CHAOS IN TRAVELING WAVES OF LATTICE SYSTEMS OF UNBOUNDED MEDIA*

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Abstract. We describe coupled map lattices (CMLs) of unbounded media corresponding to some well-known evolution partial differential equations (including reaction-diffusion equations and the Kuramoto-Sivashinsky, Swift-Hohenberg and Ginzburg-Landau equations). Following Kaneko we view CMLs also as phenomenological models of the medium and present the dynamical systems approach to studying the global behavior of solutions of CMLs. In particular, we establish spatio-temporal chaos associated with the set of traveling wave solutions of CMLs as well as describe the dynamics of the evolution operator on this set. Several examples are given to illustrate the appearance of Smale horseshoes and the presence of the dynamics of Morse-Smale type.

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Introduction. In this paper we deal with **lattice dynamical systems** of an unbounded medium. These are also called **coupled map lattices** (or briefly CMLs) and are described by equations of the form

$$u_{\bar{j}}(n+1) = f(u_{\bar{j}}(n)) + \varepsilon g_{\bar{j}}(\{u_{\bar{i}}(n)\}_{|\bar{i}-\bar{j}|\le s}). \tag{0.1}$$

Here $n \in \mathbb{Z}$ is the discrete time coordinate, $\bar{j} = (j_k)$, $k = 1, \dots, d$ is the discrete space coordinate, and $u(\bar{j}, n) = u_{\bar{j}}(n)$ is a characteristic of the medium (for example, its density, or distribution of the temperature, etc.). Furthermore, $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : (\mathbb{R}^d)^{2s+1} \to \mathbb{R}^d$ are smooth functions; f is called the **local map** and g the **interaction of finite size** s. Finally, ε is a parameter which is assumed to be sufficiently small.

A natural source of CMLs are discrete versions of partial differential equations of evolution type. They arise while modeling partial differential equations by a computer. In Section 1 we discuss some examples of partial differential equations and their discrete versions as CMLs. Intensive study of this topic representing various points of view on the subject can be found in [2–4, 6, 7, 14]. In general, no information on the global behavior of solutions of a partial differential equation can be derived from the study

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of its discrete versions even when the discretization is very fine. However, we believe that some methods of studying spatio-temporal chaos in CMLs described in this paper can be applied (perhaps with some modifications) to partial differential equations.

In [18, 24, 25], Kaneko and collaborators developed a new point of view on CMLs as phenomenological models to be used to describe the behavior of media with high level of energy pumping (corresponding to large Reynolds numbers). They observed the appearance of particle-like localized structures, i.e. distinct spatial structures obeying individual dynamics and interacting with nearest neighbors. Moreover, if the medium is spatially homogeneous, then the individual dynamics are identical. Thus, the behavior of the medium obeys Equation (0.1) (with the local map f representing the individual dynamics). The discovery by Kaneko *et al.* drew the attention of many physicists and mathematicians to CMLs and led to a great interest to this area. CMLs have now become a popular subject of study in both pure and applied mathematics.

The dynamical systems approach to the study of CMLs was originated by Bunimovich and Sinai (see [14]) and since then has become a core technique in the theory (see, for example, [3, 4, 6–10, 23, 27, 37]). The main achievement of this approach is the description of the global behavior of solutions of Equation (0.1) for a broad class of local maps exhibiting a greater or lesser degree of hyperbolicity. Let us point out that in order to solve Equation (0.1) one should fix *initial* and *boundary* conditions. The initial condition is uniquely determined by fixing values $(u_{\bar{j}}(0))$. Since the medium is unbounded the boundary conditions are given by fixing the rate of increase (or decrease) of solutions at infinity.

In this paper we consider the case when solutions grow at infinity at an exponential rate. The corresponding infinite-dimensional dynamical system which governs the behavior of solutions of Equation (0.1) (i.e., the group of time translations generated by the evolution operator) is described in Section 2. Since the medium is unbounded one can introduce the group of space translations. We assume that they commute with the evolution operator (this depends on the interaction and is always the case when the CML is obtained as a discretization of a partial differential equation). This leads to an action of the $\mathbb{Z}^{p+1}$-lattice on the infinite-dimensional phase space by time and space translations. The main objective of this paper is to describe hyperbolic, topological, and ergodic properties of this action.

In particular, we reveal the mechanism for appearance of finite-dimensional spatial and/or temporal chaos associated with various special classes of solutions (including steady-state, spatio-homogeneous, and traveling wave solutions; see Section 3). Although the chaotic behavior occurs only on a "tiny" finite-dimensional subset it may (and often does) influence the behavior of a *physically observable* set of solutions of the CML, i.e., solutions which are typical in a sense. In particular, we establish spatio-temporal chaos associated with traveling wave solutions of the CMLs (see Section 4). This class of solutions was studied first by Afraimovich and Pesin in [6] and provides the only known class of solutions that may generate finite-dimensional spatio-temporal chaos. The dynamics of the evolution operator restricted to traveling wave solutions is completely determined by the *traveling wave map.* In Sections 5 and 6 we present all known results describing hyperbolic, topological, and ergodic properties of this map as well as consider some interesting examples.

1. Lattice dynamical systems as discretizations of partial differential equations. There are many partial differential equations of evolution type whose discrete versions are lattice dynamical systems of the form 0.1.

Among them we consider some nonlinear reaction-diffusion equation as well as then Swift-Hohenberg, Kuramoto-Sivashinsky, and Ginzburg-Landau equations.

1.1. A nonlinear reaction-diffusion equation is a partial differential equation of the form:

$$\frac{\partial u}{\partial t} = h(u) + \kappa A \Delta u, \tag{1.1}$$

where $u = u(x,t)$ is a function of two variables (the space coordinate x and time t) with values in the d-dimensional Euclidean space $\mathbb{R}^d$ ($d = 1, 2, 3$); A is the coupling matrix and κ is the diffusion coefficient. Equation 1.1 describes a large variety of phenomena in different fields. Examples are heat conductivity in physics, chemical diffusion processes in chemistry, enzyme kinetics in biology, and propagation of voltage impulses through nerve axosn in neurophysiology (see [29] and [31] for more applications).

One can obtain a number of well-known particular cases of reaction-diffusion equation 1.1 by an appropriate choice of the nonlinear term h. Among them are:

1.1.1. The Kolmogorov-Petrovsky-Piskunov (KPP) equation for which the nonlinear term is a quadratic polynomial,

$$h(u) = \alpha u(1-u), \tag{1.2}$$

where $\alpha > 0$ is a parameter. This equation appeared in genetics as a model for the spread of an advantageous gene through a population. The solution $u(x,t)$ measures the proportion of the population possessing this gene during the evolution of the system (the so-called Fisher model), see [11, 21, 28, 31].

1.1.2. The Huxley equation for which the nonlinear term is a cubic polynomial,

$$h(u) = \alpha u(1-u)(u-a), \tag{1.3}$$

where $0 < a < 1$ and $\alpha > 0$ are parameters.

1.1.3. The FitzHugh-Nagumo equation where the nonlinear term is a two-dimensional map of the plane,

$$h(u,v) = (d\varphi(u) - av, bu - cv)). \tag{1.4}$$

Here $\varphi(u)$ is a cubic polynomial, $\varphi(u) = u(u-\theta)(1-u)$ with $\theta \in (0,1)$ and $a, b, c, d > 0$ are real parameters.

The Huxley equation and FitzHugh-Nagumo equation are used to model the propagation of voltage impulse through a nerve axon (see [29]).

1.2. The Swift-Hohenberg equation is a partial differential equation of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha u - (1+\partial_x^2)^2 u - u^3 \\ &= (\alpha - 1)u - u^3 - \frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^2 u}{\partial x^2},\end{aligned}$$

where u is a real function and α a real parameter. For some physical phenomena modeled by this equation see [22, 35]; for an extensive mathematical study of the Swift-Hohenberg equation see [15].

1.3. The Kuramoto-Sivashinsky equation was introduced by Kuramoto in one space dimension for the study of phase turbulence in the Belousov-Zhabotinsky reactions. A two-dimensional extension of this equation was later used by Sivashinsky in studying the propagation of flame fronts in the case of mild combustion (see [17, 22]). We will consider only the one-dimensional version which is a partial differential equation of the form

$$\frac{\partial u}{\partial t} = -\eta u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - u\frac{\partial u}{\partial x}, \tag{1.5}$$

where u is a real function and η a real parameter.

1.4. The Ginzburg-Landau (amplitude) equation is a one-dimensional partial differential equation of the form

$$\frac{\partial u}{\partial t} = h(u) + (\lambda + i\alpha)\frac{\partial^2 u}{\partial x^2}, \tag{1.6}$$

where $h(u) = (\kappa + i\beta)u|u|^2 - \gamma u$ and λ, α, β, γ, κ are real parameters. The complex-valued function $u(x,t)$ is defined on $\Omega \times \mathbb{R}^+$ where Ω is an open domain in $\mathbb{R}^d$.

This equation governs the finite amplitude evolution of instability waves in a large variety of dissipative systems. Various forms of this equation arise in hydrodynamic instability theory. For example, one can use it to describe the nonlinear growth of convection rolls in Rayleigh-Bénard problem or the appearance of Taylor vortices in the flow between counter rotating circular cylinders (see [36]).

In this paper we will deal with the **real version of Ginzburg-Landau equation**

$$\frac{\partial u}{\partial t} = h(u) + \frac{\partial^2 u}{\partial x^2}, \tag{1.7}$$

where $h(u) = u(\gamma - \delta u^2)$, $(u \in \mathbb{R})$, and γ, δ are real parameters.

When $\gamma = \delta = 1$ this equation becomes the **nonlinear heat equation.**

1.5. Discretizations. One can obtain discrete versions of the above partial differential equations replacing derivatives by appropriate differences.

