ANATOLE KATOK'S WORKS ON HYPERBOLICITY, ENTROPY AND GEODESIC FLOWS

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ABSTRACT. This chapter contains Anatole Katok's works on Smooth Ergodic theory, which includes his works on hyperbolicity, entropy and geodesic flows. I will provide some overview of these works and describe some further developments which were stimulated by Katok's work.

1. INTRODUCTION

In this chapter the reader finds Anatole Katok's works on Smooth Ergodic theory. They cover a range of topics which I split in 5 groups:

(1) The Katok map, [2, 6]
(2) Bernoulli diffeomorphisms on compact manifolds, [2, 3, 5];
(3) General hyperbolic measures [4, 8];
(4) Entropy of dynamical systems [1, 7, 9, 12, 13, 15]
(5) Other topics [10, 11, 14].

Each group corresponds to a separated section in which I briefly discuss each paper in the group. In the end of many sections I add a subsection discussing further developments in the area stimulated and inspired by Katok's work.

2. PRELIMINARIES ON NON-UNIFORM HYPERBOLICITY

Originated in the works of Lyapunov [34] and Perron [46, 47] the nonuniform hyperbolicity theory has emerged as an independent discipline in the works of Oseledets [45] and Pesin [49]. Since then it has become one of the major parts of the general theory of dynamical systems and one of the main tools in studying highly sophisticated behavior associated with "deterministic chaos" – appearance of chaotic (turbulent-like) behavior in otherwise purely deterministic dynamical systems. We refer the reader to the article [25] by Hasselblatt and Katok for a discussion on the role of nonuniform hyperbolicity theory, its relations to and interactions with other areas of dynamics. See also the book by Barreira and Pesin [18] for a detailed presentation of the core of the nonuniform hyperbolicity theory.

2.1. **Lyapunov Exponents.** Nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. Let $f\colon M\to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M. Recall that given $x\in M$ and a vector $v\in T_xM$, the *Lyapunov exponent* of v at x is defined as

$$\chi^+(x,v)=\limsup_{t\to+\infty}\frac{1}{t}\log\|d_xf^tv\|.$$

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For a fixed $x \in T_xM$ the function $\chi^+(x,\cdot)$ attains only finitely many values $\chi_1^+(x) < \cdots < \chi^+_{p^+(x)}(x)$ each with multiplicity $k_i^+(x)$. The functions $\chi_i^+(x)$, $p^+(x)$, and $k_i^+(x)$ are invariant under f and Borel measurable.

Assume now that f preserves a Borel probability measure ν on M. This measure is called *hyperbolic* if for ν-a.e. $x \in M$ there exists a number $s = s(x)$, $1 \le s < p(x)$ such that

$$\chi_1(x) < \cdots < \chi_s(x) < 0 < \chi_{s+1}(x) < \cdots < \chi_{p(x)}(x).$$

In the case ν is ergodic the values of the Lyapunov exponent are constant almost everywhere, i.e., $k_i(x) = k_i^\nu$ and $\chi_i(x) = \chi_i^\nu$ for $i = 1, \dots, p(x) = p^\nu$.

2.2. **Non-uniform Hyperbolicity.** It follows from the Multiplicative Ergodic theorem that there is a subset $\Lambda \subset M$ of full measure ν consisting of points $x \in M$ called *Lyapunov-Perron regular* (or simply *regular*) for which there are numbers $0 < \lambda < 1$ and a sufficiently small $\varepsilon > 0$, as well as Borel functions $C(x) > 0$, $K(x) > 0$ such that

(NUH1) there is an invariant splitting $T_xM = E^s(x) \oplus E^u(x)$ into *stable* and *unstable subspaces* which depend Borel measurably on x;

(NUH2) for $v \in E^s(x)$ and $n > 0$,

$$\|d_xf^n v\| \le C(x)\lambda^n e^{\varepsilon n}\|v\|;$$

(NUH3) for $v \in E^u(x)$ and $n > 0$,

$$\|d_xf^{-n} v\| \le C(x)\lambda^n e^{\varepsilon |n|}\|v\|;$$

(NUH4) the angle $\angle(E_1(x), E_2(x)) \ge K(x)$;

(NUH5) for $n \in \mathbb{Z}$,

$$C(f^n(x)) \le C(x)e^{\varepsilon|n|}, \quad K(f^n(x)) \ge K(x)e^{-\varepsilon|n|}.$$

The inequalities in (NUH5) mean that estimates in (NUH2), (NUH3), and (NUH4) may deteriorate along the trajectory with sub-exponential rate. We stress that the rates of contraction along stable subspaces and expansion along unstable subspaces are exponential and hence, prevail.

2.3. **Regular Sets.** Given $\ell \ge 1$, consider the *regular* (or *level*) set Λ_ℓ given by

$$\Lambda_\ell = \left\{x \in \Lambda : C(x) \le \ell, K(x) \ge \frac{1}{\ell}\right\}.$$

It is easy to see that the sets Λ_ℓ are nested and exhaust Λ that is

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_\ell \subset \cdots \quad \text{and} \quad \bigcup_{\ell \ge 1} \Lambda_\ell = \Lambda.$$

2.4. **Stable and Unstable Local Manifolds.** The stable and unstable distributions can be locally "integrated" into local *stable* and *unstable manifolds* $V^s(x)$ and $V^u(x)$ such that $x \in V^{s/u}(x)$, $T_xV^{s/u}(x) = E^{s/u}(x)$, and for $y \in V^{u/s}(x)$ $n \ge 0$,

$$d(f^n(x), f^n(y)) \le T(x)\lambda^n e^{\varepsilon n} d(x,y), \quad y \in V^s(x),$$
$$d(f^{-n}(x), f^{-n}(y)) \le T(x)\lambda^n e^{\varepsilon n} d(x,y), \quad y \in V^u(x),$$

where d is the Riemannian distance in M and $T(x)$ is a Borel function satisfying

$$T(f^m(x)) \le T(x)e^{10\varepsilon|m|}, \quad m \in \mathbb{Z}.$$

For every regular set Λ_ℓ unstable and stable local manifolds $V^{u/s}(x)$ depend continuously on x and there is a number r_ℓ such that these manifolds can be constructed in such a way that their "size" is r_ℓ. For $x, y \in \Lambda_\ell$ let

$$[x, y] = V^u(x) \cap V^s(y). \tag{1}$$

There is $\delta_\ell > 0$ such that if $d(x, y) < \delta_\ell$, then the above intersection is transversal and consists of a single point.

2.5. **Smooth Invariant Measures.** We now consider the particular case when f preserves a smooth measure ν on M, i.e., a measure which is equivalent to volume. In this case it is shown in [49] (see also [18]) that ν has a ergodic decomposition into countably many ergodic components of positive volume. On each such a component f is Bernoulli up to a rotation that is if f is weakly-mixing, then f is Bernoulli. Similar results hold if ν is a Sinai-Ruelle-Bowen (SRB) measure on M, i.e., a measure whose conditional measures it generates on unstable manifolds are absolutely continuous with respect to volume, see [30].

