TIME RESCALING OF LYAPUNOV EXPONENTS

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Abstract. For systems with zero Lyapunov exponents we introduce and study the notion of scaled Lyapunov exponents which is used to characterize the sub-exponential separation of nearby trajectories. We briefly discuss the abstract theory of such exponents and discuss some examples.

Dedicated to Michael Rabinovich on the occasion of his 75th birthday.

1. Introduction.

Lyapunov exponents are classical characteristics of instability of trajectories and in the presence of nonzero Lyapunov exponents the system is expected to exhibit a certain level of chaotic behavior. This is indeed the case if the system preserves a smooth measure or more generally a Sinai-Ruelle-Bowen (SRB) measure. This is one of the manifestations of the classical non-uniform hyperbolicity theory (see [2]). For system preserving SRB measures the Kolmogorov-Sinai (metric) entropy of the system can be computed using Pesin's entropy formula: the entropy is the mean over the phase space of the system of the sum of positive Lyapunov exponents. In particular, the entropy of the system is positive.

On the other hand, if a measure invariant under the system has all its Lyapunov exponent zero, then by the Margulis-Ruelle inequality, the entropy of the measure is zero. When entropy of the measure is positive it characterizes the complexity of the system (with respect to this measure) but in the case when entropy is zero little if any meaningful information about the complexity can be recovered.

This observation is crucial, since there are many examples of physical systems which exhibit sub-exponential instability of trajectories and hence, have zero Lyapunov exponents with respect to some "natural" invariant measures. Such systems include some models with sequential dynamics studied by M. Rabinovich [9] in connection to his work on dynamics of neural and

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cognitive systems, see [10, 11]. It also includes a model of weak transient chaos considered by V. Afraimovich and A. Neiman (see their paper in this volume).

To study models with sub-exponential instability of trajectories, one may introduce a more appropriate sub-exponential scale in which Lyapunov exponents should be computed. This is a "better adapted" or "internal" scale of the system. The classical notion of Lyapunov exponent is based on the exponential scale and if it happens to be the "internal" scale of the system which has positive entropy, one obtains positive Lyapunov exponents. Otherwise, one should switch to a different scale (e.g., the polynomial scale) with respect to which the Lyapunov exponents and entropy may become positive. This would allow one to evaluate the level of complexity of the system.

Finding an internal scale for a given system or proving that it exists may be difficult if at all possible. However, if such a scale is found one hopes to use the corresponding *scaled Lyapunov exponents* to recover at least some part of non-uniform hyperbolicity theory. In particular, one hopes to establish a version of the Margulis-Ruelle inequality (or in some cases even Pesin's entropy formula) to connect an appropriately rescaled metric entropy with scaled Lyapunov exponents.

In [8] using the general Carathéodory construction as described in [7], the concept of scaled entropy was introduced in both topological and metric settings. While the standard approach to topological entropy defines it as the exponential growth rate of the number of periodic points, the definition of the scaled topological entropy allows asymptotic rates of the general form $e^{\alpha a(n)}$, where $\alpha > 0$ is a parameter and $a(n)$ is a *scaling sequence*. Similar idea was used in [8] in defining scaled metric entropy.

Measures with zero Lyapunov exponents often appear as infinite invariant measures for dynamical systems on compact phase spaces. A classical example is the Manneville-Pomeau map $x \to x + x^{1+\alpha}(\pmod 1)$, where α controls the degree of intermittency at the neutral fixed point. If $\alpha \in (0,1)$ then the systems preserves a finite measure which is absolutely continuous with respect to the Lebesgue measure. However, for $\alpha > 1$ this measure becomes infinite and the corresponding Lyapunov exponents are zero. After the rescaling $t \to t^{\alpha}$ the scaled Lyapunov exponent becomes positive and one recovers the "rescaled" version of Pesin's entropy formula as well, see [3, 4].