For the time derivative we assume the following discretization:

$$\frac{\partial u}{\partial t} \longrightarrow [(u(x, t + \Delta t) - u(x, t)]/\Delta t,$$

where Δt is the discretization step. Thus, we obtain the following local maps:

1. for the Kolmogorov-Petrovsky-Piskunov equation,

$$f(u) = u + \Delta t h(u) = u + \nu u(1 - u) \tag{1.8}$$

   is a quadratic polynomial, where $\nu = \alpha\Delta t$ is a parameter;
2. for the Huxley equation,

$$f(u) = u + \Delta t h(u) = u + \nu u(1 - u)(u - a) \tag{1.9}$$

   is a cubic polynomial, where $\nu = \alpha\Delta t$ is a parameter;
3. for the FitzHugh-Nagumo equation,

$$f(u, v) = (u + A\varphi(u) - \alpha v, \beta u + \gamma v) \tag{1.10}$$

   is a two-dimensional map, where $A = d\Delta t$, $\alpha = a\Delta t$, $\beta = b\Delta t$, $\chi = c\Delta t$, and $\gamma = 1 - \chi$ are positive numbers and $0 < \theta < 1$;
4. for the Swift-Hohenberg equation,

$$f(u) = (1 + \nu)u - \mu u^3 \tag{1.11}$$

   is a cubic polynomial, where $\nu = (1 - \alpha)]\Delta t$ and $\mu = \Delta t$ are parameters.
5. for the Kuramoto-Sivashinsky equation,

$$f(u) = \nu u \tag{1.12}$$

   is a linear function, where $\nu = (1 - \eta)\Delta t$ is the parameter;
6. for the (real) version of the Ginzburg-Landau equation,

$$f(u) = u + \Delta t h(u) = u + \nu u(1 - qu^2) \tag{1.13}$$

   is a quadratic polynomial, where $\nu = \gamma\Delta t$ and $q = (\delta/\gamma)\Delta t$ are parameters.

1.6. For the space derivatives we allow any discretization scheme involving an arbitrary number of points. For example, we choose the following discretizations:

$$\frac{\partial u}{\partial x} \longrightarrow [u(x+\Delta x,t) - u(x,t)]/\Delta x,$$

$$\frac{\partial^2 u}{\partial x^2} \longrightarrow [u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)]/(\Delta x)^2,$$

$$\frac{\partial^4 u}{\partial x^4} \longrightarrow [u(x+2\Delta x,t) - 4u(x+\Delta x,t) + 6u(x,t)$$
$$-4u(x-\Delta x,t) + u(x-2\Delta x,t)]/(\Delta x)^4,$$

where Δx is the discretization step. The interaction g is then a function of $2s+1$ variables, where

$s = 1$ for Kolmogorov-Petrovsky-Piskunov equation,
Huxley equation,
FitzHugh-Nagumo equation,
Ginzburg-Landau equation,
$s = 2$ for Swift-Hohenberg equation,
Kuramoto-Sivashinsky equation.

2. Lattice systems as infinite-dimensional dynamical systems. We provide a formal mathematical description of lattice systems in terms of dynamical systems. Consider the direct product of finite-dimensional Euclidean spaces $\mathcal{R} = \bigotimes_{\mathbb{Z}^p} \mathbb{R}^d = (\mathbb{R}^d)^{\mathbb{Z}^p}$. We introduce a special norm called **norm with weights**. Namely, given $q_1 = (q_{1,i})$, $q_2 = (q_{2,i})$ with $q_{1,i} > 1$, $q_{2,i} > 1$, $i = 1, \dots, p$ and $u = (u_{\bar{j}}) \in \mathcal{R}$, $\bar{j} \in \mathbb{Z}^p$, we set

$$\|u\|_{q_1,q_2} = \sum_{j_1} \cdots \sum_{j_p} \frac{|u(\bar{j})|}{q(j_1)\cdots q(j_p)}, \tag{2.1}$$

where $\bar{j} = (j_i)$ and $q(j_i) = (q_{1,i})^{j_i}$ if $j_i \geq 0$ and $q(j_i) = (q_{2,i})^{j_i}$ if $j_i < 0$. Norms with weights were used by Sattinger in [34] to study the stability of traveling waves for some partial differential equations and by Afraimovich and Pesin in [6] to study spatio-temporal chaos in traveling waves for some lattice systems.

The lattice dynamical system 0.1 can now be described as the infinite-dimensional dynamical system (M_{q_1,q_2}, Φ) with the underlying phase space

$$M_{q_1,q_2} = \{u = (u_{\bar{j}}) : \|u\|_{q_1,q_2} < \infty\} \tag{2.2}$$

and the non-linear evolution operator $\Phi = \Phi_\varepsilon$

$$(2.3) \qquad (\Phi u)_{\bar{j}}(n+1) = f(u_{\bar{j}}(n)) + \varepsilon g_j(\{u_{\bar{i}}(n)\}_{|\bar{i}-\bar{j}|\leq s}).$$

Let us point out that the dynamical system (M_{q_1,q_2}, Φ) corresponds to solutions of the CML which satisfy the initial condition $(u_{\bar{j}}(0)) \in M_{q_1,q_2}$ and boundary condition $\|u\|_{q_1,q_2} < \infty$ (i.e., the solution may grow at infinity at an exponential rate).

We observe that M_{q_1,q_2} is a Banach space with respect to the metric $\|\cdot\|_{q_1,q_2}$. Moreover, the operator Φ has a special form $\Phi = F+\varepsilon G$ where $F = \bigotimes_{\mathbb{Z}^p} f$ is the direct product of copies of the local map f and $G = (g_{\bar{j}})_{\bar{j}\in\mathbb{Z}^p}$. F is called **the uncoupled map** and G the **coupling** or **interaction**. If ε is sufficiently small the operator Φ is a small perturbation of the map F.

We assume that the following main assumptions hold:

A1. There exists $K > 0$ such that for $l = 1, 2$:

$$(2.4) \qquad \sup_{x\in\mathbb{R}^d} \|D^l f_x\| \leq K,$$

$$(2.5) \qquad \sup_{1\leq i\leq 2s+1} \sup_{x\in\mathbb{R}^{d2s+1}} \left\| \frac{\partial^l g}{\partial x_i{}^l} \right\| \leq K;$$

A2. For any $x = (x_i) \in (\mathbb{R}^d)^{2s+1}$

$$(2.6) \qquad \det\left(\frac{\partial g(x)}{\partial x_1}\right) \neq 0,$$

and

$$(2.7) \qquad \sup_{x\in\mathbb{R}^{d2s+1}} \det\left(\frac{\partial g(x)}{\partial x_1}\right)^{-1} < \infty.$$

We emphasize that we do not require that the maps f and g be bounded but only their first and second derivatives. Under assumptions $A1$ and $A2$, the dynamical system (M_{q_1,q_2}, Φ) is correctly defined. More precisely the following statement holds:

PROPOSITION 2.1 (see [6]). *Under Assumptions $A1$ and $A2$ for any $q_1 = (q_{1,i}), q_2 = (q_{2,i})$ as above the operator Φ maps M_{q_1,q_2} into itself.*

Let us stress that the operator Φ is not differentiable in the sense of Frechét but in the sense of Gâteaux (the latter means that it is differentiable only along finite-dimensional subspaces). One can show that the Gâteaux differential of Φ is given by the following linear map:

$$(2.8) \qquad d_u\Phi(\xi)_{\bar{j}} = f'(u_{\bar{j}})\xi_{\bar{j}} + \sum_{|\bar{i}-\bar{j}|\leq s} a_{\bar{i}} \cdot \xi_{\bar{i}},$$

where $u = (u_{\bar{j}})$, $\xi = (\xi_{\bar{j}}) \in M_{q_1,q_2}$ and $a_{\bar{i}} = \frac{\partial g}{\partial u_{\bar{i}}}(\{u_{\bar{i}}\}_{|\bar{i}-\bar{j}|\leq s})$.

The space M_{q_1,q_2} admits **spatial translations (or shifts)** $S^{\overline{k}} : M_{q_1,q_2} \to M_{q_1,q_2}$ given by

$$S^{\overline{k}}(u)_{\overline{i}} = u_{\overline{i}+\overline{k}}, \tag{2.9}$$

where $\overline{i} = (i_j), \overline{k} = (k_j) \in \mathbb{Z}^p$.

We now assume that the following condition holds:

A3. The maps Φ^l and $S^{\overline{k}}$ commute for any $l \in \mathbb{Z}$ and $\overline{k} \in \mathbb{Z}^p$.

Let us note that if the interaction $g_{\overline{j}}$ does not depend on $\overline{j}$ the assumption $A3$ holds. This is the case when the lattice system is obtained as a discretization of a partial differential equation.

Under assumption $A3$ the maps $\{(\Phi^l, S^{\overline{k}}) : l \in \mathbb{Z}, \overline{k} \in \mathbb{Z}^p\}$ generate an action of the $\mathbb{Z}^{p+1}$-lattice on M_{q_1,q_2}. The main goal of this paper is to describe topological, hyperbolic, and ergodic properties of this action.

3. Types of chaotic behavior in lattice systems.

3.1. In [32], Pesin and Sinai discussed the following types of chaotic behavior in lattice systems corresponding to actions of temporal and/or spatial translations. A lattice dynamical system is said to display:

1) **temporal chaos** if there exists a measure μ invariant under the $\mathbb{Z}^1$-action $\{\Phi^l\}$ which is mixing;
2) **spatial chaos** if there exists a measure μ invariant under the $\mathbb{Z}^p$-action $\{S^{\overline{k}}\}$ which is mixing;
3) **spatio-temporal chaos** if there exists a measure μ invariant under the $\mathbb{Z}^{p+1}$-action $\{\Phi^l, S^{\overline{k}}\}$ which is mixing.

In many cases the chaotic behavior of lattice systems is essentially finite-dimensional. This means that there exists a finite-dimensional (often smooth) submanifold in the infinite-dimensional phase space which is invariant with respect to time translations or space translations or both and which supports an invariant mixing measure. Such submanifolds are usually associated with special classes of solutions.

It may also happen that such a submanifold is **stable** in the infinite-dimensional phase space, i.e., solutions which start in a small neighborhood of this submanifold approach it with time.

In this case chaotic behavior is persistent and thus is *physically observable.* Otherwise, chaotic behavior occurs on a "tiny" finite-dimensional submanifold and is "invisible." In some cases the invariant submanifold is stable in a weaker sense: it possesses an infinite-dimensional separatrix which is everywhere dense in the phase space. In this case the chaotic behavior should also be considered as physically observable. However, it is essentially unstable (with respect to small perturbations of initial data) and hence, is significantly more difficult to study.