2.6. **Pseudo-orbits and Shadowing.** A sequence of points $\{x_n\}_{m<n<k}$ (where $-\infty \le m < k \le \infty$) in M is called ε*-pseudo-orbit* (where $\varepsilon > 0$ is a constant) if $d(f(x_n), x_{n+1}) < \varepsilon$ for all $m < n < k$. We say that an ε-pseudo-orbit $\{x_n\}$ is δ*-shadowed* by the orbit $\mathcal{O}(x)$ of a point x (where $\delta > 0$ is a constant), if $d(f^n(x), x_n) < \delta$.

3. The Slow-down Procedure in Hyperbolic Dynamics

3.1. **The Katok Map.** In his seminal paper [2] Katok constructed the first example of a C^∞ area preserving diffeomorphism f_{T^2} of the 2-torus T^2 with nonzero Lyapunov exponents almost everywhere, which is not an Anosov map. f_{T^2} is commonly known as the *Katok map.* Starting with a hyperbolic toral automorphism A, Katok's construction works to destroy the uniform hyperbolic structure associated with A by slowing down trajectories in the disk D^2 around the origin of small radius r.[1] This means that the time, a trajectory of f_{T^2} stays in D^2, gets larger and larger the closer the trajectory passes by the origin, while the map is unchanged outside D^2. The slow-down procedure is controlled by a "slow-down function" ψ which depends only on the distance from the point in D^2 and is such that

(K1) ψ is of class C^∞ everywhere but at the origin;
(K2) $\psi(0) = 0$ and $\psi(u) = 1$ for $u \ge r$;
(K3) $\psi'(u) > 0$ for $0 < u < r$;
(K4) $\int_0^1 \frac{du}{\psi(u)} < \infty$.

It is easy to see that along the trajectories, starting on the stable and unstable separatrices of the origin, the map f_{T^2} has zero Lyapunov exponents ensuring that it is not an Anosov map. Although a "typical" trajectory may spend arbitrarily long periods of time in D^2, the average time it stays in D^2 is proportional to the area of D^2 and hence, is small. This is the main reason the map f_{T^2} has non-zero Lyapunov exponents almost everywhere.

The Katok map f_{T^2} has some other interesting properties:

- it is topologically conjugate to A;[2]

[1] A version of a slow down procedure was used in an earlier work [48] to construct the first example of a flow with non-zero Lyapunov exponents (except for the exponent in the flow direction) on a 3-manifold (see a detailed presentation of this construction in Section 6.5 of the book [18].

[2] The conjugacy map while continuous is not Hölder continuous which is one the main obstacles in studying hyperbolic and ergodic properties of the Katok map.

- its stable and unstable distributions are continuous are integrable to continuous stable and unstable foliations with smooth leaves;
- it lies on the boundary of Anosov diffeomorphisms in $\mathrm{Diff}^1(T^2, m)$ (here m is area).

3.2. **Smooth Realizations of Pseudo-Anosov Maps.** Pseudo-Anosov maps were introduced by Thurston in his work on classifying diffeomorphisms of compact C^∞ surfaces up to isotopy (see [54]). According to this classification, a diffeomorphism of a surface M is isotopic to a homeomorphism f satisfying one of the following properties:

(1) f is of finite order and is an isometry with respect to a Riemannian metric of constant curvature on M;
(2) f is a "reducible" diffeomorphism, that is, a diffeomorphism leaving invariant a closed curve;
(3) f is a *pseudo-Anosov map*, i.e., it is a surface homeomorphism that is differentiable except at most at finitely many points called *singularities*.

A pseudo-Anosov map has some interesting properties: 1) it minimizes both the number of periodic points (of any given period) and the topological entropy in their isotopy classes; 2) it preserves an absolutely continuous measure with C^∞ density, which is positive except at the singularities and it is Bernoulli with respect to this measure (see [24, Exposé 10]).

If M is a torus, then any pseudo-Anosov map is an Anosov diffeomorphism (see [24, Exposé 1]). However, if M has genus greater than 1, then Gerber and Katok [6] show that a pseudo-Anosov map cannot be conjugated to a diffeomorphism, which is smooth outside the singularities or even outside a sufficiently small neighborhood of the singularities. Thus in order to find smooth models of pseudo-Anosov maps one may have to apply some nontrivial construction, which is global in nature. In [6], Gerber and Katok constructed, for every pseudo-Anosov map f, a C^∞ diffeomorphism g, which is topologically conjugate to f through a homeomorphism which is isotopic to the identity, preserves a smooth measure on M and is Bernoulli with respect to this measure.

Starting with a pseudo-Anosov map f they obtain the diffeomorphism g applying a slow-down procedure similar to the one used in the construction of the Katok map of the 2-torus. They modified and adopted this procedure to reflect on the structure of singularities of f which do not admit a locally stable or unstable subspace forming a curve, but rather forming the prongs that meet at the singularity. Furthermore, whereas the slow-down function used in the construction of the Katok map depends only on the distance from a point in D^2 to the origin, the choice of slow-down function for a pseudo-Anosov homeomorphism depends on the number of prongs of the singularity.[3] This substantially affects the analysis of the behavior of the trajectories near the singularities.

4. Bernoulli Diffeomorphisms on Compact Manifolds

4.1. **Construction of Bernoulli Diffeomorphisms I: The Two-dimensional Case.** Katok's construction of the map f_{T^2} was motivated by the following problem:

[3]This number depends only on the genus of the surface.

Problem 1. *Does any smooth compact Riemannian manifold carry a C^∞ volume preserving diffeomorphism which is Bernoulli?*

In [2] this problem was solved by Katok in the two dimensional case. First, using results of non-uniform hyperbolicity theory, one can show that the Katok map f_{T^2} is Bernoulli. Next Katok used a well-known topological procedure that, given a surface M, allows one to first map the torus T^2 onto the unit disk D^2 and then "embed" D^2 into M. He then proved that if the slow-down function ψ is chosen in such a way that its inverse is sufficiently flat at the origin,[4] then f_{T^2} can be carried over to a C^∞ area preserving Bernoulli diffeomorphism f_{D^2} of the disk D^2 which is identity on the boundary of the disk and is sufficiently flat near the boundary. Finally, using these properties of f_{D^2} Katok showed that this map can in turn be carried over to a C^∞ area preserving Bernoulli diffeomorphism f_M of the surface M. This diffeomorphism also has non-zero Lyapunov exponents almost everywhere. This gives a solution of Problem 1 in the two-dimensional case.

4.2. **Construction of Bernoulli Diffeomorphisms II: The Multi-dimensional Case.** Turning to higher dimensions, in [5] Katok (jointly with Brin and Feldman) showed that any smooth compact Riemannian manifold of dimension ≥ 5 carries a C^∞ volume preserving Bernoulli diffeomorphism.