The goal of this article is to examine the dependence of Lyapunov exponents on the scale in which they are computed and to outline the abstract theory of scaled Lyapunov exponents. We will do this in the general setting of cocycles over dynamical systems.

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2. Scaled Lyapunov exponents for cocycles

In this section, we describe the general theory of scaled Lyapunov exponents for cocycles over dynamical systems. Consider an invertible measurable transformation f of a measure space X.

2.1. Linear multiplicative cocycles.

Let $f : X \to X$ be an invertible measurable transformation of a measure space X. We call the function $\mathcal{A} : X \times \mathbb{Z} \to GL(d, \mathbb{R})$ a *linear multiplicative cocycle over* f or simply a *cocycle* if it has the following properties:

(1) $\mathcal{A}(x, 0) = Id$ for every $x \in X$ and $\mathcal{A}(x, m+n) = \mathcal{A}(f^n(x), m)\mathcal{A}(x, n)$ for all $m, n \in \mathbb{Z}$;
(2) the function $\mathcal{A}(\cdot, n) : X \to GL(d, \mathbb{R})$ is measurable for each $n \in \mathbb{Z}$.

Every cocycle is generated by a measurable function $A : X \to GL(d, \mathbb{R})$, which is called the *generator*. In fact, every such function determines a cocycle by the formula

$$\mathcal{A}(x, n) = \begin{cases} A(f^{n-1}(x)) \cdots A(f(x))A(x) & \text{if } n > 0, \\ Id & \text{if } m = 0, \\ A(f^n(x))^{-1} \cdots A(f^{-2}(x))^{-1} A(f^{-1}x)^{-1} & \text{if } n < 0. \end{cases}$$

On the other hand, a cocycle $\mathcal{A}$ is generated by the matrix function $A = \mathcal{A}(\cdot, 1)$.

A simpler way to describe a cocycle $\mathcal{A}$ over f is by considering a *linear extension* $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ of f that is induced by the cocycle (for simplicity of presentation we only consider the trivial bundle in this paper). It is given by

$$F(x, v) = (f(x), A(x)v).$$

If $\pi : X \times \mathbb{R}^d \to X$ is the projection defined by $\pi(x, v) = x$, then it is easy to see that $\pi \circ F = f \circ \pi$.

If f is a differentiable map of a compact phase space M, then f generates a *differential* cocycle $\mathcal{A}(x, n)$ over f whose generator is $A(x) = df_x$ and it acts on the tangent bundle TM of M. Therefore, the results in this and next sections are applicable to smooth maps.

2.2. **Definition of scaled Lyapunov exponents.** Let $\mathcal{A}$ be a cocycle over an invertible measurable transformation f of a measure space X. Using the notion of the standard Lyapunov exponent (see [2] for details), we introduce the notion of the scaled Lyapunov exponent for the cocycle.

Given $x \in X$, we call a sequence of positive numbers $\mathbf{a} = \{a(x,n)\}_{n\geq 1}$ a *scaled sequence* if

(1) it is monotonically increasing to infinity, e.g., $a(n) = n^\alpha$, $\log n$, etc.;
(2) for each n the function $a(x,n)$ is Borel;
(3) $a(f(x),n) = a(x,n+1)$, in other words the function $a(x,n)$ depends on the entire trajectory of the point x.

Given a point $x \in X$, a scaled sequence $\mathbf{a} = \{a(x,n)\}$ and a vector $v \in \mathbb{R}^d$, we call the following quantity

$$\chi(x,v,\mathbf{a}) = \limsup_{n\to+\infty} \frac{1}{a(x,n)} \log \|\mathcal{A}(x,n)(v)\|$$

the *scaled Lyapunov exponent* of (x,v) (with respect to the scaled sequence $\mathbf{a}$ and the cocycle $\mathcal{A}$). With the convention that $\log 0 = -\infty$ this extends the definition of the *standard Lyapunov exponent* corresponding to the scaled sequence $a(n) = n$ for each n.