3.2. One can observe temporal chaos in the space of solutions which are **spatially-homogeneous**, i.e., do not depend on the spatial coordinate, $u_{\overline{j}}(n) = u(n)$. We assume that the coupling $g_j(x_1 \dots, x_{2s+1})$ does not depend on j, i.e., $g_j = g$. It is easy to see that the function $u(n)$ should satisfy the following equation:

$$u(n+1) = f(u(n)) + \varepsilon g(\underbrace{u(n), \dots, u(n)}_{2s+1}). \tag{3.1}$$

This equation reproduces orbits of the map $H_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ given by $H_\varepsilon(x) = f(x) + \varepsilon \widehat{g}(x)$ where $\widehat{g}(u) = g(\underbrace{u, \dots, u}_{2s+1})$. The map H_ε is a small perturbation of the local map f and completely determines the behavior of the evolution operator Φ_ε on the space of spatially-homogeneous solutions. To illustrate this let us introduce the **embedding** map $\chi : \mathbb{R}^d \to M_{q_1,q_2}$, where $\chi(x) = (u_{\overline{j}})$ with $u_{\overline{j}} = x$ for any $\overline{j} \in \mathbb{Z}^p$. Obviously, this map is linear.

PROPOSITION 3.1.

1. *The space of spatially-homogeneous solutions $\mathcal{H} = \chi(\mathbb{R}^d)$ is a d-dimensional linear subspace of the Banach space M_{q_1,q_2}.*
2. *$\mathcal{H}$ is invariant under both space and time translations, i.e. under the $\mathbb{Z}^{p+1}$-action $(\Phi_\varepsilon^l, S^{\overline{k}})$.*
3. *The action of the evolution operator on the space of spatially-homogeneous solutions, $\Phi_\varepsilon|\mathcal{H}$, is linearly conjugate to the map H_ε, i.e. the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{R}^d \arrow[r, "\chi"] \arrow[d, "H_\varepsilon"'] & \mathcal{H} \subset M_{q_1,q_2} \arrow[d, "\Phi_\varepsilon"] \\
\mathbb{R}^d \arrow[r, "\chi"] & \mathcal{H} \subset M_{q_1,q_2}
\end{tikzcd}$$

D1. *Embedding of Spatially-homogeneous Solutions.*

Proof. The first statement follows immediately from the fact that the map χ is linear. Since every point in $\mathcal{H}$ is spatially-homogeneous, $\mathcal{H}$ is invariant under space translations. The invariance of $\mathcal{H}$ under time translations and commutativity of the diagram follows from the definitions of $\mathcal{H}$ and H_ε. □

As an immediate consequence of Proposition 3.1 we obtain that if μ is an invariant mixing measure for the map H_ε then the measure $\chi_*\mu$ (defined by $\chi_*\mu = \mu \circ \chi$) is an invariant mixing measure for the evolution operator $\Phi_\varepsilon|M_{q_1,q_2}$. Thus, the latter displays temporal chaos.

The problem of finding an invariant mixing measure for H_ε is purely finite-dimensional and can be solved within classical perturbation theory. For example, if the local map is hyperbolic (i.e, possesses a hyperbolic invariant set) then so is the map H_ε for sufficiently small ε. This guarantees the existence of mixing measures (see a more detailed description of hyperbolic sets in Section 5 below; see also the Appendix).

It is unknown whether the space $\mathcal{H}$ is stable in M_{q_1,q_2}, in the above mentioned sense, and therefore, whether the associated temporal chaos is physically observable.

3.3. In order to demonstrate the phenomenon of spatial chaos we consider the space of **steady-state (stationary) solutions**, i.e., solutions which do not depend on time: $u_{\bar{j}}(n) = u_{\bar{j}}$. It is easy to see that these solutions form a finite-dimensional linear subspace of the phase space M_{q_1,q_2}. We have the following equation for $u_{\bar{j}}$:

$$u_{\bar{j}} = f(\psi_{\bar{j}}) + \varepsilon g_j(\{u_{\bar{i}}\}_{|\bar{i}-\bar{j}|\leq s}). \tag{3.2}$$

Steady-state solutions have been studied by many authors. For example, in [2], Afraimovich and Chow considered the case of a one-dimensional lattice ($p = 1$) and assumed that $g = g(u_1, u_2, u_3) = u_1 - 2u_2 + u_3$ (this form of function g corresponds to the spatial discretization of the reaction-diffusion equation described above). In this case Equation 3.2 becomes:

$$h(u_j) + a(u_{j-1} - 2u_j + u_{j+1}) = 0 \tag{3.3}$$

with j integer. Solving (3.3) for u_{j+1} yields

$$u_{j+1} = -\frac{1}{a}h(u_j) + 2u_j - u_{j-1}. \tag{3.4}$$

Solutions to equations 3.4 are determined by the map $G_a : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ given by

$$G_a(x, y) = \left(y,\ -\frac{1}{a}h(y) + 2y - x\right), \tag{3.5}$$

where $x, y \in \mathbb{R}^d$.

PROPOSITION 3.2. *There exists an embedding $\chi : \mathbb{R}^{2d} \to M_{q_1,q_2}$ such that*

1. *the space of steady-state solutions $\mathcal{S} = \chi(\mathbb{R}^{2d})$ is a smooth 2d-dimensional submanifold of M_{q_1,q_2};*
2. *the space $\mathcal{S}$ is invariant under both space and time translations;*
3. *the action of the evolution operator on the space of steady-state solutions, $\Phi_\varepsilon|\mathcal{S}$, is conjugated to the map G_a, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{R}^{2d} & \xrightarrow{\ \chi\ } & \mathcal{S} \subset M_{q_1,q_2} \\ {\scriptstyle G_a}\downarrow & & \downarrow{\scriptstyle S_j} \\ \mathbb{R}^{2d} & \xrightarrow{\ \chi\ } & \mathcal{S} \subset M_{q_1,q_2} \end{array}$$

D2. *Embedding of Steady-state Solutions.*

Proof. For any $(x, y) \in \mathbb{R}^{2d}$, set $\chi(x, y) = (u_j)$, where

$$u_j = \begin{cases} (G_a^n)_1(x, y) & \text{for } n > 0 \\ x & \text{for } n = 0 \\ ((G_a^{-1})^{-n})_1(x, y) & \text{for } n < 0 \end{cases}$$

and $G_a^{-1}(x, y) = (-\frac{1}{a}h(x)+2x-y,\ x)$ (here we use the index 1 to denote the first component of the maps G_a and G_a^{-1}, i.e. the corresponding projection to $\mathbb{R}^d$). The first statement now follows. By the definition of steady-state solutions every point in $\mathcal{S}$ is fixed under time translations. The invariance of $\mathcal{S}$ under space translations S_j and the commutativity of the diagram follow from the definitions of G_a and χ. □

In [2], Afraimovich and Chow studied topological properties of steady-state solutions. A solution $u^* = \{u_j^*\}$ of equation 3.4 is said to be a homoclinic point for space translations on $\mathcal{S}$ if there exists $u^0 \in \mathbb{R}^d$ such that

$$\lim_{|j| \to \infty} u_j^* = u^0. \tag{3.6}$$

One can show that $u^0 \in \mathcal{S}$. In [2], the authors studied the problem of existence of a homoclinic point with bounded l^2 (or l^∞)-norm. If such a point exists, then the map G_a possesses a locally maximal hyperbolic set and hence, has many invariant mixing measures (see Appendix). They also discussed the stability of homoclinic points with respect to small linear perturbations.

4. Traveling wave solutions. We describe another class of solutions known as traveling wave solutions. We shall show that traveling wave solutions running with a given velocity form a finite-dimensional smooth submanifold in the infinite-dimensional phase space which is invariant under both time and space translations. We also observe finite-dimensional spatio-temporal chaos associated with traveling waves.

Let m be an integer and $\bar{l} = (l_i) \in \mathbb{Z}^p$ an integer p-tuple. For $\bar{j} = (j_i), \bar{k} = (k_i) \in \mathbb{Z}^p$, we set $(\bar{j}, \bar{k}) = \sum_{i=1}^p j_i k_i$. A **traveling wave solution** of Equation 0.1 is of the form:

$$u(\bar{j}, n) = \psi((\bar{l}, \bar{j}) + mn), \tag{4.1}$$

where $\psi : \mathbb{Z} \to \mathbb{R}^d$ is a function. The numbers m and l_i determine the **velocity** of the wave. From now on we assume that

1. the numbers l_i are relatively prime (i.e., their least common divisor is 1);
2. $m > s \sum_{i=1}^p l_i$.

The last condition means that the velocity of the wave is large.

For simplicity we consider only one-dimensional lattices (i.e., $p = 1$; for the general case we refer the reader to [6, 7]). The norm in the phase space $\mathcal{M}_{q_1,q_2}$ is determined by two numbers $q_1 > 1$ and $q_2 > 1$ in the following way

$$\|u\|_{q_1,q_2} = \sum_{j<0} \frac{|u_j|}{q_1{}^{-j}} + \sum_{j\varepsilon 0} \frac{|u_j|}{q_2{}^{j}}. \tag{4.2}$$

We fix $m, l \in \mathbb{Z}$ which satisfy Conditions 1 and 2, i.e. m, l are relatively prime numbers and $m > ls + 1$.

We also assume that the coupling $g_j(x_1 \ldots, x_{2s+1})$ does not depend on j, i.e., $g_j = g$.

The function ψ must satisfy the following **traveling wave equation**:

$$\psi(k) = f(\psi(k-m)) + \varepsilon g(\{\psi(k-m+li)\}_{|i|\leq s}), \tag{4.3}$$

where $k = lj + mn + m$ is called the **traveling coordinate.**

Denote by $\Psi = \Psi(\varepsilon, q_1, q_2, m, l, s)$ the set of solutions of the traveling wave equation which is a subset in $(\mathbb{R}^d)^{\mathbb{Z}}$. In view of our assumptions on m and l, Equation 4.3 is an equation with delay argument (i.e., $k - m + ls \leq k - 1 < k$). Therefore, a traveling wave is uniquely determined by the values:

$$x_p = \psi(-m - ls + p + 1) \quad \text{for } p = 1, \ldots, ls + m. \tag{4.4}$$

These relations define a (non-linear) map $\alpha : (\mathbb{R}^d)^{ls+m} \to \Psi$.

