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NON-ABSOLUTELY CONTINUOUS FOLIATIONS

BY

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ABSTRACT

We consider a partially hyperbolic diffeomorphism of a compact smooth manifold preserving a smooth measure. Assuming that the central distribution is integrable to a foliation with compact smooth leaves we show that this foliation fails to have the absolute continuity property provided that the sum of Lyapunov exponents in the central direction is not zero on a set of positive measure. We also establish a more general version of this result for general foliations with compact leaves.

1. Introduction

Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a *smooth* measure μ (i.e., a measure that is equivalent to the Riemannian volume on M). The tangent bundle TM can be split into three df-invariant continuous subbundles

$$TM = E^s \oplus E^c \oplus E^u$$

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such that df contracts uniformly over $x \in M$ along the *stable* subspace $E^s(x)$, it expands uniformly along the *unstable* subspace $E^u(x)$, and it may act either as non-uniform contraction or expansion with weaker rates along the *central* subspace $E^c(x)$. See Section 2 for definitions.

It is well-known that the distributions E^s and E^u are (uniquely) integrable to stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ that possess the crucial *absolute continuity property* (see the definition in the next section). On the other hand, the central distribution may not be integrable and even if it is, the corresponding foliation $\mathcal{W}^c$ may fail to satisfy the absolute continuity property in some very strong way — the phenomenon known as "Fubini nightmare" (see [12, 15, 16]). Our goal is to show that the failure of absolute continuity is a generic phenomenon in a sense. More precisely, we describe some general conditions that guarantee non-absolute continuity of the central foliation.

The absolute continuity property can be understood in a variety of ways. The most common and most strong interpretation of absolute continuity is obtained through holonomy maps associated with the foliation. The others require that the conditional measures generated by the Riemannian volume on leaves of the foliation are equivalent to the leaf volume. This can also be stated in a weaker or stronger sense leading to two other definitions of absolute continuity. We shall discuss all these three interpretations in the next section.

This work was initiated when A. Wilkinson discovered and sent to the second author a copy of a hand-written note [10] from R. Mañé to M. Shub in which the case of one-dimensional central distributions was considered (indeed, Mañé considered more general one-dimensional foliations whose leaves have finite length; see below). Our approach is an elaboration and generalization of Mañé's approach.

We say that a partially hyperbolic diffeomorphism preserving a Borel probability measure μ is **center-dissipative** if $\chi^c(x) \neq 0$ for μ-almost every $x \in M$, where $\chi^c(x)$ denotes the sum of the Lyapunov exponents of f at the point x along the central subspace $E^c(x)$. Recall that μ is a smooth measure if it is equivalent to the Riemannian volume in M.

THEOREM 1.1: *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that*

(1) *the central distribution E^c is integrable to a foliation $\mathcal{W}^c$ with smooth compact leaves;*
(2) *f is center-dissipative on a set of full μ-measure.*

Then the central foliation $\mathcal{W}^c$ is not absolutely continuous.

We say that a partially hyperbolic diffeomorphism preserving a Borel probability measure μ has **negative central exponents** if for μ-almost every $x \in M$, all the Lyapunov exponents along the central distribution are negative. The definition of having **positive central exponents** is analogous. Observe that in the case of one-dimensional central distribution center-dissipativity is equivalent to having negative (positive) central exponents.

As an immediate corollary of Theorem 1.1 we obtain the following result.

THEOREM 1.2: *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that*

(1) *the central distribution E^c is integrable to a foliation $\mathcal{W}^c$ with smooth compact leaves;*
(2) *f has negative (positive) central exponents.*

Then the central foliation $\mathcal{W}^c$ is not absolutely continuous. Moreover, if μ is ergodic then the conditional measures induced by μ on leaves of $\mathcal{W}^c$ are atomic.

For a general center-dissipative diffeomorphism, the conditional measure on central leaves, though not smooth, may not be atomic as the following example illustrates.

Example: Let A be a hyperbolic automorphism of the torus $\mathbb{T}^2$ and F the direct product map of A and the identity map of the circle $\mathbb{T}^1$. F is partially hyperbolic and so is any of its perturbation. Shub and Wilkinson [16] showed that arbitrarily close to F in the C^1 topology there is a C^2 volume-preserving ergodic diffeomorphism G with negative central exponents. The central foliation $\mathcal{W}^c_G$ consists of invariant circles and is not absolutely continuous: The conditional measure generated by volume on almost every circle is atomic (indeed, has only one atom). Consider now a C^2 volume-preserving diffeomorphism H that is the direct product of G and the identity map on $\mathbb{T}^1$. It is partially hyperbolic and its central foliation $\mathcal{W}^c_H$ consists of invariant tori. Clearly, H is center-dissipative on a set of full volume and has one negative and one zero Lyapunov exponents along the central direction. The central foliation $\mathcal{W}^c_H$ is not absolutely continuous and the conditional measure generated by volume on almost every torus is not atomic. In this example the invariant Lebesgue measure is not ergodic. However, if one considers the direct product of G and the irrational rotation of $\mathbb{T}^1$ the resulting map is ergodic (since G is weakly mixing), the central foliation is not absolutely continuous, and the conditional measures on central leaves are not atomic.