Starting with the Katok map f_{D^2} of the unit disk D^2, one can construct a skew-product diffeomorphism $F: D^2 \times T^{m-2} \to D^2 \times T^{m-2}$ (here T^{m-2} is the torus of dimension $m-2$ and m is the dimension of the manifold M) given by $F(x, y) = (f_{D^2}(x), h(x)(y))$, where $h: D^2 \to T^{m-2}$ is a C^∞ map, which is the identity in a small neighborhood of the boundary of D^2. The map F is a T^{m-2}-extension of the base map f_{D^2} which is a Bernoulli diffeomorphism. To prove that F is Bernoulli the authors first show that it is weakly mixing and then use a result of Rudolph that claims that a weakly mixing compact group extension over a Bernoulli shift is metrically isomorphic to a Bernoulli shift.

To obtain the desired map on the manifold M they used the standard topological procedure which allows one to first move $D^2 \times T^{m-2}$ to the unit ball D^m in $\mathbb{R}^m$ and then "embed" D^m into M. Since the Katok map is infinitely flat near the boundary of D^2 and the map h is the identity near this boundary, one can show that the map F is carried over to a map on M which has all the desired properties. Note that F has 2 non-zero Lyapunov exponents while other $m-2$ exponents are all zero.

4.3. **Smooth Non-Bernoulli K-automorphisms.** Examples of measurable non-Bernoulli but K-transformations preserving some "natural" measures have long been known in ergodic theory (see for for example [44]), but a first smooth example of this kind was constructed by Katok in [3]. More precisely, he presented an example of a C^∞ volume preserving K-diffeomorphism, which is not Bernoulli. The desired map f is a skew-product map $F: M \times N \to M \times N$ given by $F(x, y) = (f(x), g_{\varphi(x)}(y))$ where f is a volume preserving C^∞ Anosov diffeomorphism of a compact smooth manifold M, g_t a volume preserving ergodic C^∞ flow on a compact smooth manifold N, and φ is a C^∞ function on M. The map F can be arranged to be: 1) partially hyperbolic with stable and unstable foliations to be lifts of stable and unstable foliations of f respectively, and 2) accessible with respect to these foliations. Using ergodicity of the flow g_t, one can show that F is a K-diffeomorphism. On the other

[4] The level of flatness depends on the surface M.

hand, if the flow g_t is chosen to be not loosely Bernoulli, then one can verify that so is the map F. Hence, F is not Bernoulli.

4.4. **Further Developments.** The construction of Bernoulli diffeomorphisms described above left two questions unanswered whether: 1) this construction can be modified to obtain a volume preserving Bernoulli diffeomorphism with **all** Lyapunov exponents non-zero almost everywhere, and 2) dimensions 3 and 4 can be covered.

In [19] Brin used a different construction than presented above to obtain a C^∞ volume preserving Bernoulli diffeomorphism. The map he constructed has all but one non-zero Lyapunov exponents almost everywhere on any compact smooth Riemannian manifold of dimension ≥ 5. Later Dolgopyat and Pesin [22] have shown that the map in Brin's construction can be modified in such a way that it has all Lyapunov exponents non-zero. They also constructed C^∞ volume preserving Bernoulli diffeomorphism with n on-zero Lyapunov exponents on any manifold of dimension 3 or 4, thus obtaining a complete affirmative solution of Problem 1.

In another direction, note that a Bernoulli map is mixing, and one would naturally wonder what is the rate of mixing, i.e., the rate of decay of correlations. Katok has long conjectured that the map f_{T^2} described above has polynomial decay of correlations (with respect to area). This was proved in a recent paper [51], in which following footsteps of Katok's construction, the authors showed that any surface M admits an area preserving diffeomorphism f_M with non-zero Lyapunov exponents almost everywhere, which is Bernoulli and has polynomial decay of correlations [5] (it also satisfies the Central Limit Theorem and has Large Deviations). It should be mentioned that while Katok's construction requires that the inverse of the slow-down function ψ should be arbitrary flat at zero to ensure that the map f_{T^2} is of class C^∞, to achieve polynomial decay of correlations, one needs to choose the function ψ to be polynomial at zero.[6] As a result the map f_M turns out to be only of class $C^{1+\beta}$ for some $\beta = \beta(\alpha) > 0$. It is still an open problem to construct a C^∞ area preserving Bernoulli diffeomorphism f_M on a given surface M which has non-zero Lyapunov exponents almost everywhere and has polynomial decay of correlations.

5. General Hyperbolic Measures

Here we discuss Katok's work on topological properties of general hyperbolic measures, which are subject of his seminal paper [4] and his talk at ICM [8].[7] In the paper he discussed two topics: 1) existence and asymptotic growth of the number of periodic hyperbolic points; and 2) approximations of hyperbolic measures by horseshoe-type hyperbolic invariant sets. At his ICM talk he stated more results on periodic hyperbolic points as well as briefly mentioned some other topological properties of hyperbolic measures.

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[5]More precisely, f_M admits a polynomial upper bound for the class of Hölder continuous observables and a polynomial lower bound for the class of Hölder continuous observables each of which vanishes in a small neighborhood of a compact set on which the map is identity.

[6]This means that the requirement (K4) above on the function ψ should be replaced with the requirement that $\psi(u) = (u/r)^\alpha$ for $0 \leq u \leq \frac{r}{2}$.

[7]I would like also to mention Katok's other paper [27] which contains a brief overview of some results in non-uniform hyperbolicity.

5.1. **Hyperbolic Periodic Points and Approximations by Horseshoes I.** Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M preserving a hyperbolic Borel measure ν. Denote by $\mathrm{Per}(f)$ the set of all periodic points of f, by $P_n(f)$ the set of periodic points of period n,[8] and by $P_n^h(f)$ the set of hyperbolic periodic points of period n. The main result of [4] claims that

$$p(f) := \limsup_{n\to\infty} \frac{\log P_n(f)}{n} \geq \limsup_{n\to\infty} \frac{\log P_n^h(f)}{n} \geq h_\nu(f), \tag{2}$$

where $h_\nu(f)$ is the entropy of ν.

In the two dimensional case any measure of positive entropy must be hyperbolic and (2) yields

$$p(f) \geq h(f), \tag{3}$$

where $h(f)$ is the topological entropy of f. This result is remarkable, since it bounds below the asymptotic growth rate of the number of periodic points of f of period n by its topological entropy without explicitly using any invariant measure. It also gives negative answer to a question by Herman on whether positive topological entropy is compatible with minimality or unique ergodicity of f.

The proof of (2) is an extension to the non-uniformly hyperbolic case of an argument (due to Anosov, see [16]), which allows one to construct periodic points for Anosov diffeomorphisms. To outline this argument recall that a point x is said to be recurrent if for every $\delta > 0$, there is $n > 0$ such that $d(f^n(x), x) < \delta$.

