(y_1)) + C_9 d(E^k_{y_1}, E^k_z) \leqslant C_{10}(|W^1| + |W^2|)(\eta\nu^{-1})^k, \tag{3.2.34}$$

where $C_{10} = C_{10}(l, s, r)$ is a constant. Obviously there exists a $J_0 > 0$ such that

$$J_0^{-1} \leqslant J(k, z) \leqslant J_0. \tag{3.2.35}$$

The first inequality in (3.2.31) follows from (3.2.34) and (3.2.35). The second inequality follows similarly. The lemma is proved.

We denote by μ^i_m the Riemannian volume in $f^m(W^i)$ induced by the Riemannian metric of the manifold M, $i = 1, 2$.

LEMMA 3.2.6. *There exists $C_{11} = C_{11}(l, s, r) > 0$ such that for any $m \in \mathbf{Z}^+$, $a > 0$, $\delta_0, \delta_1 \leqslant \frac{1}{2}(|W^1| + |W^2|)$ and any two measurable sets $X^1 \subset W^1_m(w, y^1, \delta_0)$ and $X^2 \subset W^2_m(w, y^2, \delta_1)$ for which $|\mu^1_m(X^1)(\mu^2_m(X^2))^{-1} - 1| \leqslant a$, we have*

$$\left| \frac{\mu^1_0(f^{-m}(X^1))}{\mu^2_0(f^{-m}(X^2))} - 1 \right| \leqslant C_{11}(|W^1| + |W^2| + a). \tag{3.2.36}$$

PROOF. Denote

$$A_i(z) = |df_z^{-m}|_{T_z W^i_m}|, \quad i = 1, 2.$$

For $i = 1, 2$ we have

$$\mu^i_0(f^{-m}(X^i)) = \int_{X^i} A_i(z)\, d\mu^i_m(z) = A_i(z_i)\, \mu^i_m(X^i),$$

where $z_i \in X^i$. Therefore from the conditions of the lemma and (3.2.31) it follows that

$$\left| \frac{\mu^1_0(f^{-m}(X^1))}{\mu^2_0(f^{-m}(X^2))} - 1 \right| \leqslant \left| \frac{A_1(z_1)}{A_2(z_2)} - 1 \right| \frac{\mu^1_m(X^1)}{\mu^2_m(X^2)} + \left| \frac{\mu^1_m(X_1)}{\mu^2_m(X^2)} - 1 \right| \leqslant \left| \frac{A_1(z_1)}{A_1(y^1)} \frac{A_2(y^2)}{A_2(z_2)} \frac{A_1(y^1)}{A_2(y^2)} - 1 \right| \times \frac{\mu^1_m(X^1)}{\mu^2_m(X^2)} + a \leqslant C_{11}(|W^1| + |W^2| + a).$$

The lemma is proved.

We will now show how to cover the "nice" parts of $f^m(W^1)$, i.e. the sets $f^m(\hat{\Lambda}^l_{k,s,r} \cap W^1)$, by certain sets lying in the submanifolds $W_m(w_j, y^i, \delta_0)$ with centers at certain points w_j. This covering will not only have finite multiplicity not depending on m (such a cover was constructed, for example, in [3]), but will be almost a decomposition; that is, the sum of the measures of the intersections of elements of the cover is sufficiently small by comparison with the value $|W^1| + |W^2|$. Here certain "superfluous" pieces of the submanifold $f^m(W^1)$ are covered and we must take care that the measure of these pieces should be sufficiently small. For this we use a method proposed in [3].

Let B_t be a closed ball in W^1 of radius t and $A = \hat{\Lambda}^l_{k,s,r} \cap B_t$. We select a number $\beta > 0$ such that

$$\left| \frac{\mu^1(B_{t+\beta})}{\mu^1(B_t)} - 1 \right| \leqslant \frac{1}{2}(|W^1| + |W^2|). \tag{3.2.37}$$

LEMMA 3.2.7. *There exist δ_0, $0 < \delta_0 \leqslant \frac{1}{2}(|W^1| + |W^2|)$, and for each $m \in \mathbf{Z}^+$ sets of points $\{w_j\}$ and $\{y_j^1\}$, $j = 1, \ldots, n_1$, satisfying the following conditions:*

1. $w_j \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$, $y_j^1 \in U(x, q^l_{k,s,r}) \cap V(w_j)$, $j = 1, \ldots, n_1$.