Remark 1: The assumption that the leaves of the central foliation are compact is important. Indeed, consider a hyperbolic automorphism A of the three-dimensional torus $\mathbb{T}^3$ with eigenvalues λ_1, λ_2, and λ_3 such that $0 < |\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. The tangent bundle is split $T\mathbb{T}^3 = E_{1,A} \oplus E_{2,A} \oplus E_{3,A}$, where $E_{i,A}$ is the eigensubspace corresponding to λ_i, $i = 1, 2, 3$. One can view A as a partially hyperbolic diffeomorphism with $E_{2,A}$ as its central distribution. Clearly, A is center-dissipative and the distribution $E_{2,A}$ is integrable to a foliation, which is *smooth* (hence, absolutely continuous). If g is close to A in the C^1 topology then g is an Anosov diffeomorphism and the tangent bundle is split $T\mathbb{T}^3 = E_{1,g} \oplus E_{2,g} \oplus E_{3,g}$ so that g is partially hyperbolic. The central distribution $E_{2,g}$ is integrable to a foliation $\mathcal{W}_{2,g}$ with smooth non-compact leaves. This foliation is, in general, not smooth (but Hölder continuous). It is an open problem whether *g can be perturbed (in the C^1 or C^2 topology) to a diffeomorphism h for which the foliation $\mathcal{W}_{2,h}$ is not absolutely continuous.* We believe that indeed a stronger conjecture holds true: *For a "typical" g in a small neighborhood of A the foliation $\mathcal{W}_{2,g}$ is not absolutely continuous.*

Theorem 1.1 is a particular case of a more general result, which we now describe. It turns out that partial hyperbolicity, more precisely, the fact that f is normally hyperbolic to the foliation $\mathcal{W}^c$, is not important. Therefore, let us consider a C^2 diffeomorphism f of a compact smooth Riemannian manifold M preserving a Borel probability measure μ and a foliation $\mathcal{W}$ of M with smooth leaves, which is invariant under f. We say that f is **$\mathcal{W}$-dissipative** if there exists an invariant set $\mathcal{A}$ of positive μ-measure such that $\chi_{\mathcal{W}}(x) \neq 0$ for μ-almost every $x \in \mathcal{A}$, where $\chi_{\mathcal{W}}(x)$ denotes the sum of the Lyapunov exponents of f at the point x along the subspace $T_x\mathcal{W}(x)$.

For $x \in M$ we denote by $\mathrm{Vol}(\mathcal{W}(x))$ the volume of the leaf $\mathcal{W}(x)$. We say that the foliation $\mathcal{W}$ has **finite volume leaves almost everywhere** if the set $\mathcal{B}$ of those points $x \in M$, for which $\mathrm{Vol}(\mathcal{W}(x)) < \infty$, has full Riemannian volume.

An example of a foliation whose leaves have finite volume almost everywhere is a foliation with smooth *compact* leaves. If $\mathcal{W}$ is such a foliation then for every $x \in M$ the function $x \to \mathrm{Vol}(\mathcal{W}(x))$ is well-defined (finite) but may not be bounded (see [7]). In this connection one can wonder if there is a foliation of a compact manifold whose almost all (but not all) leaves are compact or if there is a foliation which is invariant under a diffeomorphism f of M, normally hyperbolic, and such that all leaves have finite volume but some (or all) are not compact.[1]

1 We would like to thank C. Pugh who mentioned these problems to us.

THEOREM 1.3: *Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Let also $\mathcal{W}$ be an f-invariant foliation of M with smooth leaves. Assume that $\mathcal{W}$ has finite volume leaves almost everywhere. If f is $\mathcal{W}$-dissipative almost everywhere then the foliation $\mathcal{W}$ is not absolutely continuous.*

The center-dissipativity property is typical in a sense. More precisely, the following two statements hold.