Closing Argument. *Let x be a recurrent point for an Anosov diffeomorphism f. Then for every $\varepsilon > 0$ there is a periodic point p such that $d(x, p) < \varepsilon$.*

To see this fix $\varepsilon > 0$, $\delta > 0$ and choose $n > 0$ such that $d(f^n(x), x) < \delta$. If δ is small enough, $z = [f^n(x), x]$ is well defined (see (1)) and $d(z, x) < C_1\delta$ for some universal constant $C_1 > 0$. The point $f^n(z)$ lies on the stable manifold through $f^n(x)$ and $d(f^n(x), f^n(z)) < \lambda^n d(x, z)$ for some $0 < \lambda < 1$. Consider the point $z_1^{(s)} = [f^n(z), x] \in V^s(x)$. Applying this argument repeatedly, we obtain a sequence of points $z_m^{(s)} \in V^s(x)$ such that $z_m^{(s)} = [f^n(z_{m-1}^{(s)}), x]$. One can show that this sequence converges to a point $z_0^{(s)} \in V^s(x)$ and that $d(x, z_0^{(s)}) < C_2\delta$ for some universal constant $C_2 > 0$.

Now let us consider the point $f^{-n}(z)$. It lies on the unstable manifold through x and $d(x, f^{-n}(z)) < \lambda^n d(f^n(x), z)$. Let $z_1^{(u)} = [f^{-n}(z), x] \in V^u(x)$. Applying the above argument repeatedly, we obtain a sequence of points $z_m^{(u)} \in V^u(x)$ such that $z_m^{(u)} = [f^{-n}(z_{m-1}^{(u)}), x]$. One can show that this sequence converges to a point $z_0^{(u)} \in V^u(x)$ and that $d(x, z_0^{(u)}) < C_3\delta$ for some universal constant $C_3 > 0$. It is easy to see that the point $p = [z_0^{(u)}, z_0^{(s)}]$ is periodic of period n, and, if δ is chosen sufficiently small, $d(x, p) < \varepsilon$.

This simple argument is a driving idea behind proofs of many sophisticated results on topological properties of uniformly hyperbolic diffeomorphisms.

[8] That is the set of fixed points of f^n.

Moving to the non-uniformly hyperbolic case consider a regular set Λ_ℓ of positive ν-measure. Then almost every point $x \in \Lambda_\ell$ is recurrent, so one can try to use the above argument to obtain a periodic point p near x. However, in doing so one faces the following obstacle: the point $z_m^{(s)}$ for some m, while in Λ, may no longer lie in Λ_ℓ, and hence, its local unstable manifold may not a priori, be long enough to intersect the stable manifold through x. While this obstacle may seem technical, it reveals a principle difference between uniform and non-uniform types of hyperbolicity and overcoming this obstacle requires some special and sophisticated argument which is one of the main achievements of Katok's work [4] (see the Main Lemma).

Furthermore, Katok shows that periodic points are dense in Λ_ℓ and hence, there are at least two distinct periodic points p and q which are homoclinically related.[9] By the classical Smale-Birkhoff theorem, there is a horseshoe that contains both p and q, i.e., a set A satisfying

(H1) A is a locally maximal hyperbolic set which is totally disconnected;
(H2) $f|A$ is topologically transitive and is topologically conjugate to a subshift of finite type.

5.2. **Hyperbolic Periodic Points and Approximations by Horseshoes II.** In [7] Katok states a stronger than (2) property of periodic points which describes their distribution in $\operatorname{supp}\nu$. Namely, he shows that for any $\delta > 0$, any $x \in \operatorname{supp}\nu$, any neighborhoods V of x and W of $\operatorname{supp}\nu$, and any collection of continuous functions $\varphi_1, \dots, \varphi_m$ there exists a hyperbolic periodic point $z \in V$ of period n such that the orbit of z is contained in W and for $i = 1, \dots, m$,

$$\left|\frac{1}{n}\sum_{k=1}^{n-1} \varphi_i(f^k(z)) - \int_M \varphi_i \, d\nu\right| < \delta.$$

Furthermore, Katok claims that the horseshoes that he constructed in [4] have the following "approximation" property:

(H3) for every $\varepsilon > 0$, the horseshoe $A = A_\varepsilon$ can be constructed in such a way that $h(f|A) > h_\nu(f) - \varepsilon$.

5.3. **Entropy and growth of expanding periodic points for one-dimensional maps, [42].** This paper (written jointly with Mezhirov) was inspired by the seminal work of Misiurewicz and Szlenk [41, 42] where they proved that for any continuous, piecewise monotone map f of the circle or an interval into itself one has $h(f) \le p(f)$ (recall that $h(f)$ is the topological entropy of f and $p(f)$ is the exponential growth rate of the number of periodic orbits, see (2); compare to (3) which holds in the two-dimensional case).

In their paper Katok and Mezhirov showed that for smooth or piecewise smooth maps a large number of periodic orbits are expanding with exponent at least almost as large as entropy.

(1) *Let $f\colon S^1 \to S^1$ be a monotone C^1 map without critical points and $|\deg f| = k$ for $k > 1$ (and hence, $h(f) = \log k$). Then for each $\varepsilon > 0$ and any large enough n one can find at least $(1-\varepsilon)k^n$ periodic points x_i^n of period n for which*

$$|(f^n)'(x_i^n)| \ge (k - \varepsilon)n.$$

[9] This means that there are numbers $k > 0$ and $\ell > 0$ such that the stable manifold of $f^k(p)$ intersects the unstable manifold of $f^\ell(q)$ transversally.

(2) *Let f be a continuous, piecewise C^1 map of S^1 or I into itself with finitely many critical points and with entropy $h(f) = h > 0$. Then for each $\varepsilon > 0$ one can find a subsequence $\{n_k\}$ such that for each n_k the function f has at least $e^{(h-\varepsilon)n_k}$ periodic points $x_i^{n_k}$ of period n_k for which*
$$|(f^{n_k})'(x_i^{n_k})| > e^{(h-\varepsilon)n_k}.$$

After stating their results the authors made a crucial comment on the low regularity of the map f in the above results (I omitted references to the papers cited in this quote):

"Our results can be viewed as a simple model case for the still unknown C^1 versions of results connecting entropy and the growth of periodic orbits for $C^{1+\varepsilon}$ diffeomorphisms in dimension two and flows in dimension three, specifically. For the proofs of those results the $C^{1+\varepsilon}$ assumption is crucial since they heavily rely on Pesin theory essential elements of which fail in the C^1 case."

Indeed, for $C^{1+\varepsilon}$ maps the first result becomes a simpler version of the results for the two-dimensional invertible case and the proof uses existence of orbits, which are Lyapunov-Perron regular with respect to a measure with high metric entropy, and having the property that they return very close to the initial condition; then one applies a non-uniform version of the Anosov closing lemma for hyperbolic systems.