2.3. **Choices of scaled sequences.** The above definition of the scaled Lyapunov exponent allows any scaled sequence $\mathbf{a} = \{a(x,n)\}_{n\geq 1}$, which grows slower than n. Depending on the choice of $\mathbf{a}$ the value of the scaled Lyapunov exponent can be positive, negative or zero. It can also be $\pm\infty$.

Given a point $x \in X$, consider the sequence of positive numbers

$$\mathbf{b} = \{b(x,n) = \max_{0\leq k\leq n} |\log \|\mathcal{A}(x,k)\|\,|\}.$$

This sequence is non-decreasing and if it is bounded, then for any scaled sequence $\mathbf{a}$ the corresponding values of the Lyapunov exponent are all zero. However, if the sequence $\mathbf{b}$ is unbounded, then the corresponding values of the Lyapunov exponent are all finite (except for $v = 0$).

Consider the collection of vectors for which $\chi(x,v,\mathbf{b}) = 0$. By the properties of the scaled Lyapunov exponents described in the next section, this collection is a linear subspace of $\mathbb{R}^d$, which we denote by $\mathbb{R}^{d_1}$. Consider the "restricted" cocycle $\mathcal{A}_1(x,n)$ with values in $GL(d_1,\mathbb{R})$. Repeating the above argument we find the non-decreasing sequence of positive numbers $\mathbf{b_1}$ and if this sequence is unbounded then we can use it to rescale the Lyapunov exponents for vectors $v \in \mathbb{R}^{d_1}$. The above procedure produces a filtration

$$\mathbb{R}^d = \mathbb{R}^{d_0} \supset \mathbb{R}^{d_1} \supset \cdots \supset \mathbb{R}^{d_k},$$

a collection of scaled sequences $\mathbf{b_i}$ and the corresponding collection of scaled Lyapunov exponents $\chi(x, v, \mathbf{b_i})$, $i = 1, \dots, k-1$ such that $\chi(x, v, \mathbf{b_i}) \neq 0$ for every $v \in \mathbb{R}^{d_{i-1}} \setminus \mathbb{R}^{d_i}$ and $i = 1, \dots, k-1$. If the sequence $\mathbf{b_k}$ is unbounded, then $\mathbb{R}^{d_k} = 0$.

3. Basic properties of scaled Lyapunov exponents

The function χ has the following basic properties of a Lyapunov exponent, see [2, Chapter 2] or [1, Proposition 1] for detailed proofs.

Proposition 3.1. *For each $x \in X$, $v, w \in \mathbb{R}^d$ and $c \in \mathbb{R} \setminus \{0\}$,*

(1) $\chi(x, cv, \mathbf{a}) = \chi(x, v, \mathbf{a})$*;*
(2) $\chi(x, 0, \mathbf{a}) = -\infty$*;*
(3) $\chi(x, v + w, \mathbf{a}) \leq \max\{\chi(x, v, \mathbf{a}), \chi(x, w, \mathbf{a})\}$*.*

It follows from the abstract theory of Lyapunov exponents (see [2, Theorem 2.1]) that:

(1) $\chi(x, v + w, \mathbf{a}) = \max\{\chi(x, v, \mathbf{a}), \chi(x, w, \mathbf{a})\}$ for any $v, w \in \mathbb{R}^d$ whenever $\chi(x, v, \mathbf{a}) \neq \chi(x, w, \mathbf{a})$;
(2) if for some nonzero vectors $v_1, \dots, v_m \in \mathbb{R}^d$, the numbers $\chi(x, v_1, \mathbf{a})$, $\dots$, $\chi(x, v_m, \mathbf{a})$ are distinct, then these vectors are linearly independent;
(3) the function $\chi(x, \cdot, \mathbf{a})$ attains only finitely many values on $\mathbb{R}^d \setminus \{0\}$, which we denote by $\chi_1(x, \mathbf{a}) < \cdots < \chi_{s(x,\mathbf{a})}(x, \mathbf{a})$, where $s(x, \mathbf{a}) \leq d$; note that, in general, χ_1 may be $-\infty$ and $\chi_{s(x,\mathbf{a})}$ may be $+\infty$.