2. $f^m(A) \subset \widetilde{W}^1_m = \bigcup_{j=1}^{n_1} W^1_m\left(w_j, y_j^1, \frac{1}{2}\delta_0\right) \subset \hat{W}^1_m = \bigcup_{j=1}^{n_1} \hat{W}^1_m(w_j, y_j^1, \delta_0)$ $\subset f^m(B_{t+\beta})$.

PROOF. Let $w \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$ and $y^1 \in W^1 \cap V(w)$, and let $Q(y^1, \gamma)$ be the ball in the submanifold W^1 with center at y^1 and radius γ. There exists $t = t(\gamma) \leqslant \gamma$, not depending on w and y, such that

$$W_0^1(w, y^1, t(\gamma)) \subset Q(y^1, \gamma). \tag{3.2.38}$$

Put

$$\delta_0 = \min\left\{t\left(\frac{\beta}{4}\right), \ \frac{1}{2}(|W^1| + |W^2|)\right\}. \tag{3.2.39}$$

Fix $m > 0$ and choose from the covering set

$$\bigcup_{w \in \Lambda^l_{k,s,r}} \overline{W}^1\left(w, y^1, \frac{\delta_0}{2}\right)$$

a finite subcover. The corresponding sets we denote by $W^1_m(w_j, y_j^1, \frac{1}{2}\delta_0)$, $j = 1, \ldots, n_1$. We must show that $\hat{W}^1_m \subset f^m(B_{t+\beta})$. If this is not so, then there exists a point

$$p \in \partial(f^m(B_{t+\beta})) \cap W^1_m(w_j, y_j^1, \delta_0)$$

for some j. By (3.2.38), (3.2.39) and (3.2.26), $f^{-m}(p) \in Q(y_j^1, \frac{1}{4}\beta)$ and, by the same token, $f^{-m}(p) \notin \partial(B_{t+\beta})$. This contradiction proves the lemma.

The following lemma on covers with bounded (independent of m) multiplicity is proved by a simple modification of the argument of [3] (§2, Lemma 7).

LEMMA 3.2.8. *There exist $d_0 > 0$, an integer $L > 0$ and for every $m \in \mathbf{Z}^+$ a set of balls $Q(z_j, d_m)$, $j = 1, \ldots, n_2$, in the submanifold $f^m(W^1)$ with centers at points z_j and radii $d_m = d_0\xi^m$ satisfying the following conditions:*

1) $\widetilde{W}^1_m \subset \bigcup_{j=1}^{n_2} Q(z_j, d_m) \subset \bigcup_{j=1}^{n_2} Q(z_j, 2d_m) \subset \hat{W}^1_m$.

2) *For each $j = 1, \ldots, n_2$ there is an index i, $1 \leqslant i \leqslant n_1$, such that*

$$Q(z^j, 2d_m) \subset W^1_m(w_i, y_i^1, \delta_0).$$

3) *The multiplicity of the cover of $\widetilde{W}^1_m$ by the sets $Q(z_j, 2d_m)$ is equal to L.*

Fix $m > 0$ and consider any ball $Q(z_j, 2d_m)$. For some i, $1 \leqslant i \leqslant n_1$,

$$Q(z_j, 2d_m) \subset W_m^1(w_i, y_i^1, \delta_0).$$

We denote

$$\hat{Q}_j = \exp_{w_i}^{-1} Q(z_j, 2d_m), \quad \hat{z}_j = \exp_{w_i}^{-1} z_j, \quad \tilde{z}_j = t_{w_i}^m(\hat{z}_j),$$

where $t_{w_i}^m$ is the projection onto $E_{2f^m(w_i)}$ parallel to $E_{1f^m(w_i)}$.

LEMMA 3.2.9. *For each* ϵ, $0 < \epsilon \leqslant 1$, *there exists* $m_2 = m_2(\epsilon, l, s, r) \in \mathbf{Z}^+$ *such that for any* $m \geqslant m_2$

$$\rho(\partial(t_{w_i}^m(\hat{Q}_j)), \partial B(\tilde{z}_j, 2d_m)) \leqslant \varepsilon d_m.$$

The proof follows from (3.2.1) of Lemma 3.2.2, (3.2.19) and (3.2.25).