To deal with the case when the map f is only of class C^1 the authors used Markov approximations for orbits which avoid a neighborhood of the critical set.

5.4. **Further Developments.** About the time of his ICM talk, Katok drafted proofs of the results on topological properties of hyperbolic measures, which he discussed at the talk. At some point these proofs were included in a joint work of Katok and Mendoza published as a supplement to the book [28].[10] This supplement contains a description of some core results in nonuniform hyperbolicity including some topological properties of hyperbolic measures. To state them consider a $C^{1+\alpha}$ diffeomorphism f of a compact smooth Riemannian manifold M preserving a Borel measure ν.

(1) *Oseledets-Pesin ε-reduction theorem*: Let $A : X \to GL(n,\mathbb{R})$ be a measurable cocycle over a measure preserving transformation of the Lebesgue space (X,ν). If $\log^+ \|A^{\pm 1}(x)\| \in L^1(X,\nu)$, then there exist a measurable f-invariant function $k(x)$ and measurable functions $\chi_i(x)$ and $\ell_i(x)$, $i = 1,\dots,k(x)$ with $\sum \ell_i(x) = n$ such that for every $\varepsilon > 0$ there is a tempered map $C_\varepsilon : X \to GL(n,\mathbb{R})$[11] satisfying: for almost every $x \in X$ the cocycle
$$A_\varepsilon(x) = C_\varepsilon^{-1}(f(x))A(x)C_\varepsilon(x)$$
is block-diagonal with diagonal blocks $A_\varepsilon^i(x)$ to be $\ell_i(x) \times \ell_i(x)$-matrix for which
$$e^{\chi_i(x)-\varepsilon} \le \|A_\varepsilon^i(x)^{-1}\|^{-1}, \quad \|A_\varepsilon^i(x)\| \le e^{\chi_i(x)+\varepsilon}.$$

(2) *Closing Lemma*: For every $d > 0$ and $\ell > 1$ there exists $\beta = \beta(d,\ell)$ such that if $x \in \Lambda_\ell$, $f^n(x) \in \Lambda_\ell$ for some $n > 0$ and $d(x, f^n(x)) < \beta$, then there is a hyperbolic periodic point p of period n such that $d(x,z) < d$ and
$$d(f^i(x)), f^i(p)) < Cd \max_{1\le i\le n} \{e^{\varepsilon(i-n)}, e^{\varepsilon i}\}.$$

[10] More details of this story can be found in the article by Boris Hasselblatt in "Anatole Katok's works" in these Works.

[11] This means that $\lim_{n\to\pm\infty} \frac{1}{n}\|C_\varepsilon(x)\| = 0$ for almost every $x \in X$.

(3) *Shadowing Lemma*: For every sufficiently small $\beta > 0$ and any $\ell \geq 1$, there exists $\varepsilon = \varepsilon(\beta, \ell)$ such that given an ε-pseudo-orbit $\{x_n\}_{m<n<k}$ located in a sufficiently small neighborhood of Λ_ℓ, there exists a point $y \in M$ such that its orbit β-shadows $\{x_n\}$.

(4) *The Livshitz Theorem*: Let φ be a Hölder continuous function on M such that for each periodic point p with $f^m(p) = p$, we have that

$$\sum_{i=0}^{m-1} \varphi(f^i(p)) = 0.$$

Then there exists a Borel measurable function ψ such that for almost every x,

$$\varphi(x) = \psi(f(x)) - \psi(x).$$

Some stronger versions of the shadowing property were established by Hirayama [26] and, in the two-dimensional case, by Climenhaga, Luzzatto, and Pesin [20]. These versions provide a powerful tool in studying some intricate problems in non-uniform hyperbolicity, e.g., constructions of Sinai-Ruelle-Bowen measure or more general equilibrium measures.

On another direction, Avila, Crovisier, and Wilkinson [17] substantially generalized Katok's construction of approximating horseshoes (see Statements (H1)-(H3)) by proving the following result:

Let f be a C^r diffeomorphism preserving an ergodic hyperbolic probability measure μ. Then given $\delta > 0$ and a neighborhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M (endowed with the weak*-topology), there exists a horseshoe $A \subset M$ such that

(1) *A is δ-close to the support of μ in the Hausdorff distance;*

(2) $h_{top}(A, f) > h(\mu, f) - \delta$;

(3) *if $\chi_1 > \cdots \chi_\ell$ are the distinct Lyapunov exponents of μ, with multiplicities $n_1, \ldots, n_\ell$, then there exists a dominated splitting $T_A M = E_1 \oplus \cdots \oplus E_\ell$, with $\dim(E_i) = n_i$;*

(4) *there exists $n \geq 1$ such that for each $i = 1, \ldots, \ell$, each $x \in A$ and each unit vector $v \in E_i(x)$,*

$$\exp((\chi_i - \delta)n) \leq \|Df^n(v)\| \leq \exp((\chi_i + \delta)n).$$

A similar statement for multidimensional expanding maps was proved by Cao, Pesin, and Zhao [21], who used a different approach in which the horseshoe A was constructed via a Cantor-like procedure.

6. Entropy of Dynamical Systems

In this section we discuss some of Katok's work on topological and metric entropies of smooth dynamical systems.

6.1. **Shub's Entropy Conjecture.** In the paper [1] Katok discusses the entropy conjecture proposed by Shub in [53] as well as several results related to this conjecture which had been obtained during the short period of 3 years since publication of Shub's paper in 1974 and Katok's paper in 1977. This is a survey-like paper on a topic that attracted many experts in topology and dynamics but, in addition to known results, this paper contains many interesting observations as well as some new results.

Let f be a smooth map of a compact smooth manifold M of dimension p. Shub conjectured that the topological entropy of $h(f) \geq \log s(f_*)$ where f_* is the linear map induced by

f on the total homology group of M

$$H_*(M,\mathbb{R}) = \bigoplus_{i=0}^{p} H_i(M,\mathbb{R})$$

and $s(f_*)$ is the spectral radius of f_*. Observe that $s(f_*) = \max_{1\le i\le p} s(f_{*i})$, where $f_{*i} = f_*|H_i(M,\mathbb{R})$ and hence, the conjecture is equivalent to the systems of inequalities $h(f) \ge \log s(f_{*i})$ for $i = 1,\ldots,p$.

Katok proceeds by considering first some general results about Shub's entropy conjecture. For example, he gives a relatively simple proof of Manning's result [35], which claims that $h(f) \ge s(f_{*1})$, and discusses its various versions. He also proves the result of Misiurewicz and Przytycki [40] that for a C^1 map of a smooth compact manifold one has $h(f) \ge \log|\deg f|$. Katok then goes on to consider some particular cases where the conjecture holds. For example, manifolds of dimension ≤ 3, n-dimensional spheres and tori. Finally, he discusses the conjecture for maps which are structurally stable.