The dynamics on the set Ψ of traveling waves is given by the **shift map** Q_ε defined by $Q_\varepsilon(\psi)(k) = \psi(k+1)$. In [6], the authors demonstrated that this dynamics is finite-dimensional and is governed by the **traveling wave map** $F_\varepsilon : (\mathbb{R}^d)^{ls+m} \to (\mathbb{R}^d)^{ls+m}$ given by

$$F_\varepsilon(x_1, \ldots, x_{ls+m}) = (x_2, \ldots, x_{ls+m}, f(x_{ls+1}) + \varepsilon g(x_{p(i)})_{i=-s}^{s}), \tag{4.5}$$

where $p(i) = l(s+i) + 1$.

THEOREM 4.1 (see [6]). *There exist $q_1^{(0)} > 1$, $q_2^{(0)} > 1$ such that for any $q_1 \geq q_1^{(0)}$, $q_2 \geq q_2^{(0)}$ we have:*

1. *the map α is a smooth embedding of $(\mathbb{R}^d)^{ls+m}$ into M_{q_1,q_2}; the image $\Psi_{q_1,q_2} = \alpha((\mathbb{R}^d)^{ls+m})$ is a smooth submanifold of dimension $d(ls+m)$;*
2. *for any $x \in (\mathbb{R}^d)^{ls+m}$ and $\xi \in (\mathbb{R}^d)^{ls+m}$ the differential $d_x\alpha$ is given by*

$$\begin{aligned}(d_x\alpha(\xi))(k) =& df(\psi(k-m))(d_x\alpha(\xi))(k-m) \\ &+ \varepsilon \sum_{i=-s}^{s} a_i((d_x\alpha(\xi))(k-m+li)),\end{aligned} \tag{4.6}$$

where $a_i = \frac{\partial g}{\partial u_i}(\psi(k-m+li)_{|i|\leq s})$; moreover,

$$\sup_{x \in (\mathbb{R}^d)^{ls+m}} ||d_x \alpha||_{q_1 \cdot q_2} < \infty; \tag{4.7}$$

3. *the shift map Q_ε and the traveling wave map F_ε are smoothly conjugated via the embedding map α, i.e., $\alpha \circ F_\varepsilon = Q_\varepsilon \circ \alpha$; in other words the following diagram is commutative:*

$$\begin{array}{ccc} {\mathbb{R}^d}^{ls+m} & \xrightarrow{\alpha} & \Psi_{q_1,q_2} \\ {\scriptstyle F_\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle Q_\varepsilon} \\ {\mathbb{R}^d}^{ls+m} & \xrightarrow{\alpha} & \Psi_{q_1,q_2} \end{array}$$

D3. *Embedding of Travelling Wave Solutions, I.*

Every solution ψ of the traveling wave equation 4.3 determines a solution $u = u_\psi = u_{\psi,j}(n)$ of the lattice system 0.1 whose values at time $n = 0$ are given as

$$u_{\psi,j}(0) = \psi(lj). \tag{4.8}$$

It follows that $u_{\psi,j}(n) = \psi(lj + mn) = \psi(k - m)$ for every time n. In particular, every traveling wave is uniquely determined by its values at countably many points of the form lj with $j \in \mathbb{Z}$.

Equation 4.8 defines the linear map $\beta(\psi) = u_\psi$ on Ψ_{q_1,q_2}.

THEOREM 4.2 (see [6]). *For any $q_1 \geq q_1^{(0)}$ and $q_2 \geq q_2^{(0)}$ (see Theorem 4.1) we have:*

1. *β is a smooth linear embedding of Ψ_{q_1,q_2} into $M_{q_1^l,q_2^l}$; the image $A_{q_1,q_2} = \beta(\Psi_{q_1,q_2})$ is a smooth submanifold of dimension $d(ls+m)$;*
2. *the map Q_ε^m and the restriction of the evolution operator Φ_ε to A_{q_1,q_2} are conjugated via the map β, i.e. $\beta \circ Q_\varepsilon^m = \Phi_\varepsilon \circ \beta$; in other words, the following diagram is commutative:*

$$\begin{array}{ccc} \Psi_{q_1,q_2} & \xrightarrow{\beta} & A_{q_1,q_2} \subset M_{q_1^l,q_2^l} \\ {\scriptstyle Q_\varepsilon^m}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\varepsilon} \\ \Psi_{q_1,q_2} & \xrightarrow{\beta} & A_{q_1,q_2} \subset M_{q_1^l,q_2^l} \end{array}$$

D4. *Embedding of Travelling Wave Solutions, II.*

Combining the above two diagrams together we obtain that the dynamics of the evolution operator Φ_ε on the set of traveling wave solutions with a given velocity m/l is completely determined by the traveling wave map F_ε. Namely, the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{R}^d)^{ls+m} & \xrightarrow{\chi} & A_{q_1,q_2} \subset M_{q_1^l,q_2^l} \\ {\scriptstyle F_\varepsilon^m}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_\varepsilon} \\ (\mathbb{R}^d)^{ls+m} & \xrightarrow{\chi} & A_{q_1,q_2} \subset M_{q_1^l,q_2^l} \end{array}$$

D5. *Embedding of Travelling Wave Solutions, III.*

where $\chi = \beta \circ \alpha$ is the (non-linear) conjugacy map.

It is immediate to check that the submanifold A_{q_1,q_2} is preserved under space translations S^k and hence, is invariant with respect to both time and space translations. We call it **the traveling wave submanifold**. We will sometimes use a more explicit notation for it, $A^{l,m}_{q_1,q_2}$, to indicate that it consists of traveling wave solutions running with the velocity m/l.

As an immediate consequence of the diagram D5 we obtain that if μ is an invariant mixing measure for the map F_ε then the measure $\chi_*\mu$ is an invariant mixing measure for the evolution operator $\Phi_\varepsilon | A_{q_1,q_2}$ (see Appendix).

Below, we shall study topological and ergodic properties of the traveling wave map and in particular, describe situations where the existence of mixing measures is guaranteed. We now show that the submanifold $A^{m,l}_{q_1,q_2}$ of traveling waves solutions is stable in some weak sense.

As we noted above the traveling wave equation is an equation with delay argument and therefore, the dynamics on the set of the traveling wave solutions is a "drift." Such systems were studied in [5, 6]. We follow the approach suggested in [6] to construct an infinite-dimensional stable manifold for A_{q_1,q_2}.

Consider the non-linear operator $\hat{Q}_\varepsilon : M_{q_1,q_2} \to M_{q_1,q_2}$ given by

$$\hat{Q}_\varepsilon((w_k)) = (f(w_{k-m}) + \varepsilon g(\{w_{k-m+li}\}_{|i|\le s})). \tag{4.9}$$

It is easy to see that the restriction of this operator to the set of traveling waves $\Psi_{q_1,q_2} \subset M_{q_1,q_2}$ coincides with the shift map Q_ε. For any fixed $\psi \in \Psi_{q_1,q_2}$, we set

$$V^k_{q_1,q_2}(\psi) = \{w = (w_k) \in M_{q_1,q_2} : w_i = \psi_i \text{ for } i \le k\}. \tag{4.10}$$

It is easy to check that for every $\psi \in \Psi_{q_1,q_2}$ the subspaces $V^k_{q_1,q_2}(\psi)$ are linear and form a filtration, i.e., $V^{k+1}_{q_1,q_2}(\psi) \subset V^k_{q_1,q_2}(\psi)$ for every k. Moreover, the linear subspace

$$V_{q_1,q_2}(\psi) = \cup_{k\in\mathbb{Z}} V^k_{q_1,q_2}(\psi) \tag{4.11}$$

is everywhere dense in M_{q_1,q_2}. One can also prove the following statements:

1. The subspaces $V^k_{q_1,q_2}(\psi)$ are invariant under the operator $\hat{Q}_\varepsilon$, more precisely, $\hat{Q}_\varepsilon(V^k_{q_1,q_2}(\psi)) \subset V^{k+m-ls}_{q_1,q_2}(\psi)$;
2. If $q_1 \geq q_1^{(0)}$ and $q_2 \geq q_2^{(0)}$ then for every k the subspace $V^k_{q_1,q_2}(\psi)$ is stable, i.e. there are constants $C = C(k, q_1, q_2) > 0$ and $0 < \gamma < 1$ such that for any $w \in V^k_{q_1,q_2}(\psi)$

$$\|\hat{Q}^n_\varepsilon(w) - \hat{Q}^n_\varepsilon(\psi)\|_{q_1,q_2} \leq C\gamma^n \|w - \psi\|_{q_1,q_2}; \tag{4.12}$$

moreover the subspace $V^k_{q_1,q_2}(\psi)$ is transverse to Ψ_{q_1,q_2} at ψ.

We now describe the embedding of the subspaces $V^k_{q_1,q_2}(\psi)$ into the space of solutions of the lattice system. Consider the map $\hat{\beta}$ on M_{q_1,q_2} which associates to every $w = (w_j)$ the solution $u = (u_j(n)) = \hat{\beta}(w)$ such that $u_j(0) = w_{lj}$. It is clear that the restriction of $\hat{\beta}$ to Ψ_{q_1,q_2} coincides with β. Moreover, the following statement holds:

PROPOSITION 4.3 (see [6]). *$\hat{\beta}$ is a linear bounded operator from M_{q_1,q_2} to $M_{q_1^l,q_2^l}$ which establishes the conjugacy between $\hat{Q}^m_\varepsilon$ and the evolution operator; more precisely the following diagram is commutative:*

$$\begin{tikzcd}
M_{q_1,q_2} \arrow[r, "\hat{\beta}"] \arrow[d, "\hat{Q}^m_\varepsilon"'] & M_{q_1^l,q_2^l} \arrow[d, "\Phi_\varepsilon"] \\
M_{q_1,q_2} \arrow[r, "\hat{\beta}"] & M_{q_1^l,q_2^l}
\end{tikzcd}$$

D6. *Embedding of Stable Manifolds.*

For any traveling wave solution $u = \beta(\psi) \in \mathcal{Y}_{q_1,q_2}$, where $\psi \in \Psi_{q_1,q_2}$ we set

$$W^k_{q_1,q_2}(u) = \hat{\beta}(V^k_{q_1,q_2}(\psi)), \qquad W_{q_1,q_2}(u) = \hat{\beta}(V_{q_1,q_2}(\psi)). \tag{4.13}$$

Clearly,

$$W_{q_1,q_2}(u) = \bigcup_{k\in\mathbb{Z}} W^k_{q_1,q_2}(u). \tag{4.14}$$

The following theorem describes stability properties of the traveling wave solutions.