Further we denote by $\mathcal{V}_{\mathbf{a}}(x)$ the *filtration* of $\mathbb{R}^d$ associated to $\chi(x, \cdot, \mathbf{a})$:

$$\{0\} = V_{\mathbf{a}}^0(x) \subsetneqq V_{\mathbf{a}}^1(x) \subsetneqq \cdots \subsetneqq V_{\mathbf{a}}^{s(x,\mathbf{a})}(x) = \mathbb{R}^d,$$

where $V_{\mathbf{a}}^i(x) = \{v \in \mathbb{R}^d \colon \chi(x, v, \mathbf{a}) \leq \chi_i(x, \mathbf{a})\}$ for $i = 1, \dots, s(x, \mathbf{a})$. It is easy to see that

$$\chi_i(x, v, \mathbf{a}) = \chi_i(x, \mathbf{a}) \text{ for all } v \in V_{\mathbf{a}}^i(x) \setminus V_{\mathbf{a}}^{i-1}(x).$$

The number $k_i(x, \mathbf{a}) = \dim V_{\mathbf{a}}^i(x) - \dim V_{\mathbf{a}}^{i-1}(x)$ is the *multiplicity* of the value $\chi_i(x, \mathbf{a})$. We have that

$$\sum_{i=1}^{s(x,\mathbf{a})} k_i(x, \mathbf{a}) = d.$$

The collection of pairs

$$\mathrm{Sp}\chi_{\mathbf{a}}(x) = \big\{(\chi_i(x,\mathbf{a}), k_i(x,\mathbf{a}))\colon 1 \le i \le s(x,\mathbf{a})\big\}$$

forms the *Lyapunov spectrum* of the scaled Lyapunov exponent χ at the point $x \in X$ with respect to the scaled sequence $\mathbf{a} = \{a(x,n)\}$.

Definition 3.1. *A basis $\mathbf{v} = (v_1, \cdots, v_d)$ of $\mathbb{R}^d$ is said to be* subordinate *to the filtration $\mathcal{V}_{\mathbf{a}}(x)$ if for every $1 \le i \le s(x,\mathbf{a})$ there exists a basis of $V^i_{\mathbf{a}}(x)$ composed of d_i vectors from $(v_1, \dots, v_d)$. A subordinate basis $\mathbf{v}$ is* ordered *if for every $1 \le i \le s(x,\mathbf{a})$ the vectors $v_1, \dots, v_{d_i}$ form a basis of $V^i_{\mathbf{a}}(x)$.*

Using the same arguments in [2, Chapter 12], we can find that there always exists a subordinate basis for a filtration.

Proposition 3.2. *Given a filtration $\mathcal{V}_{\mathbf{a}}(x)$ of $\mathbb{R}^d$, there exists a basis $\mathbf{w} = (w_1, \dots, w_d)$ of $\mathbb{R}^d$ such that*

$$\inf\Big\{\sum_{j=1}^{d} \chi(x, v_j, \mathbf{a})\colon \mathbf{v} = (v_1, \dots, v_d) \text{ is a basis of } \mathbb{R}^d\Big\} = \sum_{j=1}^{d} \chi(x, w_j, \mathbf{a}).$$

