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On the integrability of intermediate distributions for Anosov diffeomorphisms†

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Abstract. We study the integrability of intermediate distributions for Anosov diffeomorphisms and provide an example of a C^∞-Anosov diffeomorphism on a three-dimensional torus whose intermediate stable foliation has leaves that admit only a finite number of derivatives. We also show that this phenomenon is quite abundant. In dimension four or higher this can happen even if the Lyapunov exponents at periodic orbits are constant.

1. *Introduction*

Let F be a C^r-diffeomorphism of a compact smooth Riemannian manifold M, $r = 1, 2, \cdots, \infty$ and $\Gamma^0(TM)$ the space of C^0-vector fields on M. The map F induces an invertible bounded linear operator on $\Gamma^0(TM)$ by:

$$F_* v(x) = DFv(F^{-1}(x)), \quad v(x) \in \Gamma^0(TM), \quad x \in M.$$

The *Mather spectrum* $\sigma(F)$ is defined to be the spectrum of the complexification of F_*. In [**M**], Mather showed that F is an Anosov system, if and only if $1 \notin \sigma(F)$, and moreover, if the nonperiodic points of F are dense in M, then $\sigma(F)$ consists of a finite number of annuli in the complex plane $\mathbb{C}$. That is, there exist $\lambda_i, \mu_i, i = 1, \cdots, p$, $0 < \lambda_1 \le \mu_1 < \lambda_2 \le \mu_2 < \cdots < \lambda_p \le \mu_p$ $p \le \dim(M)$ such that

$$\sigma(F) = \bigcup_{i=1}^{p} \{z \in \mathbb{C} : \ \lambda_i \le |z| \le \mu_i\}.$$

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The set $\{z \in \mathbb{C} : \lambda_i \leq |z| \leq \mu_i\}$ is called a $[\lambda_i, \mu_i]$-*ring*.

The decomposition of the spectrum just mentioned induces a splitting of the tangent bundle TM into p subbundles $\{E_F^{(i)}\}, i = 1, \cdots, p$, $TM = \bigoplus_{i=1}^p E_F^{(i)}$ called an *exponential splitting*. The distribution $E_F^{(i)}$ is said to be a $[\lambda_i, \mu_i]$-*distribution* and it is invariant under DF. These distributions depend Hölder continuously on the base point and are not smooth in general.

For every sufficiently small number $\delta > 0$, one can choose a smooth Riemannian metric, known as a Lyapunov metric, such that $TM = \bigoplus_{i=1}^p E_F^{(i)}$ is an almost orthogonal splitting with respect to this metric. Moreover for any vector $v(x) \in E_F^{(i)}(x)$, we have

$$(\lambda_i - \delta)\|v(x)\| \leq \|F_* v(x)\| \leq (\mu_i + \delta)\|v(x)\|.$$

From now on, we fix a Lyapunov metric adequate for our map. It induces the metric in $\mathrm{Diff}^1(M)$ which we denote by d_{C^1}.

Let W be a partition of M. It is called a *continuous C^r–foliation* if the following conditions hold:

(1) for any $x \in M$, the element $W(x)$ of W containing x is a k-dimensional C^r-injectively immersed and connected submanifold (called a C^r-*leaf* of W passing through x);

(2) there is an admissible atlas of C^0 charts of M: $\phi : \mathbf{D}^k \times \mathbf{D}^{n-k} \to M$ such that $\phi(\mathbf{D}^k \times \{y\}) \subset W(\phi(0, y))$ where $\mathbf{D}^k$ denotes the unit ball in $\mathbb{R}^k$;

(3) the function $\phi(x, y)$ is C^r-differentiable over x for any y and is continuous over y in C^r-topology.

We can consider a distribution as an *infinitesimal version* of a foliation in which the subspaces of the tangent space play the role of *infinitesimal* leaves. Given a foliation, we can obtain a distribution by associating with every point the tangent space to the leaf passing through it. Not all distributions arise in this way and the problem of finding sufficient conditions, which guarantee a given distribution to be associated with a foliation, is usually called the problem of *integrability of the distribution*. For one-dimensional distributions this amounts to the theorem of existence and uniqueness of ordinary differential equations and, for multi-dimensional distributions, a classic solution is given by the Frobenius theorem. These results involve hypothesis on differentiability and cannot be applied to the distributions invariant under Anosov systems since the distributions are, in general, only Hölder continuous.

We recall the following well-known results (cf [**BP**], [**Fe**], [**HPS**]). *If ν is a real number satisfying $\mu_k < \nu < \min\{\lambda_{k+1}, 1\}$ for some integer k, then, for each $x \in M$,* there exists a continuous C^r-foliation $W_k^s = \{W_k^s(x) : x \in M\}$ invariant under F such that the submanifold $W_k^s(x)$ is characterized by

$$W_k^s(x) = \left\{ y : \frac{1}{\nu^n} d(F^n(x), F^n(y)) \to 0, n \to \infty \right\}.$$

$\{W_k^s(x)\}$ are integral manifolds of the distribution $E_F^{(1)} \bigoplus \cdots \bigoplus E_F^{(k)}$, i.e.,

$$TW_k^s(x) = E_F^{(1)}(x) \bigoplus \cdots \bigoplus E_F^{(k)}(x), \quad x \in M.$$

Let $\mu_t = \max\{\mu_i : \mu_i < 1\}$. Then the collection of foliations $\{W_i^s\}_{i=1}^t$ forms a *filtration* *of stable foliations* of M, i.e.,

$$W_1^s(x) \subset W_2^s(x) \subset \cdots \subset W_t^s(x) = W^s(x).$$

Applying these results to the inverse map F^{-1} one can obtain analogous results for the distributions $E_F^{(k)} \bigoplus \cdots \bigoplus E_F^{(p)}$ where k is such that $\lambda_k > 1$.

The following questions about the integrability and smoothness of a single $[\lambda_i, \mu_i]$-distribution $E_F^{(i)}$ are of a general interest:

(1) *Is $E_F^{(i)}$ integrable? If it is, how smooth are its integral manifolds?*

(2) *Assume $E_F^{(i)}$ is integrable for a diffeomorphism F and G is C^1-close to F. Does G have a corresponding exponential splitting $TM = \bigoplus_{i=1}^p E_G^{(i)}$ which is also integrable? If it does, how smooth are its integral manifolds ?*

There are few related results in the literature. The basic properties of the splittings are considered in the following two statements:

THEOREM A. ([**P**].) *For every $\epsilon > 0$, there exists a δ-neighborhood η of F in* $\mathrm{Diff}^1(M)$ *such that the Mather spectrum of any diffeomorphism $G \in \eta$ is contained in $[\lambda_i', \mu_i']$-rings with $0 < \lambda_i - \lambda_i' \leq \epsilon$ and $0 < \mu_i' - \mu_i \leq \epsilon$ $i = 1, 2, \cdots, p$ and the corresponding exponential splitting $TM = \bigoplus_{i=1}^p E_G^{(i)}$ satisfies $\rho(E_F^{(i)}, E_G^{(i)}) \leq \epsilon$, where ρ is a metric on the space of subbundles induced by the Riemannian metric on M.*

Remark. This result is not trivial at least for the following reason. Although the maps F and G are close in C^1-topology, the linear operators F_* and G_* may not be close, i.e., the norm

$$||F_* - G_*|| \equiv \sup_{\substack{x \in M \\ ||v||_{C^0}=1}} |DF(F^{-1}(x))v(F^{-1}(x)) - DG(G^{-1}(x))v(G^{-1}(x))|$$

may not be small. To see this we notice that in the difference just considered the vector field v is considered at two different points and one can find a continuous vector field of norm 1 for which the two values are completely different. Indeed, if we fix F, it is possible to find $\delta > 0$ such that $||F_* - G_*|| < \delta$ implies $F = G$.