Fix an arbitrary ϵ, $0 < \epsilon \leqslant 1$, and consider a cover of the ball $B(\tilde{z}_j, (2-\epsilon)d_m)$ by closed cubes $\tilde{D}_{ij}$, $i = 1, \ldots, N_j$, of diameter ϵd_m with disjoint interiors. The cubes are constructed in the obvious way relative to the Euclidean structure on $E_{2f^m(w_j)}$ generated by the Riemannian metric. We denote by $V(\tilde{D}_{ij})$ the Riemannian volume of the cube $\tilde{D}_{ij}$. It is easy to see that for $\alpha_m = \alpha\xi^m$, $\alpha > 0$,

$$\left| \frac{V((\tilde{D}_{ij})_{\alpha_m})}{V(\tilde{D}_{ij})} - 1 \right| \leqslant C_{12} \frac{\alpha}{\varepsilon d_0}, \tag{3.2.40}$$

where C_{12} is a constant and $(\tilde{D}_{ij})_\alpha$ denotes the α-blowing up of $\tilde{D}_{ij}$. Put (see (3.2.25))

$$D_{ij} = \exp_{w_i}\{(\psi_m^1(v), v) : v \in \tilde{D}_{ij}\}. \tag{3.2.41}$$

(We recall that the graph of ψ_m^1 coincides with $\exp_{w_i}^{-1} W_m^1$.) It is easy to see that the sets D_{ij} intersect only on the boundary, and if $m \geqslant m_2$ (see Lemma 3.2.9), then

$$Q(z_j, 2d_m) \supset \bigcup_{i=1}^{N_j} D_{ij} \supset Q(z_j, d_m). \tag{3.2.42}$$

Making similar constructions for each ball $Q(z_j, 2d_m)$, $j = 1, \ldots, n_2$, we construct sets D_{ij} which, by Lemma 3.2.8 (see assertion 1) and (3.2.42), cover the set $\tilde{W}_m^1$. The multiplicity of the cover, as before, does not exceed L. We will show how it is possible, neglecting part of the elements of this cover, to obtain a cover which at most (in measure) points has multiplicity 1. For this, passing successively from $Q(z_j, 2d_m)$ to $Q(z_{j+1}, 2d_m)$, we neglect those sets D_{ij+1} whose centers are contained in the union of the sets D_{ik}, $k \leqslant j$, retained in the preceeding steps. Let D_i, $i = 1, \ldots, N$, be the renumbered elements of the cover obtained. It is not difficult to see that

$$\bigcup_{i=1} D_i = \bigcup_{j=1}^{n_2} \bigcup_{i=1}^{N_j} D_{ij}. \tag{3.2.43}$$

LEMMA 3.2.10. *There exists a $C_{13} = C_{13}(l, s, r)$ such that for every $\epsilon > 0$ and any $m \geq m_2$ (see Lemma 3.2.9)*

1. $\widetilde{W}^1_m \subset \bigcup_{i=1}^N D_i \subset \widehat{W}^1_m$,
2.

$$\left| \frac{\sum_{i=1}^N \mu_0^1(f^{-m}(D_i))}{\mu_0^1(f^{-m}(\bigcup_{i=1}^N D_i))} - 1 \right| \leq C_{13}\varepsilon. \tag{3.2.44}$$

PROOF. The first assertion follows immediately from (3.2.43) and Lemma 3.2.8 (see assertion 1). In order to prove the second assertion we consider the cover of $\widetilde{W}^1_m$ by the sets $Q(z_j, (2 - \epsilon)d_m)$, $j = 1, \ldots, n_2$. Since the multiplicity of the cover does not exceed L, we have

$$\sum_{j=1}^{n_2} \mu_m^1(Q(z_j, (2-\varepsilon)d_m)) \leq L\mu_m^1(\bigcup_{j=1}^{n_2} Q(z_j, (2-\varepsilon)d_m)) \leq L\mu_m^1(\bigcup_{i=1}^N D_i). \tag{3.2.45}$$

We denote by $n(x)$ the number of sets D_i containing x, and put

$$D_{\varepsilon j} = \{x \in f^m(W^1) : \overline{\rho}(x, \partial Q(z_j, (2-\varepsilon)d_m)) \leq \varepsilon d_m\}.$$