THEOREM 4.4 (see [6]). *For sufficiently large q_1, q_2 and for any $u \in A_{q_1,q_2}$ the submanifold $W_{q_1,q_2}(u)$ is infinite-dimensional and everywhere dense in $M_{q_1^l,q_2^l}$; It is invariant under the evolution operator Φ_ε, transverse to A_{q_1,q_2}, and stable.*

Summarizing the above results we have established that for sufficiently large q_1 and q_2

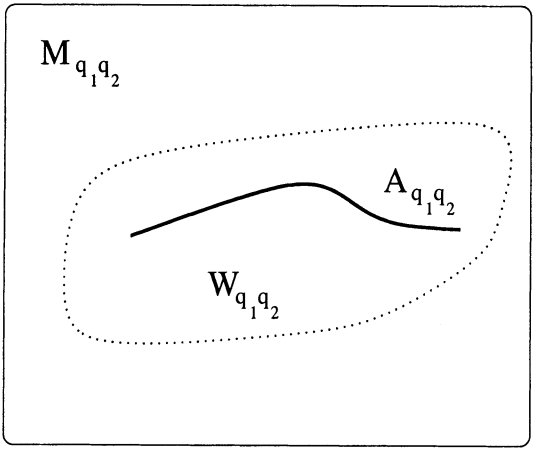

FIG. 1. ***Stable Separatrix for a Traveling Wave Submanifold.***

- the set of traveling wave solutions of the lattice system is a smooth submanifold A_{q_1,q_2} of finite dimension $d(ls+m)$ in $M_{q_1^l,q_2^l}$;
- the dynamics of the evolution operator Φ_ε on this submanifold is completely determined by the traveling wave map F_ε;
- the union of submanifolds $W_{q_1,q_2}(u)$ over $u \in A_{q_1,q_2}$ forms an infinite-dimensional everywhere dense separatrix transverse to A_{q_1,q_2}; we denote it by $W_{q_1,q_2} = W(A_{q_1,q_2})$.

Let $q_1^{(0)} > 1$ and $q_2^{(0)} > 1$ be chosen according to Theorem 4.1. Fix any $q_1 > q_1^{(0)}$ and $q_2 > q_2^{(0)}$. There exists a smallest positive integer L such that $(q_1^{(0)})^L \geq q_1$ or $(q_2^{(0)})^L \geq q_2$.

For every integer $1 \leq l < L$, let us set $\tilde{q}_i = \tilde{q}_i(l) = q_i^{1/l}$, $i = 1, 2$. We have that $\tilde{q}_i \geq q_i^0$.

Therefore, for every integer $m \geq ls+1$ such that m and l are relatively prime, Theorem 4.2 applies and provides a smooth $d(ls+m)$-dimensional submanifold $A^{m,l}_{\tilde{q}_1,\tilde{q}_2} \subset M_{q_1,q_2}$. It has an infinite-dimensional everywhere dense separatrix $W(A^{m,l}_{\tilde{q}_1,\tilde{q}_2})$.

Therefore, one has countably many submanifolds $A^{m,l}_{\tilde{q}_1,\tilde{q}_2}$ corresponding to traveling waves running with velocities m/l, such that $\tilde{q}_1^l = q_1$ and $\tilde{q}_2^l = q_2$. Each of these submanifolds possesses an everywhere dense separatrix. This indicates that there is no stability in the space M_{q_1,q_2} in a strict sense but only in a weak one (along the stable separatrices). For values of l other than mentioned above such that $q_1^{(0)} < \tilde{q}_1 < q_1$ and $q_2^{(0)} < \tilde{q}_2 < q_2$ for some $\tilde{q}_1$ and $\tilde{q}_2$, the stable separatrices $W(A^{m,l}_{\tilde{q}_1,\tilde{q}_2})$ are nowhere dense in M_{q_1,q_2}.

There is an analogy between submanifolds $A^{m,l}_{q_1,q_2}$ and inertial manifolds for differential equations $u' = F(u)$ in a Hilbert space. Recall that

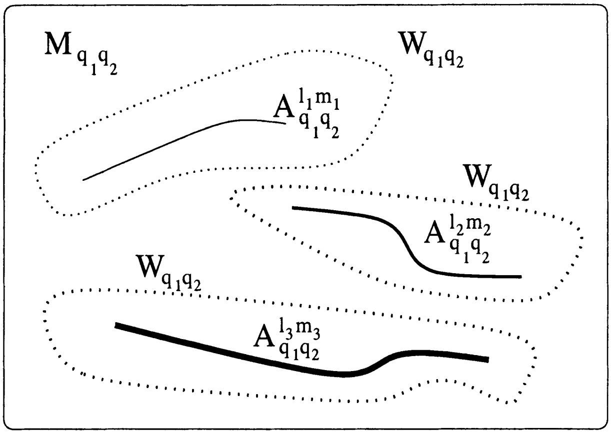

FIG. 2. ***Coexistence of Countably Many Traveling Wave Submanifolds.***

an **inertial manifold** for the semigroup $\{S(t)\}_{t\geq 0}$ associated to such an equation is a finite-dimensional smooth submanifold $\mathcal{M}$ which satisfies the following properties:

(1) $S(t)\mathcal{M} \subset \mathcal{M}$, i.e. $\mathcal{M}$ is positively invariant for the semigroup.

(2) $\mathcal{M}$ attracts exponentially **all** the orbits of the system.

While the first property obviously holds for the submanifolds $A^{m,l}_{q_1,q_2}$, the second property holds only in a weak sense, i.e., along their stable manifolds.

5. Dynamics of the traveling wave map: The case of a hyperbolic local map. As we saw above the dynamics of the evolution operator on the set of traveling wave solutions running with a given velocity is completely determined by the traveling wave map (4.5). In this section we describe hyperbolic properties of this map, i.e ., properties which characterize instability of trajectories. Our goal is to illustrate that for sufficiently small interactions these properties are dominated by the hyperbolic behavior of the local map f. In particular, we show that if the local map is hyperbolic in a strong sense (i.e., it possesses a hyperbolic set; every trajectory in this set is highly unstable; see below) then so are the traveling wave map and the restrictions of space and time translations to the submanifold of traveling wave solutions. This implies that the CML displays chaotic behavior of the highest degree, i.e., there exists a measure invariant under space and time translations which is supported on the set of traveling wave solutions and has ergodic properties of higher order (in other words, it is equivalent to the Bernoulli measure in the classical probability theory).

We begin by considering a map $F_\varepsilon : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$ of the form:

$$F_\varepsilon(x_1, \ldots, x_k, \ldots, x_n) = (x_2, \ldots, x_{k+1}, \ldots, f(x_k) + \varepsilon g(x_i)_{i=1}^n) \tag{5.1}$$

which is a generalization of the traveling wave map. Here $x_i \in \mathbb{R}^d$, $i = 1, \dots, n$ and $1 \le k < n$. Furthermore, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism and $g : (\mathbb{R}^d)^n \to \mathbb{R}^d$ is a smooth map. We assume that f and g satisfy conditions $A1$ and $A2$. This implies that the map F_ε is a local diffeomorphism, i.e., its differential $d_x F_\varepsilon$ is non-degenerate for every $x \in (\mathbb{R}^d)^n$. In particular, F_ε is locally one-to-one but it may not be globally one-to-one.

5.1. One can view the map F_ε as a small perturbation of the map F_0:

$$F_0(x_1, \dots, x_k, \dots, x_n) = (x_2, \dots, x_{k+1}, \dots, f(x_k)). \tag{5.2}$$

Let us note that the map F_0 is an endomorphism and its differential $d_x F_0$ is degenerate for every $x \in (\mathbb{R}^d)^n$. Therefore, in studying topological and ergodic properties of the map F_ε, for sufficiently small ε classical perturbation theory may not be applied directly and may need significant modifications.

We provide a useful description of F_0. Set $m = n - k + 1 < n$ and consider the map $G : (\mathbb{R}^d)^m \to (\mathbb{R}^d)^m$

$$G(x_1, \dots, x_{m-1}, x_m) = (x_2, \dots, x_m, f(x_1)). \tag{5.3}$$

This map is a diffeomorphism and its inverse is given by:

$$G^{-1}(x_1, x_2, \dots, x_m) = (f^{-1}(x_m), x_1, \dots, x_{m-1}). \tag{5.4}$$

It is easy to see that F_0 is a suspension over G, i.e., we have the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{R}^d)^n \arrow[r, "\pi"] \arrow[d, "F_0"'] & (\mathbb{R}^d)^m \arrow[d, "G"] \\
(\mathbb{R}^d)^n \arrow[r, "\pi"] & (\mathbb{R}^d)^m
\end{tikzcd}$$

D7. *The Suspension of the Endomorphism F_0 by the Diffeomorphism G.*

where $\pi : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^m$ is the natural projection to the last m d-dimensional components of $(\mathbb{R}^d)^n$.

5.2. We consider the case when the local map f possesses a locally maximal hyperbolic set Λ. Recall that a compact invariant set Λ is called **hyperbolic** if for every $x \in \Lambda$ there exists a splitting of the tangent space $T_x M$ at x into two subspaces $E^s(x)$ and $E^u(x)$, $TM_\Lambda = E^s(x) \oplus E^u(x)$ such that:

1. the splitting is invariant under the differential df, i.e.,

$$df(E^s(x)) = E^s(f(x)), \quad df(E^u(x)) = E^u(f(x)); \tag{5.5}$$

2. the subspaces $E^s(x)$ and $E^u(x)$ are respectively **stable** and **unstable** for the differential df, i.e.,

$$|df_x{}^n v| \leq c\lambda^n |v| \tag{5.6}$$

for every $v \in E^s(x)$ and

$$|df_x^n v| \geq c^{-1}\lambda^{-n}|v| \tag{5.7}$$

for every $v \in E^u(x)$, where $c > 0$ and $0 < \lambda < 1$ are constants independent of x and v.