4. The Lyapunov and Perron regularity coefficients

Consider the dual matrix $B(x) = (A(x)^*)^{-1}$ at each point $x \in X$. Given a scaled sequence $\mathbf{a} = \{a(x,n)\}_{n\ge1}$ and a point $(x, v^*) \in X \times \mathbb{R}^d$, the *dual scaled Lyapunov exponent* is given by the formula:

$$\widetilde{\chi}(x, v^*, \mathbf{a}) = \limsup_{m\to+\infty} \frac{1}{a(x,m)} \log \|\mathcal{B}(x,m)v^*\|$$

where $\mathcal{B}(x,m) = B(f^{m-1}(x)) \cdots B(f(x))B(x)$ for $m > 0$. In fact, choose dual bases $(v_1, \cdots, v_d)$ and $(v_1^*, \cdots, v_d^*)$, i.e., $\langle v_i, v_j^*\rangle = \delta_{ij}$ for each i and j (here δ_{ij} is the Kronecker symbol), and set $v_{i,m} = \mathcal{A}(x,m)v_i$ and $v^*_{i,m} = \mathcal{B}(x,m)v_i^*$. For each $m \in \mathbb{N}$ we have

$$\langle v_{i,m}, v^*_{i,m}\rangle = \langle \mathcal{A}(x,m)v_i, (\mathcal{A}(x,m)^*)^{-1}v_i^*\rangle = 1.$$

Hence, $1 \le \|\mathcal{A}(x,m)v_i\| \cdot \|\mathcal{B}(x,m)v_i^*\|$ and the exponents χ and $\widetilde{\chi}$ are *dual* at the point x, i.e., $\chi(x, v_i, \mathbf{a}) + \widetilde{\chi}(x, v_i^*, \mathbf{a}) \ge 0$ for each $1 \le i \le d$.

Arguing as above one can show that for each $v^* \in \mathbb{R}^d$, the Lyapunov exponent $\widetilde{\chi}$ can only attain finitely many values on $\mathbb{R}^d \setminus \{0\}$. We denote them by $\widetilde{\chi}_{r(x,\mathbf{a})}(x,\mathbf{a}) < \cdots < \widetilde{\chi}_1(x,\mathbf{a})$ for some integer $r(x,\mathbf{a}) \le d$. Let $\widetilde{\mathcal{V}}_{\mathbf{a}}(x) = \{\widetilde{V}^i_{\mathbf{a}}(x) : i = 1, \dots, r(x,\mathbf{a})\}$ be the filtration associated to $\widetilde{\chi}$. Note that, in general, $\widetilde{\chi}_{r(x,\mathbf{a})}$ may be $-\infty$ and $\widetilde{\chi}_1$ may be $+\infty$ and therefore, from now on we assume that the sums $\chi_1 + \widetilde{\chi}_1$ and $\chi_{s(x,\mathbf{a})} + \widetilde{\chi}_{r(x,\mathbf{a})}$ are well defined, that is $|\chi_1|$ and $\widetilde{\chi}_1$ are not both $+\infty$ and similar for $\chi_{s(x,\mathbf{a})}$ and $|\widetilde{\chi}_{r(x,\mathbf{a})}|$.

We call the quantity

$$\gamma_{\mathbf{a}}(x,\chi,\widetilde{\chi}) = \min\max\Big\{\chi(x,v_i,\mathbf{a}) + \widetilde{\chi}(x,v_i^*,\mathbf{a})\colon 1\le i\le d\Big\}$$

the *regularity coefficient* of the pair of scaled Lyapunov exponents χ and $\widetilde{\chi}$ at the point x (with respect to the scaled sequence $\mathbf{a}=\{a(x,n)\}$). Here the minimum is taken over all pairs of dual bases $(v_1,\cdots,v_d)$ and $(v_1^*,\cdots,v_d^*)$ of $\mathbb{R}^d$. We say that a point $x\in X$ is *regular* (with respect to the pair of scaled Lyapunov exponents $(\chi,\widetilde{\chi})$) if $\gamma_{\mathbf{a}}(x,\chi,\widetilde{\chi})=0$.