THEOREM B. ([**BK**], [**BP**].) *Let F be a C^1-diffeomorphism with an exponential splitting $TM = \bigoplus_{i=1}^p E_F^{(i)}$. Then, $E_F^{(i)}(x)$ is Hölder continuous for each i.*

We improve Theorem A in the following way that displays the precise relation between ϵ and δ and gives an upper bound for the angles between $E_F^{(i)}$ and $E_G^{(i)}$ in terms of δ.

THEOREM 1. *There exist positive constants L, τ, and δ_1, depending only on F, such that $\rho(E_F^{(i)}, E_G^{(i)}) \leq L\delta^\tau$ if $d_{C^1}(F, G) < \delta < \delta_1$.*

Remark. We will show that one can take τ to be equal to the Hölder exponent of the distribution $E_F^{(i)}$. This implies that if $E_F^{(i)}$ is Lipschitz continuous then $\tau = 1$. One cannot in general, obtain a better estimation in Theorem 1 than the one given by the Hölder exponent of the bundle. To see this let F be a map of a torus and $G = \mathcal{T}_\Delta \circ F \circ \mathcal{T}_{-\Delta}$ where $\mathcal{T}_\Delta$ is the translation by an amount Δ, then $d_{C^1}(F, G) \approx ||F||_{C^2}|\Delta|$ and $E_G^{(i)}(x) = E_F^{(i)}(\mathcal{T}_\Delta x)$.

THEOREM C. ([**P**].) *Assume that F is a C^r-diffeomorphism on M and $TM = \bigoplus_{i=1}^{p} E_F^{(i)}$ is the exponential splitting associated with F. For a fixed k, suppose that $[\lambda_k, \mu_k]$ satisfies one of the following non-resonance conditions:*

(1)

$$r \geq r_0 = \left[\frac{\ln \lambda_1}{\ln \mu_k}\right] + 1 \quad \text{and } [(\lambda_k)^j, (\mu_k)^j] \bigcap [\lambda_i, \mu_i] = \emptyset, 1 \leq i < k \tag{1.1}$$

for every $j = 1, \cdots, r_0$;

(2)

$$r \geq r_1 = \left[\frac{\ln \mu_p}{\ln \lambda_k}\right] + 1 \quad \text{and} \quad [(\lambda_k)^j, (\mu_k)^j] \bigcap [\lambda_i, \mu_i] = \emptyset, p \geq i > k \tag{1.2}$$

for every $j = 1, \cdots, r_1$.

Then, for every $x \in M$, there exists a local C^r-submanifold $V_F^{(k)}(x)$ tangential to $E^{(k)}$ at x (i.e., $TV_F^{(k)}(x) = E_F^{(k)}(x)$) and invariant under F (i.e., $F(V_F^{(k)}(x)) \subset V_F^{(k)}(F(x))$). Moreover, for any point x the collection of local manifolds $\{V_F^{(k)}(F^i x)\}_{i=-\infty}^{+\infty}$ is the only collection that satisfies $T_{F^i(x)} V_F^{(k)}(F^i x) = E_F^{(k)}(F^i x)$, $F(V_F^{(k)}(F^i x)) \subset V_F^{(k)}(F^{i+1}(x))$, and $\sup_{1 \leq l \leq r} \sup_{i \in \mathbb{Z}} \|d^l V_F^{(k)}(F^i x)\| \leq constant$.

Remark. In [**P**], the author claimed that the local manifold $V_F^{(k)}(x)$ can be glued together to make up a continuous foliation with smooth enough leaves. As we will see later this is not true in general. We also point out that the statement about uniqueness of $V_F^{(k)}(x)$ in Theorem C is stronger than the corresponding statement in [**P**] but can be easily proved by arguments presented there.

In the case when M is an n-dimensional torus $\mathbb{T}^n$ (or any manifold whose universal cover is $\mathbb{R}^n$) and F is a small perturbation of a linear hyperbolic automorphism A of $\mathbb{T}^n$, one can say more about the global integrability of intermediate distributions $E_F^{(i)}$. First of all, the Mather spectrum for a linear automorphism of the torus, $\sigma(A)$ consists of a finite number of circles with radii $\lambda_i, i = 1, \cdots, p$ $0 < \lambda_1 < \cdots < \lambda_p$. Let $T\mathbb{T}^n = \bigoplus_{i=1}^{p} E_A^{(i)}$ be the corresponding exponential splitting. It is easy to see that each subbundle $E_A^{(i)}$ is C^∞-integrable and the corresponding foliation $W^{(i)} = \{W^{(i)}(x) | x \in \mathbb{T}^n\}$ is a smooth C^∞-foliation of $\mathbb{T}^n$.

A combination of Theorem A and an adaptation of the pseudo-stable and pseudo-unstable manifold theorem (see [**I**]) gives us the following result about small perturbations of A.

For a fixed k, $2 \leq k \leq n-1$, and $\lambda_k < \lambda_{k+1} < 1$, we denote

$$N = \begin{cases} \left[\dfrac{\ln \lambda_k}{\ln \lambda_{k+1}}\right], & \text{if } \dfrac{\ln \lambda_k}{\ln \lambda_{k+1}} \text{ is not an integer;} \\ \left[\dfrac{\ln \lambda_k}{\ln \lambda_{k+1}}\right] - 1, & \text{if } \dfrac{\ln \lambda_k}{\ln \lambda_{k+1}} \text{ is an integer.} \end{cases}$$

If $1 < \lambda_{k-1} < \lambda_k$, we set

$$N = \begin{cases} \left[\dfrac{\ln \lambda_k}{\ln \lambda_{k-1}}\right], & \text{if } \dfrac{\ln \lambda_k}{\ln \lambda_{k-1}} \text{ is not an integer;} \\ \left[\dfrac{\ln \lambda_k}{\ln \lambda_{k-1}}\right] - 1, & \text{if } \dfrac{\ln \lambda_k}{\ln \lambda_{k-1}} \text{ is an integer.} \end{cases}$$

THEOREM D. ([**LW**].) *Let A be a linear automorphism of torus $\mathbb{T}^n$. Then, for every $\epsilon > 0$, there exists a neighborhood η of A in* $\text{Diff}^1(\mathbb{T}^n)$ *such that any C^r-diffeomorphism $G \in \eta$*

has an exponential splitting $T\mathbb{T}^n = \bigoplus_{i=1}^{p} E_G^i$ *where* $\rho(E_G^{(i)}, E_A^{(i)}) \leq \epsilon$. *If* $r > N$, *then the subbundle* $E_G^{(i)}, i = 1, \cdots, p$ *is integrable and the corresponding integral manifolds form a continuous* C^N*-foliation of* $\mathbb{T}^n$, $W_G^{(i)} = \{W_G^{(i)}(x) | x \in \mathbb{T}^n\}$. *Moreover, for any two numbers* ν_-, ν_+ *with* $\lambda_k < \nu_- < \nu_+ < \mu_k$, *we have* $y \in W_G^{(i)}(x)$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{\nu_+^n} |\tilde{G}^n(\tilde{y}) - \tilde{G}^n(\tilde{x})| = 0 \quad \textit{and} \quad \lim_{n\to-\infty} \frac{1}{\nu_-^n} |\tilde{G}^n(\tilde{y}) - \tilde{G}^n(\tilde{x})| = 0 \tag{1.3}$$

where ˜ *denotes objects in* $\mathbb{R}^n$ *lifted from the torus.*

If a continuous C^1-foliation W has the property that $T_x W(x) = E_G^{(i)}(x)$ for any $x \in \mathbb{T}^n$, then one can show that the characterization (1.3) implies $W = W_G^{(i)}$. In fact, if $W_G^{(i)}$ is of class C^{N+1}, then it is as smooth as $V_G^{(i)}$ (see Theorem C), i.e., of class C^r and $V_G^{(i)} \subset W_G^{(i)}$. In this case the local invariant manifolds $V_G^{(i)}$ can be glued together to form a continuous C^r-foliation.