THEOREM 1.4: *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that $\chi^c_f(x) < -\alpha$ for some $\alpha > 0$ and μ-almost every $x \in M$. Then any diffeomorphism g, which is sufficiently close to f in the C^1 topology, is center-dissipative on a set of positive μ-measure.*

THEOREM 1.5: *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that $\chi^c_f(x) = 0$ for μ-almost every $x \in M$. Then for any $\varepsilon > 0$ there is a C^2 diffeomorphism g such that $d_{C^1}(f,g) \le \varepsilon$ and g is center-dissipative on a set of positive measure.*

We shall present a proof of Theorem 1.4 below. Theorem 1.5 is an easy corollary of results by Baraviera and Bonatti [2].[2] These results yield that the "pathological" phenomenon described in Theorems 1.1 and 1.2 is typical in a sense. More precisely, let f be a partially hyperbolic C^2 diffeomorphism preserving a smooth measure μ. Assume that the central distribution E^c_f is integrable to a *smooth* foliation $\mathcal{W}^c_f$ with smooth compact leaves and that f is *center-bunched* (see the definition in the next section). By Theorem 1.1, f is *not* center-dissipative. A well-known example of such a diffeomorphism is a skew product map $f(x,y) = (Ax, \varphi_x(y))$, where A is an Anosov diffeomorphism of a compact smooth Riemannian manifold N and φ_x is a diffeomorphism of a compact manifold K which depends smoothly on $x \in N$ and satisfies

$$\max_{x\in N} \|dA|E^s_A(x)\| < \min_{x\in N}\min_{y\in K} \|d\varphi_x^{-1}(y)\|^{-1} \le \max_{x\in N}\max_{y\in K} \|d\varphi_x(y)\| < \min_{x\in N} \|dA^{-1}|E^u_A(x)\|^{-1}.$$

Using a result of Hirsch, Pugh and Shub [9] we conclude that there is a neighborhood $\mathcal{U}_1 \subset \mathrm{Diff}^1(M,\mu)$ of f such that any $g \in \mathcal{U}_1$ is partially hyperbolic and

2 Although in [2] the authors consider volume-preserving transformation the proof works well for any transformations preserving a smooth measure.

its central distribution E^c_g is integrable to a foliation $\mathcal{W}^c_g$ with smooth compact leaves.[3] Since the property of center-bunching is C^1 open by its definition, there exists a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ such that any $g \in \mathcal{U}_2$ is center-bunched. By a result of Dolgopyat and Wilkinson [6], there is an open and dense set $\mathcal{V} \subset \mathcal{U}_2$ such that every $g \in \mathcal{V}$ has the *accessibility property* (indeed, it is *stably accessible*, see the next section). By a theorem of Pugh and Shub [14] (see also [5]), each $g \in \mathcal{V}$ is stably ergodic with respect to μ (i.e., every h sufficiently C^1-close to g is ergodic with respect to μ) and hence, $\chi^c_g(x)$ is constant μ-almost everywhere, say $\chi^c_g(\mu)$. In particular, whether g is center-dissipative amounts to $\chi^c_g(\mu) \neq 0$. If $g \in \mathcal{V}$ and $\chi^c_g(\mu) = 0$, by Theorem 1.5, there is a C^2 diffeomorphism h, which is arbitrary C^1-close to g, center-dissipative on a set of positive and hence, by ergodicity, on a set of full μ-measure. The central distribution E^c_h of h is integrable to a foliation $\mathcal{W}^c_h$ with compact leaves and by Theorem 1.1, it is not absolutely continuous. The same holds true for any sufficiently small C^2 perturbation of h. This phenomenon was first discovered by Shub and Wilkinson [16].

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2. Preliminaries

1. Let M be a compact smooth Riemannian manifold, $f\colon M \to M$ a C^2 diffeomorphism of M. It is said to be **(uniformly) partially hyperbolic** if there are numbers $\lambda_s < \underline{\lambda}_c \leq 1 \leq \overline{\lambda}_c < \lambda_u$ such that for every $x \in M$ there exists a d_xf-invariant decomposition $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ for which

$$
\begin{aligned}
&\|d_xf(v)\| \leq \lambda_s\|v\|, \quad v \in E^s(x),\\
&\underline{\lambda}_c\|v\| \leq \|d_xf(v)\| \leq \overline{\lambda}_c\|v\|, \quad v \in E^c(x),\\
&\lambda_u\|v\| \leq \|d_xf(v)\|, \quad v \in E^u(x).
\end{aligned}
$$

$E^s(x)$, $E^c(x)$, and $E^u(x)$ are called respectively, **stable, central** and **unstable** subspaces.