Here $\overline{\rho}$ is the distance on the submanifold $f^m(W)$ induced by the Riemannian metric. It is obvious that

$$\mu_m^1(D_{\varepsilon j}) \leq C_{14}\varepsilon\mu(Q(z_j, (2-\varepsilon)d_m)), \tag{3.2.46}$$

where C_{14} is a constant. Therefore if $D_\epsilon = \bigcup_1^{n_2} D_{\epsilon j}$, then

$$\mu_m^1(D_\varepsilon) \leq \sum_{j=1}^{n_2} \mu_m^1(D_{\varepsilon j}) \leq C_{14}\varepsilon \sum_{j=1}^{n_2} \mu_m^1(Q(z_j, (2-\varepsilon)d_m)). \tag{3.2.47}$$

It is easy to see also that

$$n(x) = 1, \quad x \notin D_\varepsilon \cup (\bigcup_{i=1}^N \partial D_i). \tag{3.2.48}$$

Therefore (3.2.45)–(3.2.48) imply

$$\sum_{i=1}^N \mu(D_i) = \int_{\bigcup_{i=1}^N D_i} n(x)\,d\mu_m^1(x) \leq \mu_m^1(\bigcup_{i=1}^N D_i) + (L-1)\mu_m^1(D_\varepsilon) \leq \mu_m^1(\bigcup_{i=1}^N D_i)(1 + (L-1)LC_{14}\varepsilon).$$

Now (3.2.44) follows from Lemma 3.2.7, in which we must put $X^2 = \bigcup_1^N D_i$ and $X^1 = X^2 \setminus D_\epsilon$. The lemma is proved.

We complete the proof of the theorem. Choose $\alpha > 0$ and consider the set

$$\hat{D}_i = \exp_{w_i}^{-1}\{(\psi_m^2(v), v) : v \in (\tilde{D}_i)_\alpha\} \subset W_m^2(w_j, y_j^1, \delta_0).$$

Since

$$\bigcup_{j=1}^{N} W_m^2(w_j, y_j^2, \delta_0+\alpha) \subset \bigcup_{i=1}^{N} \hat{D}_i,$$

on the basis of Lemma 3.2.4 we have for $m \geqslant m_1$

$$\bigcup_{i=1}^{N} \hat{D}_i \supset f^m(p(A)). \tag{3.2.49}$$

In addition, from the continuity of the Riemannian metric on M, the inequalities (3.2.19) and (3.2.40) and the definition of μ (see [10]), it follows that there exists a constant $C_{16} = C_{16}(l, s, r)$ such that

$$\left| \frac{\mu_m^1(D_i)}{\mu_m^2(\hat{D}_i)} - 1 \right| \leqslant C_{16}\left(\frac{\alpha}{\varepsilon d_0} + |W^1| + |W^2|\right). \tag{3.2.50}$$

It follows from (3.2.49), (3.2.50), (3.2.36), (3.2.44), (3.2.37) and Lemmas 3.2.6, 3.2.7, 3.2.9 and 3.2.10 that for $m \geqslant \max\{m_1(\alpha), m_2(\epsilon)\}$

$$\begin{aligned} \mu_0^2(p(A)) &\leqslant \sum_{i=1}^{N} \mu_0^2(f^{-m}(\hat{D}_i)) \\ &\leqslant \left(1 + C_{11}(|W^1| + |W^2|) + C_{16}\left(\frac{\alpha}{\varepsilon d_0} + |W^1| + |W^2|\right)\right) \\ &\times \sum_{i=1}^{N} \mu_0^1(f^{-m}(D_i)) \leqslant \left(1 + C_{17}\left(|W^1| + |W^2| + \frac{\alpha}{\varepsilon d_0} + \varepsilon\right)\right) \\ &\times \mu_0(B_{t+\beta}) \leqslant \left(1 + C_{18}\left(|W^1| + |W^2| + \frac{\alpha}{\varepsilon d_0} + \varepsilon\right)\right)\mu_0^1(B_t). \end{aligned}$$

Here $C_{17} = C_{17}(l, s, r)$ and $C_{18} = C_{18}(l, s, r)$ are constants. Since the ball B_t is chosen arbitrarily, the mapping p is absolutely continuous. Now let z be a point of density of $B = W^1 \cap \Lambda_{k,s,r}^l$, and $B_t(z)$ the ball in W^1 with center at z and radius t. For sufficiently small t

$$\left| \frac{\mu^1(B \cap B_t(z))}{\mu^1(B_t(z))} - 1 \right| \leqslant \min\left\{\frac{1}{2}, \frac{1}{2}(|W^1| + |W^2|)\right\}.$$