We note that Theorem D is complementary to the well known theorem on existence of strong stable and unstable foliations (see [**BP**]) and it claims the existence of foliations for intermediate annuli of the Mather spectrum that are bounded from both sides by other annuli. Let us also point out that Theorem D is proved on manifolds whose universal cover is $\mathbb{R}^n$. The foliations in Theorem D are invariant under topological coordinate change whereas the strong stable and unstable foliations are not. The leaves of the strong stable and unstable foliations are as smooth as the map is, whereas the smoothness of $W_G^{(i)}$, claimed in the Theorem D, depends on the gaps between the eigenvalues and cannot be substantially improved.

THEOREM 2. *Let* A *be a linear automorphism of* $\mathbb{T}^3$ *with eigenvalues* $\lambda_i, i = 1, 2, 3$ $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$. *Assume that* $\ln \lambda_1 / \ln \lambda_2$ *is not an integer. Let* $N = [\ln \lambda_1 / \ln \lambda_2]$. *Then, in any neighborhood* η *of* A *in* $\mathrm{Diff}^1(\mathbb{T}^3)$, *there exists* $G \in \eta$ *such that:*

(1) *the map* G *is a* C^∞*-diffeomorphism and topologically conjugate to* A;
(2) *the map* G *induces an exponential splitting of* $\mathbb{T}^3$, $T\mathbb{T}^3 = \bigoplus_{i=1}^{3} E_G^{(i)}$ *with* $E_G^{(i)}$ *close to* $E_A^{(i)}$ *and integrable. The integral manifold* $W_G^{(2)}(x)$ *passing through* x *is of class* C^N *but not* C^{N+1} *for some point* $x \in \mathbb{T}^3$;
(3) *the set of points* $S = \{x : W_G^{(2)}(x) \notin C^{N+1}\}$ *is a residual set in* $\mathbb{T}^3$.

In fact, we show that diffeomorphisms with the properties stated in Theorem 2 are dense in a small neighborhood of A.

THEOREM 3. *There exists a neighborhood* η *of* A *in* $\mathrm{Diff}^1(\mathbb{T}^3)$ *such that in any neighborhood of a* C^∞*-diffeomorphism* $F \in \eta$, *there is a* C^∞*-diffeomorphism* G *satisfying statements (1), (2) and (3) in Theorem 2.*

Remark. If $x \in \mathbb{T}^3$ is a periodic point for F of period p than the map $DF^p(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear automorphism of $T_x\mathbb{T}^3$. Its eigenvalues are called the eigenvalues at the periodic point x. If G is a small perturbation of F then G is conjugate to F by a homeomorphism h. The map G is said to have the same eigenvalues at periodic points if for any F-periodic point $x \in \mathbb{T}^3$ the eigenvalues at x for F are the same as the eigenvalues at $h(x)$ for G. From the point of view of rigidity theory it would be interesting to know

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whether one can modify the construction used in the proof of theorems 2 and 3, so that the resulting diffeomorphism of $\mathbb{T}^3$ would have the same eigenvalues at periodic points.

For tori of dimension 4 and higher one can construct examples where the perturbed map G has the same eigenvalues at periodic points as F but the leaves of the appropriate foliation $W_G^{(i)}$ are not as smooth as the map G is and, hence, satisfy statements 2 and 3 of Theorem 2. This answers a question posed to us by A. Katok.

THEOREM 4. *For any $n \geq 4$ there exists a linear Anosov automorphism A of $\mathbb{T}^n$ with the following properties:*

(1) *there exists a λ_i-ring where $\lambda_1 < \lambda_i < 1$ is an eigenvalue of A that satisfies the non-resonance condition* (1.1)*;*
(2) *in any C^∞-neighborhood of A one can find a diffeomorphism G of $\mathbb{T}^n$ such that*
   (a) *for any $x \in \mathbb{T}^n$ and for any $\epsilon > 0$, there exists $y \in V_G^{(i)}(x), d(x, y) < \epsilon$ such that $T_y V_G^{(i)}(x) \neq E_G^{(i)}(y)$;*
   (b) *the maps G and A have same eigenvalues at periodic points; the Mather spectra of G and A coincide.*

Remark. We will show that the examples can be constructed in infinite-dimensional (and infinite-codimensional) manifolds whose boundary includes the linear automorphism.

2. *Proofs*

Proof of Theorem 1. Let $\Gamma^0(TM) = H_1 \bigoplus H_2$ be a splitting of $\Gamma^0(TM)$ where H_1 and H_2 are F_*-invariant. With respect to such a splitting, one can write G_* in the form $G_* = (G_{ij})$, where $G_{ij} : H_i \to H_j, i, j = 1, 2$, are bounded linear operators, $||G_{11}|| < \lambda$, $||G_{22}^{-1}|| < \mu$, and $\lambda\mu < 1$ for some λ, μ. In [**P**], the author proved the following result (see Lemma 1): for any $\epsilon > 0$ there is $\delta' > 0$ such that if $||G_{12}|| \leq \delta'$ and $||G_{21}|| \leq \delta'$, then there exists a splitting $\Gamma^0(TM) = H_1' \bigoplus H_2'$ relative to which

$$G_* = \begin{pmatrix} G'_{11} & 0 \\ 0 & G'_{22} \end{pmatrix},$$

where $||G'_{11}|| < \lambda$, $||G'^{-1}_{22}|| < \mu$ and $\rho(H_i, H_i') < \epsilon$. In fact, H_1' is the graph of a transformation $A : H_1 \to H_2$, where A is the unique fixed point of the map Q:

$$QA = G_{22}^{-1} A G_{11} + G_{22}^{-1}(-G_{21} + A G_{12} A).$$

The map Q is contracting and transforms the unit ball of the space of all bounded and continuous maps from H_1 to H_2 into itself.

To prove Theorem 1 it suffices to show that $||A|| \leq L\delta^r$. We first show that there exist δ_0 and a constant L' such that for any $\delta' < \delta_0$ and $||G_{12}|| \leq \delta'$, $||G_{21}|| \leq \delta'$, we have $||A|| \leq L'\delta'$.

Since Q is a contracting map, for any map P with $||P|| < 1$ we have $||Q^n P - A|| \to 0$. We consider $P = 0$. Then $Q0 = -G_{22}^{-1} G_{21}$ and $||Q0|| \leq \mu \cdot \delta'$. We wish to have $||Q^n 0|| \leq L'\delta'$ for some constant L'. Let us assume that this is true for $n = k - 1$, i.e., $||Q^{k-1}0|| \leq L'\delta'$. Then,

$$\begin{aligned} ||Q^k 0|| &\leq ||G_{22}^{-1} Q^{k-1} 0 G_{11}|| + ||G_{22}^{-1}(-G_{21} + Q^{k-1} 0 G_{12} Q^{k-1} 0)|| \\ &\leq \lambda\mu L'\delta' + \mu(\delta' + L^2 \delta'^3). \end{aligned}$$

Thus, if we choose

$$L' = \frac{\mu + 1}{1 - \lambda\mu} \quad \text{and} \quad \delta_0 = \frac{1}{2L'\sqrt{\mu}},$$

we obtain

$$||Q^k 0|| \leq \lambda\mu L'\delta' + \mu\delta' + \frac{1}{4}\delta' \leq L'\delta'.$$

By induction $||Q^n 0|| \leq L'\delta'$ for all $n = 1, 2, \ldots$. Therefore,

$$||A|| \leq \lim_{n\to\infty} ||Q^n 0 - A|| + ||Q^n 0|| \leq L'\delta'.$$

Now, Theorem B implies that $\delta' \leq L'' \cdot \delta^\tau$ where $\delta = d_{C^1}(F, G)$, $L'' > 0$ is a constant and τ is the Hölder exponent. Then $||A|| \leq L'L''\delta^\tau$. This finishes the proof of Theorem 1.