3 The result of Hirsch, Pugh and Shub [9] requires the foliation $\mathcal{W}^c_f$ be smooth or have a weaker property of being *plaque expansive*. It is an open problem whether this requirement can be dropped if the leaves of the foliation are compact.

2. A partition $\mathcal{W}$ of M is said to be a **foliation with smooth leaves** or simply a **foliation** if there exist $\delta > 0$ and $k \in \mathbb{N}$ such that for each $x \in M$:

(1) The element $\mathcal{W}(x)$ of the partition $\mathcal{W}$ containing x is a smooth k-dimensional immersed manifold called the **global leaf** of the foliation at x; the connected component of the intersection $\mathcal{W}(x) \cap B(x, \delta)$ (here $B(x, \delta)$ is the ball in M centered at x of radius δ) that contains x is called the **local leaf** at x and is denoted by $\mathcal{V}(y)$ (the number δ is called the **size** of $\mathcal{V}(y)$).

(2) There exists a continuous map φ_x: $B(x, \delta) \to C^1(\mathbb{D}, M)$, where $\mathbb{D}$ is the unit ball in $\mathbb{R}^k$, such that for every $y \in B(x, \delta)$ the local leaf $\mathcal{V}(y)$ is the image of the map $\varphi_x(y)$: $\mathbb{D} \to M$ of class C^1.

A continuous distribution E on TM is called **integrable** if there exists a foliation $\mathcal{W}$ of M such that $E(x) = T_x\mathcal{W}(x)$ for every $x \in M$. It is known that the stable and unstable distributions $E^s(x)$ and $E^u(x)$ are integrable to **stable** and **unstable foliations** $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. The central distribution E^c, however, may not be integrable (see [12]).

3. Given a foliation $\mathcal{W}$ of M, consider the measurable partition ξ of the ball $B(x, r)$ into local leaves $\mathcal{V}(y)$. We denote by m Riemannian volume on M and by $m_{\mathcal{V}(y)}$ Riemannian volume on $\mathcal{V}(y)$.

The foliation $\mathcal{W}$ of M is said to be **absolutely continuous** (see [4, 12, 13]) if there exists $r > 0$ such that for every $x \in M$, any Borel subset $X \subset B(x, r)$ of positive volume and almost every $y \in X$ we have

$$m_{\mathcal{V}(y)}(X \cap \mathcal{V}(y)) > 0. \tag{2.1}$$

It is easy to see that the foliation $\mathcal{W}$ is absolutely continuous if and only if for any Borel subset $X \subset M$ of positive volume and almost every $y \in X$,

$$m_{\mathcal{W}(y)}(X \cap \mathcal{W}(y)) > 0 \tag{2.2}$$

(since M is compact it can be covered by finitely many balls of radius $r \leq \delta$ where δ is the number in the definition of the foliation). Furthermore, absolute continuity of the foliation $\mathcal{W}$ is equivalent to the following property: for any Borel subset $X \subset M$ of positive volume there is $y \in M$ such that (2.2) holds. Indeed, the set $Y = \{y \in X\colon m_{\mathcal{W}(y)}(X \cap \mathcal{W}(y)) = 0\}$ must have zero volume (otherwise, one can find a Borel subset $Z \subset Y$ of positive volume and hence, a point $z \in M$ for which $m_{\mathcal{W}(z)}(Z \cap \mathcal{W}(z)) > 0$; therefore, for some point $y \in Z \subset X$ we have $m_{\mathcal{W}(y)}(X \cap \mathcal{W}(y)) > 0$, thus leading to a contradiction).

We stress that absolute continuity is a property of the foliation with respect to volume and does *not* require the presence of any dynamics (or any invariant measure).

Remark 2: The "usual" definition of absolute continuity property is stronger than the one given above (see, for example, [1, 3]). It imposes some requirements on the Radon–Nikodym derivatives of the conditional measures generated by volume on $\mathcal{V}(y)$ viewed as elements of the partition ξ. More precisely, let T be a local transversal through x to local manifolds $\mathcal{V}(y)$, $y \in B(x,r)$. Clearly, T can be identified with the factor-space $B(x,r)/\xi$.