A hyperbolic set Λ is said to be **locally maximal** if there exists an open neighborhood V of Λ such that

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(V). \tag{5.8}$$

For every point $x \in \Lambda$ one can construct local stable and unstable manifolds. We denote them by $W^s_{loc}(x, f)$ and $W^u_{loc}(x, f)$ respectively. The local stable manifold consists of those points y whose forward trajectories follow the trajectory of x within the distance ε for sufficiently small ε. The local unstable manifold is characterized in a similar way by reversing time.

Assume that Λ is a locally maximal hyperbolic set for the local map f. Consider the set $\tilde{\Lambda}_0 = \bigotimes_{i=1}^n \Lambda_i$, where Λ_i are copies of Λ, $i = 1, \ldots, n$. We have that $F_0(\tilde{\Lambda}_0) \subset \tilde{\Lambda}_0$ and $\tilde{\Lambda}_0$ is hyperbolic.

The set $\tilde{\Lambda}$ may not survive under small perturbations of F_0 (even for local diffeomorphisms). Therefore, we introduce the set

$$\Lambda_0 = \bigcap_{j\geq 0} F_0{}^j \tilde{\Lambda}_0. \tag{5.9}$$

The set Λ_0 is invariant under F_0, i.e., $F_0(\tilde{\Lambda}_0) = \tilde{\Lambda}_0$. It is locally maximal and hyperbolic and admits the following characterization:

LEMMA 5.1 (see [6]). *For any $y \in \Lambda_0$ there exists a sequence $y_k \in \Lambda_0$ such that $y_0 = y$ and $F_0(y_k) = y_{k+1}$ for $k \in \mathbb{Z}$.*

In the case of endomorphisms, for every point x in a locally maximal hyperbolic set one can construct a local stable manifold. It is uniquely defined and consists of those points y whose forward trajectories follow the trajectory of x within the distance ε for sufficiently small ε. On the other hand there are many local unstable manifolds, each corresponding to a branch of preimages of x. We will specify later the choice of unstable manifolds.

For the endomorphism F_0, the local stable manifold at a point $x = (x_i) \in \Lambda_0$ is given by

$$W^s_{loc}(x, F_0) = \prod_{i=1}^{k-1} B(x_i, \varepsilon) \times \prod_{i=k}^{n} W^s_{loc}(x_i, f), \tag{5.10}$$

where $\varepsilon > 0$ is sufficiently small. We will define the local unstable manifold at x by

$$W^u_{loc}(x, F_0) = \prod_{i=1}^{k-1} \{x_i\} \times \prod_{i=k}^{n} W^u_{loc}(x_i, f). \tag{5.11}$$

In [6], the authors studied hyperbolic properties of the map F_ε for sufficiently small ε using a modification of the classical perturbation theorem for hyperbolic sets.

THEOREM 5.2 (see [6]). *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is an invariant locally maximal hyperbolic set Λ_ε for F_ε.*

5.3. We consider the traveling wave map F_ε. By Theorem 5.2, for sufficiently small ε there exists a locally maximal hyperbolic set Λ_ε for F_ε. Let μ be an invariant mixing measure on Λ_ε. Using the map χ we can push this measure on A_{q_1,q_2} (see Diagram D5). Hence, we obtain the measure $\mu_{q_1,q_2} = \chi_*\mu$.

THEOREM 5.3 (see [6]). *The measure μ_{q_1,q_2} is invariant under time and space translations and is mixing.*

Therefore, the lattice dynamical system displays spatio-temporal chaos.

5.4. In this section we show that the local map corresponding to the FitzHugh-Nagumo equation possesses a locally maximal hyperbolic set. The FitzHugh-Nagumo equation is the only example known to the authors of a partial differential equation whose local map is hyperbolic.

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (x + Ah(x) - \alpha y, \beta x + \gamma y), \tag{5.12}$$

where $h(x) = x(x-\theta)(1-x)$, A, $0 < \gamma, \theta < 1$ and α, β are positive numbers. Let $R = [\theta, 1] \times [r, s]$ be a rectangle, where r and s are positive numbers.

PROPOSITION 5.4. *Assume that numbers α, β, γ and θ satisfy the following conditions:*

$$\gamma < \frac{1}{3}, \quad \frac{1-\theta}{\theta} < \frac{\alpha\beta}{1-\gamma}, \quad \gamma < \frac{1-\theta}{2}. \tag{5.13}$$

Then for all sufficiently large A there exists a rectangle $R = [\theta, 1] \times [r, s]$ such that the intersection $R \cap f(R)$ consists of two connected components R_1 and R_2 (see Figure 3).

Based on Proposition 5.4 one can now develop a "horseshoe-type construction" to obtain an invariant subset Λ for f. One can then use the standard "cone technique" to show that Λ is hyperbolic. More precisely, the following statement holds.

PROPOSITION 5.5. *The set*

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(R) \tag{5.14}$$

is a locally maximal hyperbolic set for f.

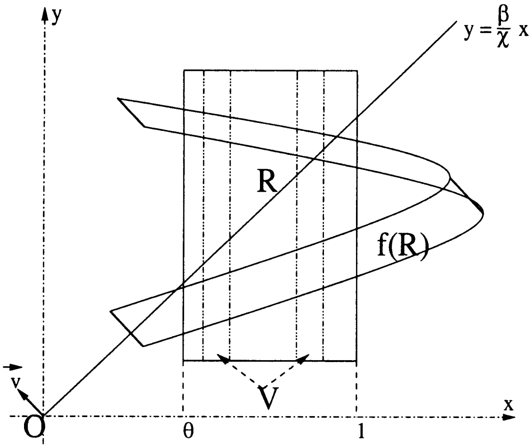

FIG. 3. *The Horseshoe for the Local Map of the FitzHugh-Nagumo Equation.*

6. Dynamics of the traveling wave map: The case of a local map of Morse-Smale type.

6.1. In this section we consider the case when the local map f is a Morse-Smale diffeomorphism. It was studied in [30]. Morse-Smale diffeomorphisms are hyperbolic in a weak sense: there exist finitely many hyperbolic periodic points which determine the behavior of all other orbits (the formal definition is given below). These systems are not chaotic, in fact, the behavior of every single trajectory is well understood: all of them (except hyperbolic periodic points) move towards attracting periodic points as time increases and towards repelling periodic points as time decreases. Our goal in this section is to show that for sufficiently small interactions the traveling wave map F_ε is also of Morse-Smale type and so are space and time translations restricted to the set of traveling wave solutions. Thus, the topological behavior of the CML is completely understood and is not chaotic. We show that the compactification of the local map associated to some partial differential equations considered in Section 1 is Morse-Smale.

We first provide some necessary background.

Let M be a smooth compact Riemannian manifold and $\mathcal{F} : M \to M$ a C^1-diffeomorphism. A point $x \in M$ is called **nonwandering** if for any neighborhood U of x there exists a positive integer n such that $\mathcal{F}^n(U) \cap U \neq \emptyset$. $\Omega(\mathcal{F})$ denotes the set of all nonwandering points of $\mathcal{F}$.

A point $x \in M$ is periodic if $\mathcal{F}^p(x) = x$ for some positive integer p. The set of all periodic points of $\mathcal{F}$ is denoted by $\mathrm{Per}(\mathcal{F})$. A periodic point x is hyperbolic if $d_x\mathcal{F}^p$ is a hyperbolic linear map (i.e., $|\lambda| \neq 1$ for any eigenvalue λ). If x is a hyperbolic periodic point for $\mathcal{F}$ one can construct the local stable and unstable manifolds $W^s_{loc}(x)$ and $W^u_{loc}(x)$ as well as the global stable and unstable manifolds $W^s(x)$ and $W^u(x)$.

A diffeomorphism $\mathcal{F}$ is called a **Morse-Smale diffeomorphism** if it satisfies the following properties:

1. $\Omega(\mathcal{F}) = \mathrm{Per}(\mathcal{F})$;
2. every periodic point is hyperbolic;
3. the global stable and unstable manifolds of periodic points intersect transversally.

We now consider the case when $\mathcal{F}$ is an endomorphism. Given a hyperbolic periodic point x one can construct the local stable and unstable manifolds as well as the global unstable manifold; however, the global stable manifold may not be constructed (see [12]). Therefore, we modify the above definition in the following way.

An endomorphism $\mathcal{F}$ is called **Morse-Smale** if it satisfies the following properties:

1. $\Omega(\mathcal{F}) = \mathrm{Per}(\mathcal{F})$;
2. Every periodic point is hyperbolic;
3. The local stable and global unstable manifolds of periodic points intersect transversally.

We point out that an invertible Morse-Smale endomorphism is a Morse-Smale diffeomorphism and so is its inverse.

6.2. We assume that the following conditions hold:

MS1. The local map f has ∞ as a repelling or attracting fixed point; one can define the **compactification map** $\tilde{f} : S^d \to S^d$ by $\tilde{f} = P \circ f \circ P^{-1}$, where $P : S^d \setminus \{N\} \to \mathbb{R}^d$ is the stereographic projection and N "the North Pole" of S^d;

MS2. The map $\tilde{f}$ is a C^1 Morse-Smale diffeomorphism;

MS3. The map g and all its first order derivatives "vanish at infinity", i.e. for any $\alpha > 0$ there exists a ball $B(0, \mathcal{R}) \subset {\mathbb{R}^d}^n$ centered at O of some radius $\mathcal{R}$ such that $\|g(x)\|, \|(g_{x_i}(x))\| < \alpha$ for every $x \notin B(0, \mathcal{R})$ and $i = 1, \ldots, n$ (here (g_{x_i}) denotes the $d \times d$ matrix of partial derivatives of g with respect to x_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, d$).

Let us remark that the map g corresponding to a space discretization of a partial differential equation is linear (see Section 1). Therefore, it can be changed outside a large ball in $(\mathbb{R}^d)^n$ to satisfy Assumption MS3. Below we will show that the local maps arising from our list of examples satisfy Assumptions MS1 and MS2.

It is easy to check that under Assumptions MS1-MS3 for $\varepsilon \geq 0$, the map F_ε induces a map $\tilde{F}_\varepsilon : (S^d)^n \to (S^d)^n$ where $(S^d)^n$ is the compactification of $(\mathbb{R}^d)^n$ "along each d-dimensional component." Assumption MS3 insures that the map $\tilde{F}_\varepsilon$ is as smooth as the map F_ε and is a small perturbation of $\tilde{F}_0$ in the C^1-topology.