Proof of Theorem 2. We define the perturbation G in the form $G = \pi \cdot (\tilde{A} + \tilde{h})$, where π is the usual covering map from $\mathbb{R}^3$ onto $\mathbb{T}^3$ and $\tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3$ is a map. We describe $\tilde{h}$ in $\{\tilde{e}_A^{(1)}, \tilde{e}_A^{(2)}, \tilde{e}_A^{(3)}\}$-coordinate system. Let $p_0 = (0, \beta_0, 0) \in \mathbb{R}^3$ be given in this coordinate system with $0 < \beta_0 < 1$ sufficiently small and $B_\epsilon(p)$ be an open ball centered at $p \in \mathbb{R}^3$ of radius $\epsilon > 0$. We choose p_0 close to the origin and $0 < \epsilon < 1$ to be small enough such that the sets $\pi(B_\epsilon(0))$, $A\pi(B_\epsilon(p_0))$, $\pi(B_\epsilon(p_0))$, $A^{-1}\pi(B_\epsilon(p_0)), \cdots,$ $A^{-K}\pi(B_\epsilon(p_0))$ are all disjoint. Moreover, if ϵ and β_0 are sufficiently small, K can be arbitrarily large and we can assume that

$$\left(\frac{\lambda_2}{\lambda_1}\right)^K \frac{1}{8\lambda_1} > L. \tag{2.1}$$

We can also require $\tilde{A}(B_\epsilon(p_0))$ and $(B_\epsilon(p_0))$ to be contained in the same fundamental domain D. Define a smooth function $f\colon D \to \mathbb{R}$ such that:

(1) $|f(\alpha, \beta, \gamma)| \leq \delta$ and $||Df(\alpha, \beta, \gamma)|| \leq \delta$, where δ is a constant satisfying

$$0 < \delta < \max\left(\frac{\lambda_1\epsilon}{4}, \frac{1}{4}(\lambda_2 - \lambda_1)\right);$$

(2) $\left|\frac{\partial f}{\partial \beta}(\alpha, \beta_0, 0)\right| \geq \frac{\delta}{4}$ for $|\alpha| \leq \frac{\epsilon}{3}$;

(3) $f(\alpha, \beta, \gamma) \equiv 0$ for $(\alpha, \beta, \gamma) \notin B_\epsilon(p_0)$.

Let us set for all $(\alpha, \beta, \gamma) \in \mathbb{R}^3$

$$\tilde{h}(\alpha, \beta, \gamma) = (f(\alpha, \beta, \gamma), 0, 0) \pmod 1$$

and then

$$\tilde{G} = \tilde{A} + \tilde{h}, \quad G = \pi \cdot (\tilde{A} + \tilde{h}).$$

If δ is sufficiently small, it is clear that G is a C^∞-diffeomorphism of $\mathbb{T}^3$. Moreover, for any $0 < t < 1$, the function tf satisfies conditions 1–3 with the constant $t\delta$ instead of δ. Thus, for any neighborhood η of A in $\mathrm{Diff}^1(\mathbb{T}^3)$, we can choose δ so small that $G \in \eta$ and the following two statements are true:

(a) *the map G is an Anosov diffeomorphism, topologically conjugate to A;*

(b) *the map G possesses an exponential splitting of $T\mathbb{T}^3 = \bigoplus_{i=1}^3 E_G^{(i)}$ (we also have the corresponding splitting of $T\mathbb{R}^3 : T\mathbb{R}^3 = \bigoplus_{i=1}^3 \tilde{E}_G^{(i)}$, where $\tilde{E}_G^{(i)}$ is $\tilde{G}$ invariant and $D\pi(\tilde{E}_G^{(i)}) = E_G^{(i)}$) and the Mather spectrum of G satisfies the non-resonance conditions* (1.1) *and* (1.2).

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We will verify the following properties of diffeomorphisms $\tilde{G}$ and G:

I $\quad G(x) = A(x)$ *if* $x \notin \pi(B_\epsilon(p_0))$;

II $\quad \tilde{G}^n(B_\epsilon(p_0)) = \tilde{A}^n(B_\epsilon(p_0))$ *for any* $|n| \leq K$;

III $\quad \tilde{E}_G^{(2)}(0) = \{t\tilde{e}_A^{(2)} : t \in \mathbb{R}\}$;

IV $\quad$ *the map G has a local smooth manifold* $\tilde{V}_G^{(2)}(0)$ *at* $p = 0$ *such that* $\tilde{V}_G^{(2)}(0) = \{t\tilde{e}_A^{(2)} : t \in (-a, a)$ *for some* $a > 0\}$;

V $\quad \tilde{E}_G^{(2)}(A(p_0)) \neq \tilde{E}_A^{(2)}(A(p_0)) = \{t\tilde{e}_A^{(2)} : t \in \mathbb{R}\}$;

VI $\quad \tilde{E}_G^{(2)}(A^n(p_0)) \neq \tilde{E}_A^{(2)}(A^n(p_0))$, *for* $n \geq 1$.

The properties I and II are obvious. To show III and IV, it is sufficient to observe that $p = 0$ is a fixed point of $\tilde{A}$ and $\tilde{G} \equiv \tilde{A}$ near $p = 0$. Property VI follows from V. In fact, $\tilde{G} = \tilde{A}$ near $\tilde{A}^n(p_0)$, $n \geq 1$. $\tilde{E}_G^{(2)}(\tilde{A}^n(p_0))$ are $\tilde{G}$-invariant and $\tilde{E}_A^{(2)}(\tilde{A}^n(p_0))$ are $\tilde{A}$-invariant for all $n \in \mathbb{Z}$. Therefore, if $\tilde{E}_G^{(2)}(\tilde{A}(p_0)) \neq \tilde{E}_A^{(2)}(\tilde{A}(p_0))$ then $\tilde{E}_G^{(2)}(\tilde{A}^n(p_0)) \neq \tilde{E}_A^{(2)}(\tilde{A}^n(p_0))$ for all $n \geq 1$.

Let us now prove property V. We show that if V is not true, the angle between $\tilde{E}_G^{(2)}(p)$ and $\tilde{E}_A^{(2)}(p)$ will exceed the upper bound given in Theorem 1 at some point.

To estimate the angle, we assume that

$$\tilde{E}_G^{(2)}(p) = \{t(u(p)\tilde{e}_A^{(1)} + \tilde{e}_A^{(2)} + w(p)\tilde{e}_A^{(3)}) : t \in \mathbb{R}\},$$

where $u(p)$ and $w(p)$ are two continuous functions in $\mathbb{R}^3$ and $u^2(p) + w^2(p) \leq 1$. The differential of $\tilde{G}$ is given by:

$$D\tilde{G} = \begin{pmatrix} \lambda_1 + \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta} & \frac{\partial f}{\partial \gamma} \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Since $\tilde{E}_G^{(2)}(p)$ is invariant under $D\tilde{G}$ we have

$$C\begin{pmatrix} u(p) \\ 1 \\ w(p) \end{pmatrix} = \begin{pmatrix} \lambda_1 + \frac{\partial f}{\partial \alpha}(\tilde{G}^{-1}(p)) & \frac{\partial f}{\partial \beta}(\tilde{G}^{-1}(p)) & \frac{\partial f}{\partial \gamma}(\tilde{G}^{-1}(p)) \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} u(\tilde{G}{-}1(p)) \\ 1 \\ w(\tilde{G}{-}1(p)) \end{pmatrix}.$$