We say that the foliation $\mathcal{W}$ of M is **absolutely continuous (in the strong sense)** if for almost every $x \in M$, any $r > 0$, and any Borel subset $X \subset B(x,r)$ of positive volume,

$$m(X) = \int_T h(y) \int_{\mathcal{V}(y)} I_X(y,z) g(y,z) dm_{\mathcal{V}(y)}(z) dm_T(y), \tag{2.3}$$

where I_X is the characteristic function of the set X, h a Borel function in $y \in T$, g a Borel function in $y \in T$ and $z \in \mathcal{V}(y)$, and m_T the Riemannian volume on T. In other words,

$$g(y,z) = \frac{d\tilde{m}_{\mathcal{V}(y)}}{dm_{\mathcal{V}(y)}}(z) \quad \text{and} \quad h(y) = \frac{d\tilde{m}_T}{dm_T}(y),$$

where $\tilde{m}_{\mathcal{V}(y)}$ is the conditional measure generated by volume on $\mathcal{V}(y)$ as an element of the partition ξ and $\tilde{m}_T$ is the factor-measure for the partition ξ generated by volume.

Let us stress that, if f is a diffeomorphism of M and μ a smooth f-invariant measure, then (2.3) holds for μ (instead of volume m) with appropriately chosen densities h and g with respect to $m_{\mathcal{V}(y)}$ and $m_T(y)$. The stable and unstable foliations are known to be absolutely continuous (in the strong sense). Indeed, they have an even stronger property. Namely, consider the family of local leaves

$$\mathcal{L}(x) = \{\mathcal{V}(w) \colon w \in B(x,r)\}.$$

Choose two local transversal T_1 and T_2 to the family $\mathcal{L}(x)$, and define the **holonomy map** $\pi = \pi(x,\mathcal{W}) \colon T_1 \to T_2$ by setting

$$\pi(y) = \mathcal{V}(w) \cap T_2,$$

for $y \in \mathcal{V}(w) \cap T_1$, $w \in B(x,r)$. The holonomy map π is a homeomorphism onto its image. It is called **absolutely continuous** if m_{T_2} is absolutely continuous with respect to $\pi_* m_{T_1}$. It is well-known (see, for example, [3]) that a foliation $\mathcal{W}$ of M is absolutely continuous (in the strong sense) provided that for any family $\mathcal{L}(x)$ and any two local transversal T_1 and T_2, the holonomy map is absolutely continuous.

The holonomy maps associated with families of stable and unstable local manifolds are absolutely continuous and, in fact, the functions h and g are continuous (and hence bounded).

4. The point $x \in M$ is said to be **Lyapunov regular** (see [3]) if there exists numbers $\chi_1(x) > \ldots > \chi_{s(x)}(x)$ and a $d_x f$-invariant decomposition $T_x M = E_1(x) \oplus \cdots \oplus E_{s(x)}(x)$ such that for each $i = 1, \ldots, s(x)$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \chi_i(x), \quad v \in E_i(x) \setminus \{0\},$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\operatorname{Jac}(d_x f^n)| = \sum_{i=1}^{s(x)} \chi_i(x) \dim E_i(x),$$

where Jac stands for the Jacobian. We denote the set of regular points by Γ. By the Multiplicative Ergodic theorem, Γ has full μ-measure. The numbers $\chi_i(x)$ are called the **Lyapunov exponents** of f at x along the subspaces $E_i(x)$. Note that the functions $x \mapsto \chi_i(x), r(x)$ and $\dim E_i(x)$ are Borel measurable and f-invariant.

Let $x \in \Gamma$ and $\mathcal{W}$ be an f-invariant foliation of M whose leaves are smooth submanifolds of M. We define the **Lyapunov exponent along the foliation** by

$$\chi_{\mathcal{W}}(x) = \lim_{n\to\pm\infty} (1/n) \log |\operatorname{Jac}(d_x f^n | T_x \mathcal{W}(x))|.$$

Let A^+ be the set of points for which $\chi_{\mathcal{W}}(x) > 0$. Suppose $\mu(A^+) > 0$. Then for a sufficiently small $\lambda > 0$, sufficiently large integer ℓ, and every $\varepsilon \in (0, \lambda/100)$ there exists a Borel set $A^+_{\lambda,\ell,\varepsilon} \subset A^+$ of positive μ-measure such that for every $x \in A^+_{\lambda,\ell,\varepsilon}$ and $n \geq 0$,

$$|\operatorname{Jac}(d_x f^n | T_x \mathcal{W}(x))| \geq \ell^{-1} e^{\lambda n} e^{-\varepsilon n}. \tag{2.4}$$

See [3]. When the central distribution E^c is integrable to a foliation $\mathcal{W}^c$ with smooth leaves, we denote $\chi_{\mathcal{W}^c}(x)$ simply by $\chi^c(x)$. Clearly,

$$\chi^c(x) = \sum_i \chi_i(x) \dim E_i(x),$$

where the sum is taken over all i for which the associated subspaces satisfy $\bigoplus_i E_i(x) = E^c(x)$.