Let us also note that, if instead of the component-wise compactification of $(\mathbb{R}^d)^n$ we used the standard one-point compactification, the induced map $\tilde{F}_0$ need not be continuous.

The following result describes the dynamics of the compactification of the traveling wave map in the case when the local map is a Morse-Smale diffeomorphism. It was proved in [30].

THEOREM 6.1. *Under Assumptions* MS1-MS3 *there exists* $\varepsilon_0 > 0$ *such that for any* $0 \le \varepsilon < \varepsilon_0$ *the map* $\tilde{F}_\varepsilon$ *is a Morse-Smale endomorphism. Moreover,* $\tilde{F}_\varepsilon$ *is a local diffeomorphism for* $0 < \varepsilon < \varepsilon_0$.

Although the traveling map F_ε is defined on $(\mathbb{R}^d)^n$ which is not compact, using Theorem 6.1 and the standard stereographic projection we conclude that F_ε is of Morse-Smale type. More precisely it has finitely many periodic orbits, all hyperbolic, say $Per(F_\varepsilon)$. They constitute the non-wandering set. Furthermore if a trajectory remains bounded then its α- and ω-limit sets is a subset of $Per(F_\varepsilon)$.

As we saw in Section 4 the dynamics of the evolution operator on the set of traveling wave solutions is conjugate to the dynamics of the traveling wave map F_ε. Hence, it is also of Morse-Smale type.

6.3. We illustrate that local maps for some CMLs from our list (see Section 1) are of Morse Smale type.

6.3.1. Kuramoto-Sivashinsky equation. The local map is linear, $f(u) = \nu u$. For ν different from 0 and 1 it is a diffeomorphism of $\mathbb{R}$ which has O the only fixed hyperbolic.

THEOREM 6.2. *The compactification map* $\tilde{f}$ *is a Morse-Smale diffeomorphism of* S^1 *with two hyperbolic fixed points corresponding to* 0 *and* ∞ *(one of them is attracting and the other one is repelling).*

6.3.2. Huxley equation. Recall that the local map is a cubic polynomial, $f(u) = u + \nu u(1-u)(u-\theta)$.

Since $0 < \theta < 1$ the map f has three fixed points 0, θ, and 1. It is easy to check that if $\nu < 0$ is sufficiently small they are hyperbolic (the derivatives are $f'(0) = 1 - \nu\theta > 1$, $f'(\theta) = 1 - \nu(\theta^2 - \theta) < 1$, and $f'(1) = 1 - \nu(1-\theta) > 1$).

It is also easy to see that for every $u \in \mathbb{R}$ the derivative of f is, $f'(u) = -3\nu u^2 + 2\nu(\theta+1)u + (1-\nu\theta)$, strictly positive. Hence, f is strictly increasing and therefore, does not have any other periodic point. It also follows that f is a diffeomorphism of $\mathbb{R}$ with ∞ a repelling point.

Therefore, f satisfies Assumptions MS1–MS2.

THEOREM 6.3. *For* $-1 < \nu < 0$ *and* $0 < \theta < 1$ *the compactification map* $\tilde{f}$ *is a Morse-Smale diffeomorphism of* S^1 *with 4 fixed points.*

Similar arguments apply to show that the Swift-Hohenberg and Ginzburg-Landau equations (whose local maps are also cubic polynomials) have local maps of Morse-Smale type in some range of parameters.

6.3.3. KPP equation. The local map is a quadratic polynomial $f(u) = u + \alpha u(1-u)$. It has two fixed points 0 and 1.

The derivative of f is $f'(u) = 1 + \alpha(1 - 2u)$. It is easy to see that there exists $\alpha_0 > 0$ such that $f'(u) > 0$ for all $0 < \alpha < \alpha_0$ and $u <$

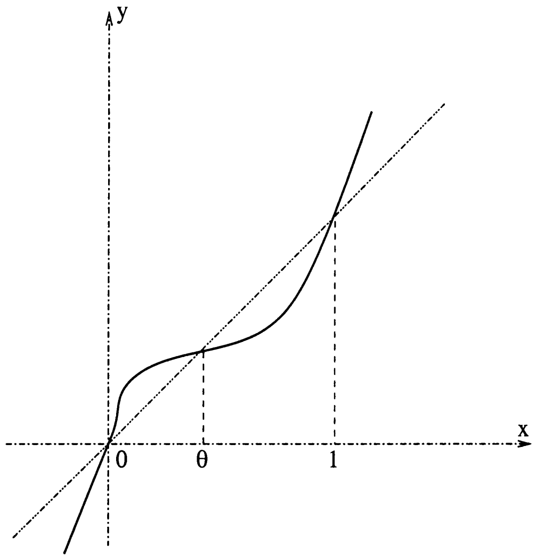

FIG. 4. *The Local Map for Huxley Equation ($\nu < 0$).*

$(1/2)(1 + 1/\alpha_0)$. We change f for $u > A = 1/(2\alpha_0)$ such that $f' > 0$ and the new map f has one more hyperbolic fixed point $p > A$ (see Figure 4).

The new map is a diffeomorphism of $\mathbb{R}$ with 4 fixed points: 0 is repelling ($f'(0) = 1 + \alpha > 1$), 1 attracting ($f'(1) = 1 - \alpha < 1$); p repelling and ∞ attracting.

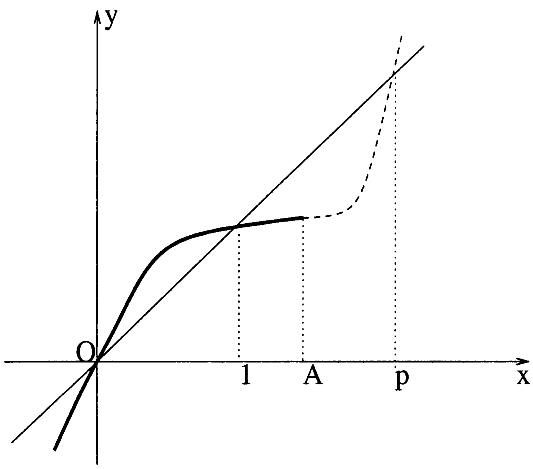

FIG. 5. *The Change of the Local Map for the KPP Equation.*

THEOREM 6.4. *For $0 < \alpha < \alpha_0$ the compactification map $\tilde{f}$ is a Morse-Smale diffeomorphism of S^1 with 4 hyperbolic fixed points.*

Let us point out that the number of fixed points of a Morse-Smale diffeomorphism of S^1 is even, alternating attractors and repellers.

We consider now the KPP equation for sufficiently large values of the parameter. Set $u_0 = (1+\alpha)/\alpha$ (α is fixed). It is clear that $f(0) = 0 = f(u_0)$ and $f(u_0/2) > u_0$ for $\alpha > 1$. Since $f'(u) = 1+\alpha-2\alpha u$ for every $u \in [0, u_0]$ we obtain that $f'(u) = 0$ for $u = u_0/2$, $f'(u) = 1$ for $u_1 = 1/2$, and $f'(u) = -1$ for $u_2 = (2+\alpha)/(2\alpha)$. One also has that $f(u_1) = f(u_2) > u_0$ if $\alpha > 1+\sqrt{5}$.

In this case one can find two intervals $U_1 = [0, u_1]$ and $U_2 = [u_2, u_0]$ such that for every $u \in U = U_1 \cup U_2$ the derivative of f is expanding, i.e. $|f'(u)| \geq \lambda > 1$ for some λ which does not depend on u (see Figure 6).

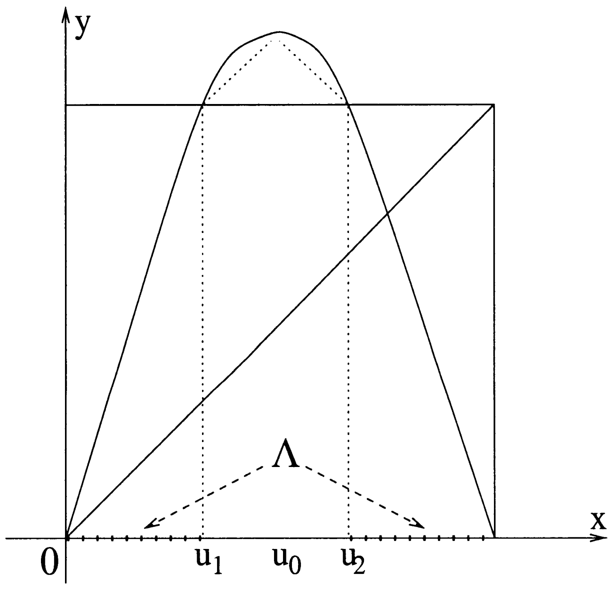

FIG. 6. ***Expanding Local Map for the KPP Equation.***

Define Λ to be the set of all points $u \in I$ whose positive orbit remains in I, i.e. $\Lambda = \bigcap_{i>0} f(I)$. The set Λ is a Cantor-like subset of U on which f is conjugate to the full shift on the space of one-sided infinite sequences. It is well known that this system is mixing. We therefore pose the following conjecture:

Conjecture. For $\alpha > 1+\sqrt{5}$ the map F_ε (for sufficiently small $\varepsilon > 0$) has an invariant mixing measure.

6.3.4. FitzHugh-Nagumo equation. Consider the following vector field:

$$X(x,y) = (dh(x) - ay, bx - cy). \tag{6.1}$$

Let X_t be the flow generated by X. A rectangle R is called invariant under X_t if for every $x \in \partial R$ the vector $X(x)$ points inside R.

PROPOSITION 6.5. *There exists a rectangle $R \subset \mathbb{R}^2$ of the form $R = [-x_0, x_0] \times [-y_0, y_0]$ with $x_0, y_0 > 0$, such that for any $r \geq 1$, the rectangle rR is invariant.*

It follows from Proposition 6.5 that for every $t > 0$ the diffeomorphism X_t admits the compactification map $\tilde{X}_t : S^2 \to S^2$ (see Section 6.2). The latter is a diffeomorphism with the North Pole (corresponding to ∞) as a repelling point.