This implies that

$$\lambda_2 u(p) = \left[\lambda_1 + \frac{\partial f}{\partial \alpha}(\tilde{G}^{-1}(p))\right]u(\tilde{G}^{-1}(p)) + \frac{\partial f}{\partial \beta}(\tilde{G}^{-1}(p)) + \frac{\partial f}{\partial \gamma}(\tilde{G}^{-1}(p))w(\tilde{G}^{-1}(p)). \tag{2.2}$$

$$\lambda_2 w(p) = \lambda_3 w(\tilde{G}^{-1}(p)). \tag{2.3}$$

From equation (2.3) we have that $w(p) = 0$ for all $p \in \mathbb{R}^3$. Thus, we obtain that

$$\tilde{E}_G^{(2)}(p) = \{t(u(p)\tilde{e}_A^{(1)} + \tilde{e}_A^{(2)}) : t \in \mathbb{R}\},$$

where $|u(p)| \leq 1$ and

$$\lambda_2 u(p) = \left[\lambda_1 + \frac{\partial f}{\partial \alpha}(\tilde{G}^{-1}(p))\right]u(\tilde{G}^{-1}(p)) + \frac{\partial f}{\partial \beta}(\tilde{G}^{-1}(p)). \tag{2.4}$$

Assume that V is not true, i.e., $\tilde{E}_G^{(2)}(\tilde{A}(p_0)) = \tilde{E}_A^{(2)}(\tilde{A}(p_0))$. There exists $q \in B_\epsilon(p_0)$ such that $\tilde{G}(q) = \tilde{A}(p_0) = (0, \lambda_2\beta_0, 0)$. We estimate $|u(q)|$ which gives the angle between $\tilde{E}_G^{(2)}(q)$ and $\tilde{E}_A^{(2)}(q)$. Suppose that $q = (\alpha', \beta', \gamma')$. Then,

$$\tilde{G}(q) = (\lambda_1\alpha' + f(\alpha', \beta', \gamma'), \lambda_2\beta', \lambda_3\gamma').$$

Thus, $\gamma' = 0$, $\beta' = \beta_0$ and $\lambda_1\alpha' + f(\alpha', \beta', \gamma') = 0$. It follows that $\alpha' = -\lambda_1^{-1} f(\alpha', \beta', \gamma')$. Therefore,

$$|\alpha'| = \frac{|f(\alpha', \beta', \gamma')|}{\lambda_1} \le \frac{\delta}{\lambda_1} \le \frac{\epsilon}{4}.$$

This implies that $q \in B_{\epsilon/3}(p_0)$. The assumption $\tilde{E}_G^{(2)}(\tilde{A}(p_0)) = \tilde{E}_A^{(2)}(\tilde{A}(p_0))$ means that $u(\tilde{A}(p_0)) = 0$. We have by equation (2.4) that

$$\left[\lambda_1 + \frac{\partial f}{\partial \alpha}(q)\right] u(q) + \frac{\partial f}{\partial \beta}(q) = 0$$

and, hence

$$u(q) = -\frac{\frac{\partial f}{\partial \beta}(q)}{\lambda_1 + \frac{\partial f}{\partial \alpha}(q)}.$$

According to the definition of the function f

$$|u(q)| = \frac{|\frac{\partial f}{\partial \beta}(q)|}{|\lambda_1 + \frac{\partial f}{\partial \alpha}(q)|} \ge \frac{\frac{\delta}{4}}{2\lambda_1} = \frac{\delta}{8\lambda_1}.$$

We consider the angles between $\tilde{E}_G^{(2)}$ and $\tilde{E}_A^{(2)}$ at points $\tilde{G}^{-1}(q), \tilde{G}^{-2}(q), \cdots, \tilde{G}^{-K}(q)$. Since all these points are outside of $B_\epsilon(p_0)$, $\partial f/\partial \alpha = \partial f/\partial \beta = 0$ at all of them. It follows

$$\lambda_2 u(\tilde{G}^{-j}(q)) = \lambda_1 u(\tilde{G}^{-j-1}(q))$$

and

$$u(\tilde{G}^{-j-1}(q)) = \left(\frac{\lambda_2}{\lambda_1}\right) u(\tilde{G}^{-j}(q)), \text{ for } j = 0, 1, \cdots, K-1.$$

Thus,

$$u(\tilde{G}^{-K}(q)) = \left(\frac{\lambda_2}{\lambda_1}\right)^{K-1} u(\tilde{G}^{-1}(q)) = \left(\frac{\lambda_2}{\lambda_1}\right)^{K} u(q).$$

$$|u(\tilde{G}^{-K}(q))| = \left(\frac{\lambda_2}{\lambda_1}\right)^{K} |u(q)| \ge \left(\frac{\lambda_2}{\lambda_1}\right)^{K} \cdot \frac{\delta}{8\lambda_1}. \tag{2.5}$$

Comparing (2.5) with what we have in Remark to Theorem 1, we obtain

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{K} \cdot \frac{1}{8\lambda_1} < L.$$

This contradicts (2.1).

We conclude the proof of Theorem 2. The first statement is obvious. Since $\tilde{A}^n p_0 = (0, \lambda_2^n \beta_0, 0) \to 0$ when $n \to +\infty$, we have $\tilde{A}^n p_0 \in \tilde{V}_2(0)$ for all sufficiently large n. Property VI means that $\tilde{E}_G^{(2)}(\tilde{A}^n(p_0))$ is never tangential to the local smooth manifold $\tilde{V}_A^{(2)}(0)$. Thus $\tilde{W}_G^{(2)}(0)$ and $\tilde{V}_A^{(2)}(0)$ do not coincide near the origin. This implies that $\tilde{W}_G^{(2)}(0)$ is not of class C^{N+1}. This proves the second statement.

To prove the third statement let $W_G^{(2)}(\pi(0))$ be the integral manifold of $E_G^{(2)}$. It is not of class C^{N+1} at $x = \pi(0)$. Denote $S = \{x \in \mathbb{T}^3 : W_G^{(2)}(x) \notin C^{N+1}\}$ and

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$I = \{x \in \mathbb{T}^3 :$ the negative semi-trajectory $\{G^{-n}(x) : n \geq 0\}$ is dense in $\mathbb{T}^3\}$. Since G is conjugate to a linear Anosov diffeomorphism of $\mathbb{T}^3$, G is transitive. Therefore I is a Baire set [**S**]. We will show that $I \subset S$. Assume that $x \in I$ but $x \notin S$. Then $W_G^{(2)}(x)$ is of class C^{N+1}, which implies $V_G^{(2)}(x) \subset W_G^{(2)}(x)$. Thus, by the invariance of $W_G^{(2)}$ and $V_G^{(2)}$ we have $V_G^{(2)}(G^{-n}(x)) \subset W_G^{(2)}(G^{-n}(x))$ and the size of $V_G^{(2)}(G^{-n}(x))$ grows as n increases. Since $\{G^{-n}(x)\}$ is dense in $\mathbb{T}^3$, there exists a subsequence n_i such that $G^{-n_i}(x) \to \pi(0)$. In [**P**], the author showed that $V_G^{(2)}(x)$ depends continuously on x in C^r-topology. It follows that $V_G^{(2)}(G^{-n_i}(x)) \to V_G^{(2)}(\pi(0))$. On the other hand, since $W_G^{(2)}$ is a continuous foliation, $V_G^{(2)}(\pi(0)) \subset W_G^{(2)}(\pi(0))$ by the uniqueness of the limit. This implies that $W_G^{(2)}(\pi(0))$ is also of class C^{N+1} as $V_G^{(2)}(\pi(0))$ is. This contradicts the assumption that $x \notin S$.