The measure μ is called **hyperbolic** if $\chi_i(x) \neq 0$ for μ-almost every $x \in M$ and every $i = 1, \ldots, s(x)$. Let $\rho > 0$ be sufficiently small, and ℓ a sufficiently large integer. For every $\varepsilon \in (0, \rho/100)$ there exists a Borel set $\Lambda_{\rho,\ell,\varepsilon} \subset M$ of positive μ-measure such that for $x \in \Lambda_{\rho,\ell,\varepsilon}$ there exists a local smooth submanifold

$\mathcal{V}^-(x)$ of M with the following property: for $y \in \mathcal{V}^-(x)$ and $n \geq 0$,

$$d_{f^n(x)}(f^n(x), f^n(y)) \leq \ell e^{-\rho n} e^{\varepsilon n} d_x(x,y), \tag{2.5}$$

where d_x denotes the induced Riemannian distance in $\mathcal{V}^-(x)$. The local manifold $\mathcal{V}^-(x)$ is tangent to

$$E^-(x) = \bigoplus_{i;\chi_i(x)<0} E_i(x).$$

For $x \in \Lambda_{\rho,l,\varepsilon}$ the size $\delta(x)$ of $\mathcal{V}^-(x)$ is uniformly bounded from below, say by δ_l. See [3, 11].

Let $x \in \Gamma$. Define the **central Lyapunov exponent** by

$$\chi_c(x) = \lim_{n\to+\infty} (1/n) \log \|d_x f^n | E^c(x)\|.$$

Clearly, f has negative central exponents if and only if $\chi_c(x) < 0$. If f has negative central exponents then $E^-(x) = E^s(x) \oplus E^c(x)$ for almost every $x \in M$.

5. See [8]. A partially hyperbolic diffeomorphism f is called **center-bunched** if

$$\max\{\lambda_s, \lambda_u^{-1}\} < \underline{\lambda}_c / \overline{\lambda}_c.$$

A diffeomorphism f is said to have the **accessibility property** if any two points $p, q \in M$ are **accessible**, i.e., there are points $z_i \in M$ with $z_0 = p$, $z_\ell = q$ and such that z_i lies on a stable or unstable local manifold through z_{i-1}. We say f is **stably accessible** if every g sufficiently C^1-close to f is accessible.

3. Proof of Theorem 1.3

Let $A^- \subset M$ be the set of points for which $\chi_{\mathcal{W}}(x) < 0$ and $A^+ \subset M$ the set of points for which $\chi_{\mathcal{W}}(x) > 0$. They both are f-invariant and either $m(A^-) > 0$ or $m(A^+) > 0$ or both (we use here the fact that the invariant measure μ is smooth and hence, equivalent to volume). Without loss of generality we may assume that $m(A^+) > 0$. Fix a sufficiently large $\ell > 0$ such that $m(A^+_{\lambda,\ell,\varepsilon}) > 0$.

Given $V > 0$, consider the set

$$Y_V = \{y \in M\colon\ \mathrm{Vol}(\mathcal{W}(y)) \leq V\}.$$

Let $x \in A^+_{\lambda,\ell,\varepsilon}$ be a density point of m. By the conditions of the theorem, one can choose $V > 0$ such that the set

$$R = A^+_{\lambda,\ell,\varepsilon} \cap B(x,r) \cap Y_V$$

has positive volume.

Assume on the contrary that the foliation $\mathcal{W}$ is absolutely continuous. Then for almost every $y \in R$ the set $R_y = R \cap \mathcal{W}(y)$ has positive volume in $\mathcal{W}(y)$. Using again the fact that the invariant measure μ is smooth and hence, equivalent to volume we find that $\mu(R) > 0$. Therefore, the trajectory of almost every point $y \in R$ returns to R infinitely often. Let y be such a point and $\{n_k\}$ the sequence of successive returns to R. We may assume that $m_{\mathcal{W}(y)}(R_y) > 0$. Observe also that $f^n(\mathcal{W}(y)) = \mathcal{W}(f^n(y))$ for every integer n.

Since $f^{n_k}(y) \in R \subset Y_V$, we have that for every $k > 0$,

$$m_{\mathcal{W}(f^{n_k}(y))}(f^{n_k}(R_y)) \leq \mathrm{Vol}(\mathcal{W}(f^{n_k}(y))) \leq V.$$

On the other hand, by (2.4),

$$\begin{aligned}\mathrm{Vol}(\mathcal{W}(f^{n_k}(y))) &\geq m_{\mathcal{W}(f^{n_k}(y))}(f^{n_k}(R_y)) \\ &= \int_{R_y} |\operatorname{Jac}(d_z f^{n_k}|T_z\mathcal{W}(z))| dm_{\mathcal{W}(y)}(z) \\ &\geq \ell^{-1} e^{(\lambda-\varepsilon)n_k} m_{\mathcal{W}(y)}(R_y) > V\end{aligned}$$

if n_k is sufficiently large. This yields a contradiction and completes the proof of Theorem 1.3.