Now we let

$$\chi_1'(x,\mathbf{a})\le\cdots\le\chi_d'(x,\mathbf{a}) \text{ and } \widetilde{\chi}_1'(x,\mathbf{a})\ge\cdots\ge\widetilde{\chi}_d'(x,\mathbf{a})$$

be respectively the values of χ and $\widetilde{\chi}$ at the point x counted with their multiplicities. Define the *Perron coefficient* of the pair χ and $\widetilde{\chi}$ at $x\in X$ (with respect to the scaled sequence $\mathbf{a}=\{a(x,n)\}$) by

$$\pi_{\mathbf{a}}(x,\chi,\widetilde{\chi}) = \max\{\chi_i'(x,\mathbf{a})+\widetilde{\chi}_i'(x,\mathbf{a})\colon 1\le i\le d\}.$$

The following theorem can be proven by the same arguments as in the proof of Theorem 2.8 in [2].

Theorem 4.1. *For a point $x\in X$, if $\chi_1+\widetilde{\chi}_1$ and $\chi_{s(x,\mathbf{a})}+\widetilde{\chi}_{r(x,\mathbf{a})}$ are well defined, then*

$$0\le\pi_{\mathbf{a}}(x,\chi,\widetilde{\chi})\le\gamma_{\mathbf{a}}(x,\chi,\widetilde{\chi})\le d\pi_{\mathbf{a}}(x,\chi,\widetilde{\chi}).$$

It follows that a point $x\in X$ is regular if and only if $\pi_{\mathbf{a}}(x,\chi,\widetilde{\chi})=0$ and also if and only if $\chi_i'(x,\mathbf{a})=-\widetilde{\chi}_i'(x,\mathbf{a})$ for $i=1,\cdots,d$.

Theorem 4.2. *If a point $x\in X$ is regular, then the filtrations $\mathcal{V}_{\mathbf{a}}(x)$ and $\widetilde{\mathcal{V}}_{\mathbf{a}}(x)$ are orthogonal, that is, $s(x,\mathbf{a})=r(x,\mathbf{a}):=s$, $\dim V_{\mathbf{a}}^i(x)+\dim\widetilde{V}_{\mathbf{a}}^{s-i}(x)=d$ and $\langle v,v^*\rangle=0$ for every $v\in V_{\mathbf{a}}^i(x)$ and $v^*\in\widetilde{V}_{\mathbf{a}}^{s-i}(x)$.*

Reversing the time, we can introduce the scaled Lyapunov exponents for negative time. The above result provides a basis to study Lyapunov-Perron regularity for scaled Lyapunov exponents. The ultimate goal is to find out whether various regularity criteria that hold in the case of standard scaling can be extended to general scaled sequences and to what extent the Multiplicative Ergodic theorem may hold for scaled Lyapunov exponents.

5. Examples

In this section we present two examples that illustrate that there are systems for which the Lyapunov exponents can be rescaled to achieve non-zero values but that in general this should not be expected.

5.1. **Existence of scaled Lyapunov exponents.** Consider the elliptic matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ whose only eigenvalue is 1. Hence, the two Lyapunov exponents for the constant cocycle generated by A are zero. In addition, the vector $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the eigenvector for A, so that $\|A^n v\| = 1$ for all integers n where $\|\cdot\|$ is the Euclidean norm. Consequently, the scaled Lyapunov exponent, $\chi(v, \mathbf{a})$ is zero for any scaling sequence $\mathbf{a}$. On the other hand, taking the vector $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we observe that $A^n w = \begin{bmatrix} n \\ 1 \end{bmatrix}$ so that the norm $||A^n w||$ grows linearly with $|n|$. Choosing $a(n) = \log n$ we obtain that

$$\chi(w, \mathbf{a}) = \lim_{n\to+\infty} \frac{1}{a(n)} \log \|\mathcal{A}(n)(w)\| = 1,$$

and the limit exists. Similar observation can be made for any non-diagonalizable elliptic matrix. In fact, the following is true:

Proposition 5.1 ([6])**.** *Suppose all eigenvalues of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ have absolute value one. Then there exists an invariant subspace $C = C(A) \subset \mathbb{R}^n$ and a norm in $\mathbb{R}^n$ such that A acts in C as an isometry and for every vector $v \in \mathbb{R}^n \setminus C$ the norm $\|A^n v\|$ grows polynomially as $|n| \to \infty$.*