Proof of Theorem 3. We will follow the same construction used in the proof of Theorem 2. Choose $\delta_0 < \frac{1}{2}\delta_1$ so small that for any $F \in \mathrm{Diff}^1(\mathbb{T}^3)$ with $d(F, A) < 2\delta_0$ the map F is topologically conjugate to A and

$$\lambda_i - \sigma \leq ||DF\xi_F^{(i)}|| \leq \mu_i + \sigma$$

for $\xi_F^{(i)} \in E_F^{(i)}$ with $||\xi_F^{(i)}|| = 1$, where $\sigma > 0$ is small. We know that $E_F^{(2)}$ is integrable and its integral manifolds form a continuous C^N-foliation $W_F^{(2)}$. If it is not of class C^{N+1} at least at some point $x \in M$, then we can choose $G = F$. Let us assume now that $W_F^{(2)}$ is of class C^{N+1}. Then it is a C^∞-foliation on M. In any neighborhood of F, we will construct a smooth map $G \in \mathrm{Diff}^1(\mathbb{T}^3)$ such that G induces a splitting of $T\mathbb{T}^3 = \bigoplus_{i=1}^3 E_G^{(i)}$ with $E_G^{(2)}$ integrable but not of class C^{N+1}.

Construction of the map G. Let $z \in T\mathbb{T}^3$ be the unique fixed point of F and $W_F^{(i)}(z)$ be the integral manifold of $E_F^{(i)}$ passing through z, $i = 1, 2, 3$. We define G by constructing a diffeomorphism $\tilde{G}$ of the universal cover $\mathbb{R}^3$. Let $p_z \in \mathbb{R}^3$ be the fixed point of the lift $\tilde{F}$. We will use '$\tilde{\ }$' to denote the objects in the covering space. Choose a point $p_0 \in \tilde{W}_F^{(2)}(p_z)$ close to p_z and $\epsilon > 0$ small enough such that the set $\tilde{F}(B_\epsilon(p_0)), B_\epsilon(p_0), \tilde{F}^{-1}(B_\epsilon(p_0)), \cdots, \tilde{F}^{-K}(B_\epsilon(p_0))$ are located in the interior of a fundamental domain D and all disjoint. The constant K will be determined later. If we choose ϵ sufficiently small and p_0 close to p_z, K can be taken arbitrarily large. We can also require that $B_\epsilon(p_z) \bigcap \tilde{F}^j(B_\epsilon(p_0)) = \emptyset$, $j = 1, 0, -1, \cdots, -K$. When ϵ is small, for any $p \in \mathbb{R}^3$, the basis $\{\tilde{e}_F^{(1)}(p), \tilde{e}_F^{(2)}(p), \tilde{e}_F^{(3)}(p)\}$ can serve as a local coordinate system in $B_\epsilon(p)$ such that $B_\epsilon(p)$ is identified with an open ball in $\mathbb{R}^3$ centered at the origin. We use $(\alpha_p(q), \beta_p(q), \gamma_p(q))$ to denote a point $q \in B_\epsilon(p)$ in this local coordinate system. In particular $\alpha_p(p) = \beta_p(p) = \gamma_p(p) = 0$. Let us fix δ, $0 < \delta < \delta_0$, $K = K(\delta)$ a positive integer and $\epsilon = \epsilon(K, \delta)$ which we will determine later. Let $f : D \to \mathbb{R}$ be a smooth function satisfying

(1) $f \equiv 0$ if $p \notin B_\epsilon(p_0)$;

(2) $|f| \leq l_1\epsilon$; $||Df|| \leq \frac{1}{2}\delta$, where l_1 is a constant;

(3) $\left|\dfrac{\partial f}{\partial \beta_{p_0}}(p)\right| > \frac{\delta}{4}$ if $p \in B_{\epsilon/3}(p_0)$.

Define $\tilde{G}(p) = \tilde{F}(p)$ if $p \notin B_\epsilon(p_0)$, $p \in D$ and $\tilde{G}(p) = \tilde{F}(p) + (f(\alpha_{p_0}(p), \beta_{p_0}(p), \gamma_{p_0}(p)), 0, 0)$ if $p \in B_\epsilon(p_0)$. The map $\tilde{G}$ is well defined in D and we set $G = \pi \cdot \tilde{G}$.

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We claim that if ϵ, l_1 and K are chosen in an appropriate way then G and $\tilde{G}$ have the following properties:

I $G \in \mathrm{Diff}^1(\mathbb{T}^3)$, C^∞ *and* $d_{C^1}(F, G) < \delta$. *Thus, G induces a splitting of* $T\mathbb{T}^3 = \bigoplus_{i=1}^3 E_G^{(i)}$ *and respectively,* $\tilde{G}$ *induces a splitting of* $T\mathbb{R}^3 = \bigoplus_{i=1}^3 \tilde{E}_G^{(i)}$ *and* $D\pi \tilde{E}_G^{(i)}(p) = E_G^{(i)}(\pi(p))$;

II $\tilde{E}_G^{(2)}(\tilde{F}(p_0)) \neq \tilde{E}_F^{(2)}(\tilde{F}(p_0))$;

III $\tilde{E}_G^{(2)}(\tilde{F}^n(p_0)) \neq \tilde{E}_F^{(2)}(\tilde{F}^n(p_0))$, *for all* $n \geq 1$.

We only need to prove II and we will follow closely the proof of property V in Theorem 2.

Proof of Property II. Suppose that II is not true, i.e., $\tilde{E}_G^{(2)}(\tilde{F}(p_0)) = \tilde{E}_F^{(2)}(\tilde{F}(p_0))$. Let $q = (\alpha_{p_0}(q), \beta_{p_0}(q), \gamma_{p_0}(q)) \in B_\epsilon(p_0)$ be a point such that $\tilde{G}(q) = \tilde{F}(p_0)$. Then, $\tilde{G}(q) = (0, 0, 0)$ in $(\alpha_{\tilde{F}(p_0)}, \beta_{\tilde{F}(p_0)}, \gamma_{\tilde{F}(p_0)})$ coordinate system in $B_\epsilon(\tilde{F}(p_0))$. This implies that $\tilde{F}(\alpha_{p_0}(q), \beta_{p_0}(q), \gamma_{p_0}(q)) + (f(\alpha_{p_0}(q), \beta_{p_0}(q), \gamma_{p_0}(q)), 0, 0) = 0$. By assumption 2 $||\tilde{F}(\alpha_{p_0}(q), \beta_{p_0}(q), \gamma_{p_0}(q))|| \leq l_1\epsilon$. Then there exist a constant l_F depending only on the map F such that $||(\alpha_{p_0}(q), \beta_{p_0}(q), \gamma_{p_0}(q))|| \leq l_F l_1 \epsilon$. If l_1 is chosen to be small enough we can assume that

$$l_F l_1 < \frac{1}{3}. \tag{2.6}$$

Thus, $q \in B_{\epsilon/3}(p_0)$. We estimate the angle between $E_G^{(2)}(q)$ and $E_F^{(2)}(q)$. For any $p \in \mathbb{R}^3$ we write

$$\tilde{e}_G^{(2)}(p) = u_1(p)\tilde{e}_F^{(1)}(p) + u_2(p)\tilde{e}_F^{(2)}(p) + u_3(p)\tilde{e}_F^{(3)}(p).$$

Applying $D\tilde{G}$, we have

$$D\tilde{G}(\tilde{e}_G^{(2)}(p)) = (D\tilde{F} + D\tilde{h})(\tilde{e}_F^{(1)}(p)\ \tilde{e}_F^{(2)}(p)\ \tilde{e}_F^{(3)}(p)) \begin{pmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \end{pmatrix}.$$