4. Proof of Theorem 1.2

The fact that the foliation $\mathcal{W}^c$ is not absolutely continuous follows from Theorem 1.3. It remains to show that if μ is ergodic then the conditional measures induced by μ on leaves of $\mathcal{W}^c$ are atomic. The argument below is a simple adaptation to our case of an argument by Ruelle and Wilkinson [15] and is presented here for the sake of completeness.

Since the foliation $\mathcal{W}^c$ is invariant with respect to f, we obtain for μ-almost every $x \in M$,

$$f_*\mu_x = \mu_{f(x)} \tag{4.1}$$

(recall that μ_x denotes conditional probability measure generated by μ on the leaf $\mathcal{W}^c(x)$). Due to ergodicity of the measure μ it suffices to show that there exists a positive μ-measure set $A \subset M$ such that for μ-almost every $x \in A$, the conditional measure μ_x has an atom. Indeed, for $x \in M$ set $d(x) = \sup_{y\in\mathcal{W}^c(x)} \mu_x(y)$. Clearly, this function is Borel measurable, invariant under f, and positive for μ-almost every $x \in A$. Since μ is ergodic we have $d(x) = d > 0$ for μ-almost every $x \in M$. Let

$$S = \{x \in M\colon \mu_x(y) \geq d \text{ for some } y \in \mathcal{W}^c(x)\}.$$

By (4.1), S is invariant under f, has measure at least d and hence, measure 1. The desired result therefore would follow if we show that the conditional measure μ_x has an atom.

Set $\Lambda_\ell = \Lambda_{\rho,\ell,\varepsilon}$. Fix a sufficiently large integer $\ell \geq 1$. Then there exists a set A of positive μ-measure such that for every $x \in A$ we obtain

$$\mu_x(\mathcal{W}^c(x) \cap \Lambda_\ell) \geq 1/2.$$

We shall show that for μ-almost every $x \in A$ the measure μ_x has an atom.

It follows from the Poincaré recurrence theorem that there exists a Borel measurable set $R \subset A$ with $\mu(R) = \mu(A)$ such that every point $x \in R$ returns infinitely often to R under iterations of f. This implies that the first return map $F = f^\tau\colon R \to R$ is well-defined, where $\tau\colon R \to \mathbb{N}$ is the first return time to R. Note that $\mu(R) = \mu(F(R))$ since the map F preserves the measure $\mu_R = \frac{1}{\mu(R)}\mu$. Furthermore, since the foliation $\mathcal{W}^c$ is invariant under f, for $x \in R$, we have

$$f^{\tau(x)}(\mathcal{W}^c(x)) = \mathcal{W}^c(f^{\tau(x)}(x)) = \mathcal{W}^c(F(x)).$$

This implies that for every $x \in R$ the extension $F(\mathcal{W}^c(x)) = f^{\tau(x)}(\mathcal{W}^c(x))$ is well-defined satisfying $F(\mathcal{W}^c(x)) = \mathcal{W}^c(F(x))$.

Let $x \in A$ and for $r \in (0, \delta_l/10)$, consider a family $\mathcal{B}_x$ of $N = N(x,r) \geq 1$ closed balls that cover $\mathcal{W}^c(x)$. Define

$$m(x) = \inf\left\{\sum_{i=1}^{k} \operatorname{diam}_x B_i\right\},$$

where the infimum is taken over all collections of closed balls B_i in $\mathcal{W}^c(x)$ such that $k \leq N$ and

$$\mu_x\left(\bigcup_{i=1}^{k} B_i\right) \geq 1/2$$

(here $\operatorname{diam}_x B$ denotes the diameter of B with respect to the intrinsic distance in $\mathcal{W}^c(x)$). Let $m = \operatorname{ess\,sup}_{x\in A} m(x)$. We will show that $m = 0$.

Otherwise, there is an integer n_0 such that for every $n \geq n_0$, we have

$$\ell \Delta N e^{-\rho n} e^{\varepsilon n} < m/2, \tag{4.2}$$

where $\Delta = \operatorname{diam}_x \mathcal{W}^c(x)$.