To see how nonzero scaled Lyapunov exponents can be obtained consider a diffeomorphism $f : S^2 \to S^2$ of the unit sphere in $\mathbb{R}^3$ which fixes the South and North poles and moves every other point along the meridian from the North pole toward the South pole. In the spherical coordinates we may write $f(\theta, \varphi, r) = (f_1(\theta, \varphi), \varphi, r)$, where for every fixed angle $\varphi \in [-\pi, \pi)$, the function $g_\varphi(\theta) := f_1(\theta, \varphi)$ satisfies: (1) $g_\varphi(\frac{\pi}{2}) = \frac{\pi}{2}$ and $g_\varphi(-\frac{\pi}{2}) = -\frac{\pi}{2}$; (2) $g_\varphi(\theta) < \theta$ for all $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

We have that $f^n(\theta, \varphi, r) = (f_n(\theta, \varphi), \varphi, r)$, where $f_n(\theta, \varphi) = g^n_\varphi(\theta)$. Consequently,

$$df^n = \begin{bmatrix} \frac{dg^n_\varphi}{d\theta} & \frac{df_n}{d\varphi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.1}$$

Observe that:

- the direction spanned by the vector $v^s = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is df-invariant,

- the subspace spanned by $w^s = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not df-invariant,
- the vector $v^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is df-invariant, that is $df^n v^0 = v^0$ for all n.

Consider the special case, when $g_\varphi(\theta)$ has the following form on the interval $[-\frac{\pi}{2}, -\frac{\pi}{2} + \delta]$ for some small $\delta > 0$:

$$g_\varphi(\theta) = \begin{cases} \left(1 + (\theta + \frac{\pi}{2})^{-\frac{1}{\alpha}}\right)^{-\alpha} - \frac{\pi}{2} & \text{for } \theta \neq -\frac{\pi}{2} \\ g_\varphi(-\frac{\pi}{2}) = -\frac{\pi}{2} & \text{for } \theta = -\frac{\pi}{2}, \end{cases}$$

where $\alpha = \alpha(\varphi) > 0$ is a smooth function such that $\alpha'(\varphi) > 0$.

One can see that $g'_\varphi(-\frac{\pi}{2}) = 1$ and hence all Lyapunov exponents (in the standard scale) for all points (except maybe for the North pole) are zero. On the other hand, for $-\frac{\pi}{2} < \theta < -\frac{\pi}{2} + \delta$ we can see that,

$$g_\varphi^n(\theta) = \left(n + \left(\theta + \frac{\pi}{2}\right)^{-\frac{1}{\alpha}}\right)^{-\alpha} - \frac{\pi}{2}.$$

Having an explicit formula for $g_\varphi^n(\theta)$ allows us to compute $\frac{dg_\varphi^n}{d\theta}$ and $\frac{df_n}{d\varphi}$ in (5.1) and conclude that:

- The norm $||df^n v^s||$ decays polynomially with exponent $-(1 + \alpha(\varphi))$ as n grows to infinity.
- The norm $||df^n w^s||$ decays polynomially with exponent $-\alpha(\varphi)$ as n grows to infinity.
- The angle $\angle(df^n(w^s), w^s)$ goes to zero as n grows to infinity.

Consequently taking the scale sequence $\mathbf{a} = \log n$, we obtain the following values of the corresponding scaled Lyapunov exponents:

- $\chi(x, v^s, \mathbf{a}) = -(1 + \alpha(\varphi(x)))$
- $\chi(x, w^s, \mathbf{a}) = -\alpha(\varphi(x)))$
- $\chi(x, v^0, \mathbf{a}) = 0$.

We therefore obtained two distinct non-zero values of the scaled Lyapunov exponents that vary with x.