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be matrices representing the linear maps which transform the basis $\{\tilde{e}_F^{(i)}(F(p_0))\}$ to the basis $\{\tilde{e}_F^{(i)}(F(q))\}$ and $\{\tilde{e}_F^{(i)}(p_0)\}$ to $\{\tilde{e}_F^{(i)}(q)\}$ respectively. i.e.,

$$(\tilde{e}_F^{(1)}(F(q))\ \tilde{e}_F^{(2)}(F(q))\ \tilde{e}_F^{(3)}(F(q))) = (\tilde{e}_F^{(1)}(F(p_0))\ \tilde{e}_F^{(2)}(F(p_0))\ \tilde{e}_F^{(3)}(F(p_0)))\mathbf{L}_1,$$

$$(\tilde{e}_F^{(1)}(q)\ \tilde{e}_F^{(2)}(q)\ \tilde{e}_F^{(3)}(q)) = (\tilde{e}_F^{(1)}(p_0)\ \tilde{e}_F^{(2)}(p_0)\ \tilde{e}_F^{(3)}(p_0))\mathbf{L}_2.$$

By virtue of the invariance of $\tilde{E}_G^{(2)}$ under the map $\tilde{G}$, we have

$$C \begin{pmatrix} u_1(\tilde{F}(p_0)) \\ u_2(\tilde{F}(p_0)) \\ u_3(\tilde{F}(p_0)) \end{pmatrix} = \left(\mathbf{L}_1 \begin{pmatrix} \lambda_1(q) & 0 & 0 \\ 0 & \lambda_2(q) & 0 \\ 0 & 0 & \lambda_3(q) \end{pmatrix} + \begin{pmatrix} \frac{\partial f}{\partial \alpha_{p_0}} & \frac{\partial f}{\partial \beta_{p_0}} & \frac{\partial f}{\partial \gamma_{p_0}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{L}_2 \right) \begin{pmatrix} u_1(q) \\ u_2(q) \\ u_3(q) \end{pmatrix}.$$

The Hölder continuity of $\tilde{E}_F^{(i)}(x)$ implies

$$||I - \mathbf{L}_1|| \leq l' \cdot \epsilon^\tau \quad \text{and} \quad ||I - \mathbf{L}_2|| \leq l'' \cdot \epsilon^\tau$$

for some constant l', l''. Since the equality $\tilde{E}_G^{(2)}(\tilde{F}(p_0)) = \tilde{E}_F^{(2)}(\tilde{F}(p_0))$ implies $u_1(\tilde{F}(p_0)) = 0$, we have

$$\left(\lambda_1(q) + \frac{\partial f}{\partial \alpha_{p_0}} + l_{11}\right) u_1(q) + \left(\frac{\partial f}{\partial \beta_{p_0}} + l_{12}\right) u_2(q) + \left(\frac{\partial f}{\partial \gamma_{p_0}} + l_{13}\right) u_3(q) = 0 ,$$

where $|l_{1i}| \le l_2 \cdot \epsilon^\tau$ for some constant l_2. By Theorem 1 we obtain

$$|u_1(q)| \ge \frac{(1 - 2L\delta^\tau)(\frac{1}{4}\delta - l_2\epsilon^\tau) - L\delta^\tau(\delta + l_2\epsilon^\tau)}{\left|\lambda_1(q) + \dfrac{\partial f}{\partial \alpha_{p_0}} + l_{11}\right|} \ge M_0\delta,$$

where M_0 is a constant depending only on the diffeomorphism A and ϵ satisfies

$$l_2\epsilon^\tau \le \tfrac{1}{8}\delta. \tag{2.7}$$

Consider the angles between $E_G^{(2)}$ and $E_F^{(2)}$ at points $\tilde{F}^{-1}(q), \tilde{F}^{-2}(q), \cdots, \tilde{F}^{-K}(q)$. By the construction, $\tilde{G} = \tilde{F}$ in the neighborhoods of these points. This implies $D\tilde{G} = D\tilde{F}$. Using $\{\tilde{e}_F^{(i)}(\tilde{F}^{-j}(q))\}$ as a basis in $B_\epsilon(\tilde{F}^{-j}(q))$, $j = 0, 1, 2, \cdots, K$ we obtain

$$C\begin{pmatrix} u_1(\tilde{F}^{-j-1}(q)) \\ u_2(\tilde{F}^{-j-1}(q)) \\ u_3(\tilde{F}^{-j-1}(q)) \end{pmatrix} = \begin{pmatrix} \lambda_1(\tilde{F}^{-j}(q)) & 0 & 0 \\ 0 & \lambda_2(\tilde{F}^{-j}(q)) & 0 \\ 0 & 0 & \lambda_3(\tilde{F}^{-j}(q)) \end{pmatrix} \begin{pmatrix} u_1(\tilde{F}^{-j}(q)) \\ u_2(\tilde{F}^{-j}(q)) \\ u_3(\tilde{F}^{-j}(q)) \end{pmatrix} .$$

Therefore,

$$Cu_1(\tilde{F}^{-j-1}(q)) = \lambda_1(\tilde{F}^{-j}(q)) \cdot u_1(\tilde{F}^{-j}(q)), \ Cu_2(\tilde{F}^{-j-1}(q)) = \lambda_2(\tilde{F}^{-j}(q)) \cdot u_2(\tilde{F}^{-j}(q))$$

Hence,

$$u_1(\tilde{F}^{-j}(q)) = \frac{\lambda_2(\tilde{F}^{-j}(q))}{\lambda_1(\tilde{F}^{-j}(q))} \cdot \frac{u_2(\tilde{F}^{-j}(q))}{u_2(\tilde{F}^{-j-1}(q))} \cdot u_1(\tilde{F}^{-j-1}(q)),$$

$$|u_1(\tilde{F}^{-j}(q))| \ge \frac{(\lambda_2 - \sigma)}{(\lambda_1 + \sigma)} \cdot \frac{(1 - 2L\delta^\tau)}{(1 + 2L\delta^\tau)} \cdot |u_1(\tilde{F}^{-j-1}(q))| .$$

We may choose σ and δ such that

$$t = \left|\frac{(\lambda_2 - \sigma)}{(\lambda_1 + \sigma)} \cdot \frac{(1 - 2L\delta^\tau)}{(1 + 2L\delta^\tau)}\right| > 1.$$

Then, we have

$$|u_1(\tilde{F}^{-K}(q))| \ge t^K M_0\delta . \tag{2.8}$$

Comparing (2.8) with Theorem 1 we obtain

$$t^K M_0\delta < L\delta^\tau. \tag{2.9}$$

Choose K such that

$$t^K M_0\delta > L\delta^\tau. \tag{2.10}$$

This is possible because δ is fixed and $t > 1$. Then we choose ϵ satisfying (2.7) and also so small such that $\tilde{F}(B_\epsilon(p_0)), B_\epsilon(p_0), \tilde{F}^{-1}(B_\epsilon(p_0)), \cdots, \tilde{F}^{-K}(B_\epsilon(p_0))$ are located in the interior of a fundamental domain D and disjoint. Choose now l_1 according to (2.6). Since (2.10) contradicts (2.9), the map G constructed above with respect to these parameters δ, ϵ, l_1, and K satisfies all the desired properties. This claim is proved and the rest of the proof of Theorem 3 is obvious and omitted.

Proof of Theorem 4. We write $\mathbb{T}^n$ as $\mathbb{T}^2 \times \mathbb{T}^{n-2}$ and denote by x_1, x_2 the components in the first and second factor respectively.