Therefore from the above it follows that

$$\mu_0^2(p(B \cap B_t(z))) \leqslant \left(1 + C_{19}\left(|W^1| + |W^2| + \frac{\alpha}{\varepsilon d_0} + \varepsilon\right)\right)\mu_0^1(B \cap B_t(z)),$$

where $C_{19} = C_{19}(l, s, r)$ is a constant. Since the numbers α, ϵ and d_0 are chosen arbitrarily, where α can be chosen after d_0 and ϵ, it follows that

$$\mu_0^2(p(B \cap B_t(z))) \leqslant (1 + C_{19}(|W^1| + |W^2|))\mu_0^1(B \cap B_t(z)). \qquad (3.2.51)$$

Applying similar assertions to the mapping p^{-1}, given on the set $p(B \cap B_t(z))$, we obtain that

$$\mu_0^1(B \cap B_t(z)) \leqslant (1 + C_{19}(|W^1| + |W^2|))\mu_0^2(p(B \cap B_t(z))). \qquad (3.2.52)$$

The inequality (3.2.1) follows from (3.2.51) and (3.2.52). The theorem is proved.

3.3. Consider a set $\Lambda^l_{k,s,r}$ of positive measure, and let x be a point of density of $\Lambda^l_{k,s,r}$. There exists a partition ξ^{sm} of the neighborhood $U(x, q^l_{k,s,r})$ (the number $q^l_{k,s,r}$ was constructed in Theorem 3.2.1), each element of which is an $(n-k)$-dimensional submanifold transversal to the submanifold $V(x)$. We denote the elements of this partition by $C_{\xi^{sm}}(y)$, $y \in V(x)$. Let $A \subset C_{\xi^{sm}}(x) \cap \hat{\Lambda}^l_{k,s,r}$ be a measurable set. Put

$$\hat{A} = \bigcup_{z \in \Lambda^l_{k,s,r}, V(z) \cap A \neq \varnothing} (V(z) \cap U(x, q^l_{k,s,r})). \qquad (3.3.1)$$

Using the absolute continuity of the families of local stable manifolds, we will show that the partition ζ of $\hat{A}$ into the submanifolds $V(z)$, $z \in \Lambda^l_{k,s,r}$, $V(z) \cap A \neq \varnothing$, has a system of conditional measures absolutely continuous relative to the measure on $V(z)$ induced by the Riemannian metric. The corresponding argument is due to Ja. G. Sinaĭ (see [2], Russian p. 153, English pp. 147–148).

Let μ_y be the measure on $C_{\xi^{sm}}(y)$ induced by the restriction of the Riemannian metric on M to $C_{\xi^{sm}}(y)$. In it we introduce another measure $\hat{\mu}_y$, putting for any measurable set A

$$\hat{\mu}_y(A) = \nu(\hat{A}).$$

Let ν_z denote the measure on the submanifold $V(z)$ induced by the Riemannian metric.

PROPOSITION 3.3.1. 1. *$\hat{\mu}_y$ is absolutely continuous relative to μ_y.*

2. *The partition ζ is measurable, and the conditional measure $\hat{\nu}_z$ on an element of the partition is absolutely continuous relative to the measure ν_z, $z \in \Lambda^l_{k,s,r}$, $V(z) \cap A \neq \varnothing$.*

3. *For almost all $y \in A$, $\nu_2(V(z)) > 0$, where $z \in \Lambda^l_{k,s,r}$ and $V(z) \cap A = y$.*

PROOF. It follows from Theorem 2.3.1 that ζ is measurable. Denote by $\tilde{\mu}_y$ the conditional measure induced by ν on the element $C_{\xi^{sm}}(y)$ and by κ the measure in the quotient space $\hat{A}/_{\xi sm} = X$. It is obvious that there exist a measurable function $t_y = t_y(s)$, $s \in C_{\xi^{sm}}(y)$, $\phi = \phi(y)$, $y \in V(z)$, $z \in \Lambda^l_{k,s,r}$, such that

$$d\tilde{\mu}_y(s) = t_y(s)\, d\mu_y(s), \quad d\kappa(y) = \varphi(y)\, d\nu_z(y).$$