6.2. **Entropy and Closed Geodesics,** [7]**.** In this paper Katok studied asymptotic growth of closed geodesies for various Riemannian metrics on a compact smooth manifold M, which carries a metric of negative sectional curvature. His two main results are as follows.

Let σ be a Riemannian metric on M. Given $T > 0$, let $P_\sigma(T)$ be the number of closed geodesics γ of length $l_\sigma(\gamma) \le T$ and let $P^s_\sigma(T)$ be the number of non-zero free homotopy classes Γ of closed curves on M whose length $L_\sigma(\Gamma) \le T$.[12] We set

$$P_\sigma = \liminf_{T\to\infty} \frac{\log P_\sigma(T)}{T}, \quad P^s_\sigma = \liminf_{T\to\infty} \frac{\log P^s_\sigma(T)}{T}.$$

We denote by g^t_σ the geodesic flow that acts on the unit tangent bundle $S_\sigma M$ and preserves the Liouville measure λ_σ on $S_\sigma M$.

Katok proved the following main result:

Let σ_1 be a Riemannian metric of negative curvature. Then for every Riemannian metric σ_2 on M one has

$$P^s_{\sigma_2} \ge ([\sigma_1;\sigma_2])^{-1} h^\lambda_{\sigma_1}(g^t_{\sigma_1}),$$

where $h^\lambda_\sigma(g^t_\sigma)$ is the metric entropy of the geodesic flow in the Riemannian metric σ and

$$[\sigma_1;\sigma_2] = \int_{S_{\sigma_1}M} \|v\|_{\sigma_2}\, d\lambda_{\sigma_1}.$$

Katok also proved a version of this result for a more general class of Riemannian metrics without focal points. His approach is based on both variational and dynamical descriptions of geodesics. From the variational point of view geodesics are the shortest (locally or globally) curves in various classes of curves on the manifold. It is considered one of the main methods in differential geometry. From the dynamical point of view the geodesies are considered as orbits of the geodesic flow and the geometric characteristics of the metric are reflected in asymptotic properties of this flow; e.g. the negativity of the curvature leads to exponential divergence of the orbits. It is considered one of the main methods in modern theory of smooth dynamical systems.

As corollaries of the above result Katok obtained various inequalities between topological and metric entropies of the geodesic flows for different metrics on M. For example, he showed that

[12] Recall that $L_\sigma(\Gamma) = \inf_{\alpha\in\Gamma} l_\sigma(\alpha)$.

(1) for any two Riemannian metrics σ_1 and σ_2 on M

$$h_{\sigma_2}(g^t_{\sigma_2}) \geq [\sigma_1;\sigma_2]^{-1} h_{\sigma_1}(g^t_{\sigma_1});$$

(2) for any Riemannian metric σ on M of non-constant negative curvature

$$h^{\lambda}_{\sigma}(g^t_{\sigma}) < \Big(\frac{4\pi(g-1)}{V_\sigma}\Big)^{\frac{1}{2}} < h_\sigma(g^t_\sigma), \tag{4}$$

where V_σ is the total volume (area if $\dim M = 2$) of M.

The above results have some special corollaries for Riemannian metrics, which are conformally equivalent to a metric of constant negative curvature. In this case one can show for example, that every metric of non-constant curvature has strictly more close geodesies of length at most T for sufficiently large T than any metric of constant curvature of the same total area; in addition, the common value of topological and metric entropies for metrics of constant negative curvature with the fixed area separates the values of two entropies for other metrics with the same area. This applies to the particular case of surfaces of negative Euler characteristics for which every Riemannian metric is conformally equivalent to a metric of constant negative curvature due to the classical regularization theorem. Moreover, in this case one can show that *for every Riemannian metric σ of negative Euler characteristic E*

$$P_\sigma \geq h_\sigma(g^t_\sigma) \geq P^s_\sigma \geq (-2\pi E/V_\sigma)^{\frac{1}{2}}$$

and the last inequality is strict for every metric of non-constant curvature. This result was the first step to what is now know as the Katok Entropy-Rigidity Conjecture, see articles by Boris Hasselblatt ”Anatole Katok's works” and by Ralf Spatzier “Anatole Katok's work on cohomology and geometric rigidity” in these Works.

Many years later Katok and Erchenko [23][13] showed that the relation (4) between the topological and metric entropies of geodesic flows gives the only possible restrictions on these invariants. More precisely, they proved the following result:

Let M be a closed orientable surface of genus $g \geq 2$ and $V > 0$ be a number. Then for any numbers a, b such that $a > \left(\frac{4\pi(g-1)}{V_\sigma}\right)^{\frac{1}{2}} > b > 0$, there exists a smooth metric σ of negative curvature such that $a = h_\sigma(g^t_\sigma)$, $b = h^\lambda_\sigma(g^t_\sigma)$ and $V_\sigma = V$.

6.3. **Differentiability and Analyticity of Topological Entropy.** The study of the dependence of topological entropy of maps and flows under small perturbations has a long history. One of the first results in this direction was obtained by Misiurewicz [37] who for every $n \geq 4$ constructed an example of C^∞ diffeomorphism φ of an n-dimensional manifold M for which the topological entropy is not continuous at φ in the C^∞ topology. More precisely, this means that the function $h_{\rm top} : \mathrm{Diff}^\infty(M) \to \mathbb{R}$ (given by $\varphi \to h_{\rm top}(\varphi)$) is not continuous at φ. On the other hand, by Yomdin [55, 56] and Newhouse [43], the function $h_{\rm top}$ is upper-semi-continuous on every manifold of dimension ≥ 3 and by Katok [4, 8, 28], it is continuous in dimension 2.

For flows Misiurewicz [38] showed that for general C^k, flows, $k < \infty$, on manifold of dimension $n \geq 3$ $h_{\rm top}$ need not be continuous.

[13]Katok passed away while the paper was in its final stage of preparation.

Moving to a particular but important case of Anosov systems we first note that structural stability of Anosov diffeomorphisms implies that the function h_{top} is locally constant and structural stability of Anosov flows implies that this function is continuous.

In the paper [12] (written jointly with Knieper, Pollicott and Weiss), Katok studied the dependence of the topological entropy of an Anosov flow on small perturbations of the flow. More precisely, let φ^t be a C^{k+1} Anosov flow for $1 \le k \le \omega$ of a compact smooth Riemannian manifold and let $\{\varphi^t_\lambda\}$, $-\varepsilon \le \lambda \le \varepsilon$ be a one-parameter family of perturbations of $\varphi^t = \varphi^t_0$ (ε is assumed to be sufficiently small). The authors proved that

(1) *$\lambda \to h_{top}(\varphi^1_\lambda)$ is a C^k function of λ.*

This result is a stronger version of results by Ruelle in [52].