Consider a linear Anosov diffeomorphism in the form $A(x_1, x_2) = (A_1 x_1, A_2 x_2)$ where $A_1 : \mathbb{T}^2 \to \mathbb{T}^2$ and $A_2 : \mathbb{T}^{n-2} \to \mathbb{T}^{n-2}$ are linear Anosov automorphisms. Let λ, μ, $|\lambda| < 1$ be eigenvalues of the matrix A_1. Since $\det A_1 = 1$, we have that $\mu = 1/\lambda$. We also assume that $|\lambda| \le \frac{1}{4}$. One can choose the matrix A_2 such that: 1) it has the only simple eigenvalue γ with $|\gamma| < \lambda$; 2) the numbers γ and λ satisfy the non-resonance condition (1.1). Consider the map G such that

$$G(x_1, x_2) = (A_1(x_1), A_2 x_2 + \Psi(x_1))$$

where Ψ is a function with values on the line generated by the eigenvector corresponding to γ. This map was introduced in **[Ll]** in order to show that there is no complete set of invariants for smooth conjugacy depending on germs around periodic orbits. In this paper we will show that the same map provides also an example of Anosov map with intermediate distributions not as smooth as the map and whose periodic points have the same eigenvalues. Namely we will show that it is possible to choose Ψ such that all the statements of Theorem 4 are satisfied.

First we note that any G in the above form has the same eigenvalues as A has. Observe that

$$DG^n(x) = \begin{pmatrix} A_1^n & R_n(x_1) \\ 0 & A_2^n \end{pmatrix} \tag{2.11}$$

and the eigenvalues of $DG^n(x)$ are the same as those of the diagonal blocks. The statement about the Mather spectrum follows since G can also be written as a block upper triangular operator in a way similar to (2.11).

We now prove Theorem 4(2)(a). Assume the contrary, given two sufficiently close points x, y, one can find a C^∞-curve ℓ_{xy} connecting them and being contained in $V_G^\lambda(x)$ such that $T_z \ell_{x.y} \in E_G^\lambda(z)$ for all z in the curve. Since $DG(x) E_G^\lambda(x) = E_G^\lambda(G(x))$ we have that $T_z G^n(\ell_{x,y}) \in E_G^\lambda(z)$ for all $z \in G^n(\ell_{x,y})$. Then, by the mean value theorem, it follows that:

$$\begin{aligned} &|\tilde{G}^{-n}(x) - \tilde{G}^{-n}(y)| \le C_\epsilon (|\lambda| - \epsilon)^{-n} \quad \text{for any } n > 0 \\ &|\tilde{G}^n(x) - \tilde{G}^n(y)| \to_{n\to\infty} 0. \end{aligned} \tag{2.12}$$

To complete the proof of Theorem 4 we will give a rather explicit description of the set $W_G^\lambda(y)$ of points y that satisfy (2.12). We will show that for certain choice of Ψ this set does not contain any C^∞-curves. This contradiction establishes the theorem.

We observe that $W_A^\lambda(x) = \{x + t(e_\lambda, 0) | t \in \mathbb{R}\}$ where e_λ is an eigenvector of A_1 corresponding to λ. If h is a conjugating homeomorphism between A and G which is homologous to the identity, i.e.,

$$h \circ A = G \circ h \tag{2.13}$$

then

$$h(W_A^\lambda(x)) = W_G^\lambda(h(x)).$$

Moreover, the sequence of matrices $A^n(x) - A^n(y)$ converges to zero if and only if $h \circ A^n(x) - h \circ A^n(y)$ converges to zero. By (2.13) this is equivalent to the convergence of

$G^n \circ h(x) - G^n \circ h(y)$ to zero. Using a lift to the universal cover, we have $\tilde{h}(x) = x + \hat{h}(x)$ with $\hat{h}$ uniformly bounded so that $|\tilde{h} \circ \tilde{A}^{-n}(x) - \tilde{h} \circ \tilde{A}^{-n}(x)| \leq |\tilde{A}^{-n}(x) - \tilde{A}^{-n}(y)| + 2||\hat{h}||_{L^\infty}$ Again using (2.13), we conclude that $G^{-n}(h(x)) - G^{-n}(h(y)$ grows with an exponent less than λ^{-1} if and only if $\tilde{A}^{-n}(x) - \tilde{A}^{-n}(y)$ does. This finishes the proof of the claim.

A simple calculation shows that if we define $h(x_1, x_2) = (x_1, x_2 + \Gamma(x_1))$ it satisfies (2.13) if and only if the function Γ satisfies

$$\Gamma(x_1) = \Psi(A_1^{-1}(x_1)) + \gamma \Gamma(A_1^{-1}(x_1)). \tag{2.14}$$

In that case, using the remark before, we obtain that the manifold $W_G^\lambda(x)$ is given parametrically by

$$(x_1 + te_\lambda, x_2 + \Gamma(x_1 + te_\lambda)).$$

Theorem 4 will be established as soon as we construct a function Ψ such that the function Γ, satisfying (2.13), is not C^∞ in any open interval in the direction of e_λ. We observe that the sum

$$\Gamma(x_1) = \sum_{n=0}^{\infty} \gamma^n \Psi(A_1^{-(n+1)}(x_1)) \tag{2.15}$$

defines a continuous function since Ψ is uniformly bounded and, hence, the series converges uniformly. The uniform convergence implies also that the function $\Gamma(x_1)$ solves (2.14). One can prove this by substituting (2.15) into (2.14) and rearranging the terms in the resulting series. If we take Ψ to be a trigonometric polynomial, the fact that the function Γ given by (2.15) is not differentiable on any interval is the celebrated analysis of the Riemann–Weirstrass examples of nowhere differentiable functions.

In our case we can choose Ψ in such a way that this analysis becomes completely elementary. Since $|\lambda| < 1/4$ one can easily construct two disjoint closed stripes Ω_1, Ω_2, in a direction transversal (and close to perpendicular) to the vector e_λ in such a way that any interval I of length $\sqrt{2}$ along the direction of e_λ intersects both Ω_1 and Ω_2 in intervals I_1, I_2 of length at least $\sqrt{2}/4$. Denote by N the largest integer smaller than $\ln \gamma / \ln \lambda$. We can arrange that $D_{e_\lambda}^N \Psi(x_1) = 1$ if $x_1 \in \Omega_1$ and $= 0$ if $x_1 \in \Omega_2$. We claim that Ψ satisfies all the statements in Theorem 4.

Given any sequence $\sigma = (\sigma_0, \cdots, \sigma_n \cdots) \in \{1, 2\}^{\mathbb{N}}$ and any interval of length one along e_λ, we can find a point x_σ such that $A^{-i-1}x_\sigma \in \Omega_{\sigma_i}$. The point is unique on each interval along e_λ. Since $\lambda^{-N}\gamma < 1$ the series

$$D_{e_\lambda}^N \Gamma(x_1) = \sum_{n=0}^{\infty} \gamma^n \lambda^{-N(n+1)} D_{e_\lambda}^N \Psi(A^{-(n+1)}(x_1))$$

converges uniformly and is indeed the derivative of the function Γ.

Let $\sigma^{(n)}$ be the sequence with all components equal to one except for the nth. In any interval I of length one along the direction e_λ, there exists a point $x_{\sigma^{(n)}}^I$. We have that $|x_{\sigma^{(n)}}^I - x_{\sigma^{(n+1)}}^I| \leq K\lambda^n$. On the other hand, $|D_{e_\lambda}^N \Gamma(x_{\sigma^{(n)}}^I) - D_{e_\lambda}^N \Gamma(x_{\sigma^{(n+1)}}^I)| \geq K(\gamma\lambda^{-N})^n$.

This shows that the function Γ cannot be C^∞ in any interval of length 1 in the direction e_λ. If Γ is not C^∞ on any interval of length 1, then $\Gamma \circ A_1$ is not C^∞ on any interval of length λ along the direction e_λ. Using (2.14), we obtain that Γ cannot be C^∞ in any interval of length λ along e_λ. We can repeat the argument to conclude that

Γ is not C^∞ on any interval of strictly positive length in the direction e_λ. This shows that W_G^λ does not contain any C^∞-curves and, hence, the leaves of V_G^λ cannot contain segments in which they are tangential to the intermediate distribution.

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