Denote $\phi = (ab)/(cd)$. One can show that if

$$(\theta - 1)^2 < 4\phi \tag{6.2}$$

then the vector field X has the origin O as the only attracting critical point and if

$$(\theta - 1)^2 > 4\phi \tag{6.3}$$

then there are three critical points O and $P_i(x_i, y_i)$ for $i = 1, 2$, where $x_i = \theta + 1 \pm \sqrt{(\theta-1)^2 - 4\phi}$ and $y_i = (b/c)x_i$. The points O and P_2 are attracting and P_1 is a saddle point.

One can prove that the flow X_t does not have any closed orbits and hence, the nonwandering set consists only of the critical points O, P_1 and P_2 which are all hyperbolic.

One can also show that the stable (unstable) manifolds of these points are transversal. We conclude with the following result:

THEOREM 6.6. *$\tilde{X}_t$ is a gradient-like Morse-Smale flow on the sphere S^2.*

Recall that a flow $\tilde{X}_t$ is a gradient-like Morse-Smale flow if it has no periodic orbits, the set of critical points is finite (and therefore coincides with the set of nonwandering points), all critical points are hyperbolic, and their stable (unstable) manifolds intersect transversally.

For every real number h we define the Euler map $\varphi_h : \mathbb{R}^2 \to \mathbb{R}^2$ generated by the flow X_t by

$$\varphi_h = Id + hX, \tag{6.4}$$

It is easy to see that

$$\varphi_h(u,v) = (u + A\varphi(u) - \alpha v, \beta u + \gamma v), \tag{6.5}$$

and hence, φ_h coincides with the local map for the FitzHugh-Nagumo equation with $A = dh$, $\alpha = ah$, $\beta = bh$ and $\chi = ch$.

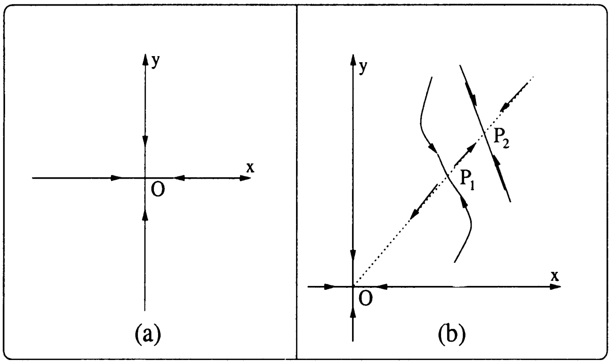

FIG. 7. *The Phase Portraits for the FitzHugh-Nagumo Equation: a) the local map has two fixed points (O is attracting, ∞ is repelling); b) the local map has four fixed points (O and P_2 are attracting, P_1 is a saddle, and ∞ is repelling).*

For h small φ_h is close to the identity map and is a diffeomorphism of $\mathbb{R}^2$. Moreover φ_h maps each rectangle rR into its interior and hence, admits a compactification diffeomorphism $\tilde{\varphi}_h : S^2 \to S^2$ with the North Pole as a repelling fixed point.

Following the proof of the Palis-Smale theorem of structural stability for Morse-Smale systems one can prove the following result

THEOREM 6.7. *There exists $h_0 > 0$ such that for any h, $|h| < h_0$ the map $\tilde{\varphi}_h$ is topologically conjugate to $\tilde{X}_h$.*

THEOREM 6.8. *For sufficiently small A, α, β, and γ the compactification map is a Morse-Smale diffeomorphism of the sphere S^2. It has two fixed points (one attracting and one repelling) if Condition 6.2 holds and four fixed points (two attracting, one saddle, and one repelling) if Condition 6.3 holds. The repelling point corresponds to ∞.*

6.4. Bifurcations. As we saw, the local map for the FitzHugh-Nagumo equation is Morse-Smale in some domain of the parameter space and has a hyperbolic set in some other domain of the parameter space. In fact one can pass from the first domain to the second one by varying parameters A (and perhaps χ) and observe a sequence of bifurcations.

We believe that a "typical" sequence of bifurcations consist of period doubling. On Figure 8 we present the **bifurcation diagram** which was obtained using the program "Dynamics" by J.A. Yorke and H.E. Nusse addapted for Unix/X11 systems by Eric J. Kostelich and Brian R. Hunt (see [38]). We wish to thank Brian Hunt for providing us with the second version of the program.

Explanation to Figure 8. Period doubling bifurcations are clearly seen when $\alpha = 0.1$, $\beta = 0.2$, $\chi = 0.4$ (and hence, $\gamma = 1 - \chi = 0.6$), $\theta = 0.5$ while parameter A varies from 0.5 to 8. The first sequence of

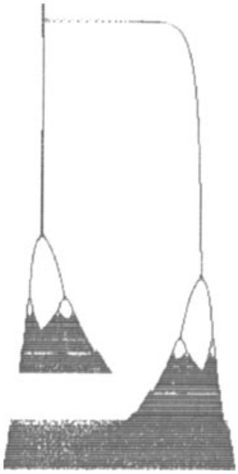

FIG. 8. ***The Bifurcation Diagram for the local map for FitzHugh-Nagumo Equation.***

period doubling bifurcations starts at point O while the second one starts from point P_2 (see Figure 7b).

7. Conclusion. We briefly summarize here some important aspects of our discussion in the paper.

1. If the local map of a CML is hyperbolic in a strong sense (i.e., it possesses a locally maximal hyperbolic set) then so is the traveling wave map F_ε (for sufficiently small ε) and in turn, space and time translations considered on the set of traveling wave solutions. In this case the dynamics of the CML is chaotic on a finite-dimensional smooth submanifold in the infinite-dimensional phase space M_{q_1,q_2} (endowed with the metric $\|\cdot\|_{q_1,q_2}$. An example is the two-dimensional local map for the FitzHugh-Nagumo equation in some range of parameters.

If the local map of a CML is Morse-Smale (i.e., it is hyperbolic in a weak sense) then so are the traveling wave map F_ε (for sufficiently small ε) and space and time translations restricted to the set of traveling wave solutions. In this case the dynamics of the CML is not chaotic and the topological behavior of individual trajectories can be completely described. Examples include one-dimensional local maps for Kuramoto-Sivashinsky equation, Huxley equation, and KPP equation as well as two-dimensional local map for the FitzHugh-Nagumo equation (in some range of parameters).

2. The chaotic behavior in the infinite-dimensional phase space M_{q_1,q_2} associated with traveling wave solutions is not stable in a strong sense with respect to initial data: A small change in the initial condition may cause the corresponding trajectory to diverge from the initial one. However, there

is stability in a weak sense: Every submanifold $A^{m,l}_{q_1,q_2}$ (which consists of traveling wave solutions running with the velocity m/l) possesses a stable everywhere dense separatrix.

The global instability is due to the metric structure of the phase space M_{q_1,q_2} and ultimately is determined by the boundary conditions we have specified (we allow solutions which can grow at an exponential rate as the spatial coordinate tends to infinity). This instability is associated with the "pathological" behavior of the metric $||\cdot||_{q_1,q_2}$ over q_1 and q_2; more precisely, the space M_{q_1,q_2} is nowhere dense in $M_{q_1',q_2'}$ if $q_1 \leq q_1'$ and $q_2 \leq q_2'$ and at least one of these inequalities is strict.

Let us note that "in practice" one can never deal with infinite space and therefore, cannot distinguish between metrics $||\cdot||_{q_1,q_2}$ with different q_1 and q_2. From this point of view the space of solutions which are bounded as the space coordinate tends to infinity (i.e., solutions for which the norm $||u|| = sup_{j\in\mathbb{Z}}|u_j|$) is more "practical." Let us point out that the traveling wave solutions that lie in the hyperbolic set for the evolution operator (assuming that the local map is hyperbolic) are indeed bounded in space. More generally, any subset which is invariant under the traveling wave map and its inverse and is bounded corresponds to a set of traveling wave solutions of the CML which is bounded in space. It is unknown whether the set of **all** traveling wave solutions running with a given velocity forms a smooth submanifold and whether this submanifold is stable in the strong sense.

3. In this paper we considered three special classes of solutions: steady-state solutions, space-homogeneous solutions, and traveling wave solutions. It would be interesting to find other classes of solutions where CMLs may display temporal or/and spatial chaos.

4. The local maps associated with most of the partial differential equations that we have considered in this paper are one-dimensional of Morse-Smale type (except for the KPP equation in some range of the parameter). The only two-dimensional example we have studied is provided by the local map associated with FitzHugh-Nagumo equation. In this case one can find both hyperbolic sets and Morse-Smale type behavior. It is a challenging problem in the area to study hyperbolic (and respectively ergodic) properties of multi-dimensional local maps associated with other well-known partial differential equations or arising in other models.

5. In this paper we considered only CMLs which correspond to partial differential equations of evolution type. Little is known about CMLs which correspond to the wave equation. In particular, it would be of great importance to understand dynamics on the space of traveling wave solutions. It is quite likely that these solutions are stable in the strong sense.

APPENDIX

A. Equilibrium states and mixing property. For the reader's convenience we collect here some basic notions from statistical physics and ergodic theory which were used in this paper. Let $T : X \to X$ be an invertible map and μ a Borel probability measure invariant under T, i.e., $\mu(A) = \mu(T(A))$ for any measurable set $A \subset X$. μ is said to be

1. **ergodic** if any invariant set $A \subset X$ has measure either 0 or 1.
2. **mixing** if for any two measurable sets $A, B \subset X$

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B). \tag{A.1}$$

It is well known that any mixing measure is ergodic.

Let X be a compact metric space, $T : X \to X$ a continuous map, and φ a continuous function on X. An invariant measure $\mu = \mu_\varphi$ is called an equilibrium measure corresponding to φ if

$$\sup_{\nu}(h_\nu(T) + \int \varphi d\nu) = h_{\mu_\varphi}(T) + \int \varphi d\mu_\varphi, \tag{A.2}$$

where $h_\nu(T)$ is the metric entropy of T with respect to ν and the supremum is taken over all T-invariant measures.

THEOREM A.1 (Corollary 20.3.8 in [26]). *Let M be a compact smooth Riemannian manifold, $T : M \to M$ a topologically transitive C^1-diffeomorphism possessing a locally maximal hyperbolic set Λ. Then for any Hölder continuous function $\varphi : \Lambda \to \mathbb{R}$ there exists a unique equilibrium measure μ_φ which is mixing.*

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