In the particular case $k = 1$ Claim (1) yields C^1 dependence of topological entropy on C^2 perturbations of a C^1 Anosov diffeomorphism. In [13] Katok (jointly with Knieper and Weiss) improved this result by allowing perturbations of class C^1. Moreover, in this setting they obtained a formula for the derivative of the topological entropy:

(2) the following formula holds

$$\frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} h(\varphi^t_\lambda) = -h(\varphi^t_0)\int_M \frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} a_\lambda(p)\,d\mu_0, \tag{5}$$

where $\mu_0 = \mu_{0,0}$ is the unique measure of maximal entropy for the Anosov flow $\varphi^t = \varphi^t_0$ and $a_\lambda(p)$ is an infinitesimal time change. In addition the authors presented a version of formula (5) for the particular case of geodesic flows on compact Riemannian manifolds of negative sectional curvature.

In the paper [12] the authors also showed that a result similar to Claim (1) holds when topological entropy is replaced with topological pressure $P(\varphi^t, f)$ where $f : M \to \mathbb{R}$ is a potential function. More precisely, let $\mu_{f,\lambda}$ be the unique equilibrium measure for f.[14]

(3) *The map $C^{k+1}(M) \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by $(f,\lambda) \to P(\varphi^1_\lambda, f)$ is of class C^k.*
(4) *For a fixed $\rho \in C^{k+1}(M)$ the map $C^{k+1}(M) \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by $(f,\lambda) \to \int_M \rho\,d\mu_{f,\lambda}$ is of class C^{k-1}.*
(5) *For a fixed $f \in C^{k+1}(M)$ the map $\lambda \to h_{\mu_{f,\lambda}}(\varphi^1_\lambda)$ is of class C^{k-1}.*

6.4. **The Brin-Katok Local Entropy Formula.** In a short paper written jointly with Brin [9] they introduced the notion of local entropy and obtained a formula that connects the local entropy with the metric entropy of the system.

Let f be a continuous map of a compact metric space X preserving a Borel probability non-atomic (not necessarily ergodic) measure m. Assuming that the metric entropy $h_m(f)$ is finite, they proved that

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{-\log m(B_n(x,\delta))}{n} = \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{-\log m(B_n(x,\delta))}{n} =: h_m(f,x)$$

and that

$$\int_X h_m(f,x)\,dm = h_m(f).$$

Here

$$B_n(x,\delta) := \{y \in X : d(f^k(x), f^k(y) < \delta, k = 0, 1, \dots, n\}$$

[14] Recall that this means that $P(f,\lambda) = h_{\mu_{f,\lambda}}(\varphi^1_\lambda) + \int_M f\,d\mu_{f,\lambda}$.

is (n,δ)-Bowen's ball at x and d is the metric in X. The quantity $h_m(f,x)$ is known as *Brin-Katok local entropy* of f and it is clearly invariant under f.

If the measure m is ergodic, then $h_m(f,x) = h_m(f)$ and hence, one obtains another "local" definition of metric entropy. In this ergodic case Katok [4] obtained another somewhat simpler proof of the above result.

As Brin and Katok pointed out their work was inspired by the discussion they had with Ledrappier and Young during the International Symposium on Dynamical Systems (Rio de Janeiro, 1981) about relations between dimension, entropy and Lyapunov exponents. By this time few results had been obtained in this direction: in the two dimensional case both Manning [36] and Young [57] found some explicit relations between Hausdorff dimension, metric entropy and Lyapunov exponents and Ledrappier [29] had some more general statement (but not an explicit formula). According to Ledrappier there were some discussions between him, Young and Mañé on this topic that led to some version of the local entropy formula. They then asked Brin and Katok to look into this who later had come up with the final result.

Another "dimension-like" interpretation of the local entropy formula involving a "dynamical" Carathéodory-like construction in geometric measure theory can be found in [50].

7. Other Papers

7.1. Lyapunov Functions and Invariant Cone Families. One of the most effective ways of verifying the conditions of nonuniform hyperbolicity for a diffeomorphism preserving a Borel measure is to show that the Lyapunov exponents are nonzero almost everywhere. To establish this one do not need to know the exact values of the Lyapunov exponent which may be difficult to compute. Here the so-called cone techniques come handy to help verify that the exponents are nonzero.

Recall that the *cone* of size $\gamma > 0$ centered around $\mathbb{R}^{n-k}$ in the product space $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ is defined by

$$C_\gamma = \left\{(v,w) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \|v\| < \gamma\|w\|\right\} \cup \{(0,0)\}.$$

Let $\mathscr{A}$ be a cocycle over an invertible measurable transformation $f\colon X \to X$ preserving a Borel probability measure ν and let $A\colon X \to GL(n,\mathbb{R})$ be its generator. The main idea underlining the cone techniques is the following. Let $Y \subset X$ be an f-invariant subset. Assume that there exist $\gamma > 0$ and $a > 1$ such that for every $x \in Y$:

(1) *invariance*: $A(x)C_\gamma \subset C_\gamma$;
(2) *expansion*: $\|A(x)v\| \geq a\|v\|$ for every $v \in C_\gamma$.

It is easy to show that the Lyapunov exponent of v is positive for every $x \in Y$ and $v \in C_\gamma \setminus \{0\}$. In other words, the $n-k$ largest values of the Lyapunov exponent are all positive.

We stress that positivity of Lyapunov exponents holds regardless of whether the set Y has positive ν-measure or not. Wojtkowski [58] made the following crucial observation. We say that the cone C_γ is *eventually strict* if there is a Borel measurable function $n(x) \geq 0$ such that for every $x \in Y$ the inclusion

$$\mathscr{A}(x,n(x))C_\gamma \subset C(\gamma)$$

is strict. One can show that if ν is a smooth measure and C_γ is eventually strict, then Lyapunov exponents are positive almost everywhere in Y and no estimate on the growth of vectors inside the cone is necessary. Another version of the cone techniques based on the notion of Lyapunov function was developed by Burns and Katok in [14].

Consider a measurable family of cones $C = \{C_x : x \in X\}$ in $\mathbb{R}^n$ and the complementary cones

$$\widehat{C}_x = \left(\mathbb{R}^n \setminus \overline{C}_x\right) \cup \{0\}.$$

The *rank* of a cone C_x is the maximal dimension of a linear subspace $L \subset \mathbb{R}^n$ which is contained in C_x. We denote it by $r(C_x)$ and we have that $r(C_x) + r(\widehat{C}_x) \le n$.

A pair of complementary cones C_x and $\widehat{C}_x$ is called *complete* if $r(C_x) + r(\widehat{C}_x) = n$.