Let $B \subset \hat{A}$ be an arbitrary set of positive measure. Then from the above and Fubini's theorem it follows that

$$\begin{aligned}\nu(B) &= \int_X d\varkappa(y) \int_{C_{\xi \mathrm{sm}}(y)} \chi_B(s)\, d\tilde{\mu}_y(s) = \int_X d\varkappa(y) \int_{C_{\xi \mathrm{sm}}(y)} \chi_B(s)\, t_y(s)\, d\mu_y(s) \\ &= \int_X d\varkappa(y) \int_{C_{\xi \mathrm{sm}}(q_0)} J_{q_0}(q)\, \chi_B(p_{q_0}(q))\, t(p_{q_0}(q))\, d\mu_{q_0}(q) \\ &= \int_{C_{\xi \mathrm{sm}}(q_0)} d\mu_{q_0}(q) \int_X J_u(q)\, \chi_B(p_u(q))\, t(p_u(q))\, d\varkappa(u) \\ &= \int_{C_{\xi \mathrm{sm}}(q_0)} d\mu_{q_0}(q) \int_{V(q)} J_u(q)\, \chi_B(p_u(q))\, t(p_u(q))\, \varphi(u)\, d\nu_q(u).\end{aligned}$$

Here $J_{q_0}(q)$ is the positive measurable function arising from the absolute continuity of the successor mapping $p_{q_0}: C_{\xi \mathrm{sm}}(q_0) \to C_{\xi \mathrm{sm}}(y)$, $p_{q_0}(q) = s$. (We note that $J_{q_0}(q) = 0$ for $q \notin C_{\xi \mathrm{sm}}(q_0) \cap \hat{A}$.) For fixed q and variable q_0 the points $p_{q_0}(q)$ run through $V(q)$. From the latter equality it follows that on the elements $C_\zeta(q)$ the conditional measure is absolutely continuous relative to the measure ν_q, and the measure on $\hat{A}/\zeta$ is absolutely continuous relative to μ_y. Assertions 1 and 2 are proved.

In order to prove assertion 3 we denote

$$Y = \{z \in \hat{A} \cap \Lambda^l_{k,s,r} : \nu_z(C_\zeta(z)) = 0\},$$

and let $Z = \bigcup_{y \in Y} C_\zeta(y)$. Obviously $Y \subset Z$. From 1 and 2 it follows that

$$\nu(Y) \leqslant \nu(Z) = \int_{Z/\zeta} \hat{\nu}_z(C_\zeta(z))\, d\hat{\mu}_x(z) = \int_{Z/\zeta} l(z)\, \nu_z(C_\zeta(z))\, d\hat{\mu}_x(z) = 0.$$

Here $l(z)$ is the measurable function arising from the absolute continuity of $\hat{\nu}_z$ relative to ν_z. The proposition is proved.

3.4. PROPOSITION 3.4.1. *Let $\nu(\Lambda_{k,s,r}) > 0$. There exists a set $N \subset M$ of measure zero such that for any $l \in \mathbf{Z}^+$ and $x \in \Lambda^l_{k,s,r} \setminus N$*

$$\nu_x(V(x) \cap \Lambda_{k,s,r}) = \nu_x(V(x))$$

(recall that ν_x is the Riemannian volume on $V(x)$ induced by the Riemannian metric on M).

PROOF. Let N_1 be the set of nonregular points in M. We have $\nu(N_1) = 0$ (see Theorem 0.3). Let $x \in \Lambda_{k,s,r}$. Put $A_x = \{y \in V(x): y \in N_1\}$ and consider the set N_2 of those points $x \in \Lambda_{k,s,r}$ for which $\nu_x(A_x) > 0$. The set N_2 is measurable. We will show that $\nu(N_2) = 0$. If this is not so, then for some $l \in \mathbf{Z}^+$ the set $N_2 \cap \Lambda^l_{k,s,r}$ has positive measure. Let x be a point of density of this set. By Proposition 3.3.1, $\nu_y(V(y) \cap N_1) = 0$ for almost all $y \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$. Thus the set of those points $y \in \Lambda^l_{k,s,r} \cap U(x, q^l_{k,s,r})$ for which $\nu_y(V(y) \cap N_1) > 0$ has zero measure. This contradiction shows that $\nu(N_2) = 0$. If $x \in \Lambda^l_{k,s,}$ [illegible] $\cup N_2)$, then almost any (relative to ν_x) point $y \in V(x)$ is a regular point, and, by Pr[illegible]ition 2.3.1, $y \in \Lambda_{k,s,r}$. The proposition is proved.

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