Let $B(y_1, r), \ldots, B(y_{k(x)}, r)$ be those balls in $\mathcal{B}_x$ for which the intersection $B(y_i, r) \cap \Lambda_\ell$ is not empty. Since these balls cover $\mathcal{W}^c(x) \cap \Lambda_\ell$ and

$$\mu_x(\mathcal{W}^c(x) \cap \Lambda_\ell) \geq 1/2,$$

we have that

$$\mu_x\Big(\bigcup_{i=1}^{k(x)} B(y_i, r)\Big) \geq \mu_x(\mathcal{W}^c(x) \cap \Lambda_\ell) \geq 1/2.$$

By (4.1), $f^n_*\mu_x = \mu_{f^n(x)}$ for every $n \in \mathbb{Z}$ and hence,

$$\mu_{f^n(x)}\Big(\bigcup_{i=1}^{k(x)} f^n(B(y_i, r))\Big) \geq 1/2. \tag{4.3}$$

Since the balls $B(y_i, r)$ intersect Λ_ℓ and have diameter less than $\delta_\ell/10$, we obtain, by (2.5), that

$$\operatorname{diam}_{f^n(x)} f^n(B(y_i, r)) \leq \ell\Delta e^{-\rho n} e^{\varepsilon n}. \tag{4.4}$$

Fix $n \geq n_0$ and let $y \in F^n(R)$. Then there is $x \in R$ such that $F^n(x) = y$. It follows from the definition of m, the F-invariance of the foliation $\mathcal{W}^c|R$, and inequalities (4.2), (4.3) and (4.4) that

$$\begin{aligned} m(y) = m(F^n(x)) &\leq \sum_{i=1}^{k(x)} \operatorname{diam}_{F^n(x)} F^n(B(y_i, r)) \\ &\leq \ell\Delta k(x) e^{-\rho\tau_n(x)} e^{\varepsilon\tau_n(x)} \\ &\leq \ell\Delta N e^{-\rho n} e^{\varepsilon n} < m/2. \end{aligned}$$

This implies that

$$\begin{aligned} m &= \operatorname{ess\,sup}_{x\in A} m(x) = \operatorname{ess\,sup}_{x \in R} m(x) \\ &= \operatorname{ess\,sup}_{y \in F^n(R)} m(y) < m/2, \end{aligned}$$

contradicting the assumption that $m > 0$.

We conclude that $m = 0$, and hence, $m(x) = 0$ for μ-almost every $x \in A$. This implies that for every such x there is a sequence of closed balls $B_1(x), B_2(x), \ldots$ with

$$\lim_{j\to\infty} \operatorname{diam}_x B_j(x) = 0$$

and $\mu_x(B_j(x)) \geq 1/2N$ for all $j \in \mathbb{N}$. Take $z_j \in B_j(x)$. Then any accumulation point of the sequence $\{z_j\}$ is an atom for μ_x.

5. Proof of Theorem 1.4

Let $f \in \mathcal{U}$ be a diffeomorphism for which $\chi^c_f(x) < -\alpha$ for some small $\alpha > 0$ and for every x in a set A_f of full μ-measure. It follows that for every $x \in A_f$,

$$\lim_{n\to+\infty} (1/n) \log |\operatorname{Jac}(df^n|E^c_f(x))| < -\alpha.$$

Integrating over M we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int_M \log |\operatorname{Jac}(df^n|E^c_f(x))| d\mu(x) < -\alpha.$$

In particular, there exists $n_0 > 0$ such that

$$\frac{1}{n_0} \int_M \log |\operatorname{Jac}(df^{n_0}|E^c_f(x))| d\mu(x) < -\frac{\alpha}{2}.$$

Without loss of generality we may assume that $n_0 = 1$ so that

$$\int_M \log |\operatorname{Jac}(df|E^c_f(x))| d\mu(x) < -\frac{\alpha}{2}.$$

Since the central distribution depends continuously on the perturbation, for a diffeomorphism g, which is sufficiently close to f in the C^1 topology, we have

$$\int_M \log |\operatorname{Jac}(dg|E^c_g(x))| d\mu(x) < -\frac{\alpha}{4}.$$

It follows from the Birkhoff ergodic theorem that there exists a g-invariant subset A_g with $\mu(A_g) > 0$ such that for every $x \in A_g$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |\operatorname{Jac}(dg|E^c_g(g^j(x)))| \leq -\frac{\alpha}{4}.$$

Hence,

$$\lim_{n\to+\infty} \frac{1}{n} \log |\operatorname{Jac}(dg^n|E^c_g(x))| \leq -\frac{\alpha}{4}$$

for every $x \in A_g$ and the desired result follows.

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