We say that the family of cones C is $\mathscr{A}$*-invariant* if for ν-almost every $x \in X$,

$$A(x)C_x \subset C_{f(x)} \quad \text{and} \quad A(f^{-1}(x))^{-1}\widehat{C}_x \subset \widehat{C}_{f^{-1}(x)}.$$

Let now $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function that is homogeneous of degree one (i.e., $Q(av) = aQ(v)$ for every $v \in \mathbb{R}^n$) and takes both positive and negative values. The set

$$C^u(Q) := \{0\} \cup Q^{-1}(0, +\infty) \subset \mathbb{R}^n$$

is called the *positive cone of* Q and the set

$$C^s(Q) := \{0\} \cup Q^{-1}(-\infty, 0) \subset \mathbb{R}^n$$

is called the *negative cone of* Q. The rank of $C^u(Q)$ (respectively, $C^s(Q)$) is called the *positive* (respectively, *negative*) *rank* of Q and is denoted by $r^u(Q)$ (respectively, $r^s(Q)$). Clearly, $r^u(Q) + r^s(Q) \le n$ and since Q takes both positive and negative values, we have $r^u(Q) \ge 1$ and $r^s(Q) \ge 1$. The function Q is said to be *complete* if the cones $C^u(Q)$ and $C^s(Q)$ form a complete pair of complimentary cones, i.e., if

$$r^u(Q) + r^s(Q) = n.$$

A measurable function $Q\colon X \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Lyapunov function* for the cocycle $\mathscr{A}$ (with respect to ν) if there exist positive integers r^u and r^s such that for ν-almost every $x \in X$:

(1) the function $Q_x = Q(x, \cdot)$ is continuous, homogeneous of degree one and takes both positive and negative values;
(2) Q_x is complete, $r^u(Q_x) = r^u$ and $r^s(Q_x) = r^s$;
(3) for every $x \in \mathbb{R}^n$ we have $Q_{f(x)}(A(x)v) \ge Q_x(v)$.

The numbers r^u and r^s are called respectively the *positive* and *negative ranks* of Q.

One can show that if Q is a Lyapunov function, then the two families of cones

$$C^u(Q_x) = \{v \in \mathbb{R}^n : Q_x(v) > 0\} \cup \{0\}, \quad C^s(Q_x) = \{v \in \mathbb{R}^n : Q_x(v) < 0\} \cup \{0\}$$

are $\mathscr{A}$-invariant.

A Lyapunov function is said to be *eventually strict* if for ν-almost every $x \in X$ there exists $m = m(x)$ depending measurably on x such that for every $v \in \mathbb{R}^n \setminus \{0\}$ we have

$$Q_{f^m(x)}(\mathscr{A}(x, m)v) > Q_x(v), \quad Q_{f^{-m}(x)}(\mathscr{A}(x, -m)v) < Q_x(v). \tag{6}$$

The main result in [14] claims that

Assume that

$$\max\{\log\|A\|, \log\|A^{-1}\| \in L^1(X, \nu)$$

and that there exists an eventually strict Lyapunov function for the cocycle $\mathscr{A}$*. Then for* ν*-almost every* $x \in X$ *the cocycle has* r^u *positive and* r^s *negative values of the Lyapunov exponent, counted with their multiplicities.*

Similar results were obtained for: 1) cocycles over measure preserving flows on Lebesgue spaces; 2) $2m$-dimensional symplectic cocycles over measure preserving transformations and flows (and the corresponding m-dimensional symplectic cones; in this case the cocycle has exactly m positive and m negative Lyapunov exponents); 3) $C^{1+\varepsilon}$ diffeomorphisms and flows on compact Riemannian manifolds (in this case the Lyapunov function is assumed to be continuous).

7.2. **Random Perturbations of Transformation of an Interval,** [10]**.** In this paper written jointly with Kifer, they introduced a class of ε-small random perturbations f_ε of a smooth map f of the interval $[0,1]$ into itself satisfying Misiurewicz's conditions (i.e., f has non-positive Schwarzian derivative, no sinks and trajectories of critical points of f stay away from critical points, see [39]). The main result of the paper claims that

Under some natural assumptions on the family of random perturbations and the map f each map f_ε has an invariant probability measure μ_ε which converges as $\varepsilon \to 0$ to the unique absolutely continuous invariant measure μ for f.

This can be expressed by saying that μ is *stable* under small random perturbations of f. A famous example, which fits into the above setting, is a unimodal map f satisfying the Collet-Eckmann condition. One can show that f satisfies Misiurewicz's conditions and possesses a unique absolutely continuous invariant measure. As a corollary of the above result one has that this measure is stable with respect to small random perturbations of f.

7.3. **Four Applications of Conformal Equivalence to Geometry and Dynamics,** [11]**.** Let M be a compact surface with negative Euler characteristic E and σ a C^∞ Riemannian metric on M. The conformal equivalence theorem from complex analysis claims that there exists a scalar positive C^∞ function ρ_σ on M, uniquely defined up to a positive constant, such that the Riemannian metric $\sigma' = \rho_\sigma \sigma$ has constant negative curvature. In other words the metrics σ and σ' are conformally equivalent.[15]

Given a Riemannian metric σ on M, let $\varphi^\sigma = \{\varphi_t^\sigma\}_{t\in\mathbb{R}}$ be the geodesic flow associated to σ. Let also $h^\sigma = h^\sigma(\varphi^\sigma)$ be the topological entropy of the geodesic flow and $h_\lambda^\sigma = h_\lambda^\sigma(\varphi^\sigma)$ the metric entropy of the geodesic flow with respect to the Liouville measure λ^σ on the unit tangent bundle $S^\sigma M$.

In this paper Katok presented the following four applications of the conformal equivalence theorem:

(1) To obtain estimates for the topological and metric entropies , i.e.,

$$h_\lambda^\sigma \le \rho_\sigma(-2\pi E V_\sigma)^{1/2}, \quad h^\sigma \ge \rho_\sigma^{-1}(-2\pi E V_\sigma)^{1/2},$$

where V_σ is the total area of the surface M, E the energy level, and the function ρ_σ comes from the conformal equivalence theorem (the first inequality requires that σ is a metric without focal points).

(2) To show that different conformally equivalent metrics σ and σ' of negative curvature on M have different geodesic length spectra, i.e., $L_\sigma(\gamma) \ne L_{\sigma'}(\gamma)$ for every non-zero free homotopy class γ (here $L_\sigma(\gamma)$ is the length of the unique closed geodesic in γ).

[15] The term comes from the fact that two conformally equivalent metrics determine the same complex structure on M.

(3) To prove that for metrics of negative curvature the geodesic and harmonic measure classes at infinity are mutually singular.
(4) To establish an upper bound for the asymptotic Cheeger isoperimetric constant

$$C_\sigma \le \rho_\sigma(-2\pi E V_\sigma)^{1/2},$$

where C_σ is the lower limit of the ratios of the length of a rectifiable closed Jordan curve on the universal bundle $\tilde{M}$ of M to the area bounded by the curve as the area goes to infinity.

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