5.2. **Non-existence of the scaled limit.** In general, one cannot expect the existence of the limit in the definition of scaled Lyapunov exponents. To see this consider a diffeomorphism f for which there exists an invariant family of one dimensional subspaces $E(x)$. Denoting by v_x a unit vector in $E(x)$ we have that

$$\log ||df^n(x)(v_x)|| = \sum_{k=0}^{n-1} \log ||df(f^k(x))(v_{f^k(x)})||.$$

Considering a function $\varphi(x) := \log ||df(x)(v_x)||$ we can rewrite the above sum as $\sum_{k=0}^{n-1} \varphi \circ f^k(x)$. Assume that the standard Lyapunov exponent, $\chi(x, v_x)$ is zero almost everywhere with respect to some ergodic measure μ. It means that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k = 0 \ \mu - a.e. \tag{5.2}$$

In that case one cannot obtain a nonzero finite limit by rescaling as the following result shows.

Theorem 5.2 ([5],[12]). *Let $(X, \mathfrak{B}, \mu)$ be a probability space, $f : X \to X$ a measure preserving ergodic transformation, and let $\varphi \in L^1(X, \mathfrak{B}, \mu)$ be such that* (5.2) *holds. If $g \in L^\infty(X, \mathfrak{B}, \mu)$ is such that for some scaled sequence $\{a(x, n)\}_{n\geq 1}$*

$$\frac{1}{a(x,n)} \sum_{m=0}^{n-1} (\varphi \circ f^m)(x) \to g(x) \ a.e.$$

as $n \to \infty$, then $g = 0$ a.e.

This result is an immediate corollary of the following stronger and more general statement.

Theorem 5.3 ([5],[12]). *Let $(X, \mathfrak{B}, \mu)$ be a probability space, $f : X \to X$ a measure preserving ergodic transformation, and let $\varphi \in L^1(X, \mathfrak{B}, \mu)$ be such that* (5.2) *holds. Then for any scaled sequence $\{a(x, n)\}_{n\geq 1}$ and almost every $x \in X$,*

$$\liminf_{n\to\infty} \frac{1}{a(x,n)} \sum_{k=0}^{n-1} \left(\varphi \circ f^k\right)(x) \leq 0$$

while

$$\limsup_{n\to\infty} \frac{1}{a(x,n)} \sum_{k=0}^{n-1} \left(\varphi \circ f^k\right)(x) \geq 0.$$

We finish this section with a positive result on shift spaces which can be easily extended to hyperbolic diffeomorphisms using their symbolic representations. Under some additional assumptions on the shift space and the function φ one can construct a set of positive Hausdorff dimension $\dim_H$ on which the scaled sum converges to an arbitrary real number a.

Theorem 5.4 ([12]). *Let (X,σ) be the full shift on the space of double sided infinite sequences on a finite alphabet. Let also φ be a Hölder continuous potential not cohomologus to a constant and such that* (5.2) *holds for $f = \sigma$. Finally, let μ be the unique equilibrium state for φ, then the Hausdorff dimension*

$$\dim_H \left\{ x \in X | \lim_{n\to\infty} \frac{1}{b(n)} \sum_{m=1}^{n-1} \varphi(\sigma^m(x)) = a \right\} \geq \dim_H \mu$$

for any $a \in \mathbb{R}$ and for any invertible, strictly increasing, continuous, positive function $b(R)$ satisfying:

(1) $\lim_{R\to\infty} b(R) = \infty$*;*
(2) $\lim_{R\to\infty} \frac{b(R)}{R} = 0$*;*
(3) $\alpha_n/\beta_n \to 1$ *implies that* $b(\alpha_n)/b(\beta_n) \to 1$ *for any two sequences* $\{\alpha_n\}, \{\beta_n\} \subset \mathbb{N}$.

References

[1] L. Barreira, C. Valls, *Growth rates and nonuniform hyperbolicity*, Discrete and Continuous Dynamical Systems, **22(3)**, (2008), 509-528.

[2] L. Barreira, Y. Pesin, *Introduction to smooth ergodic theory*, GSM, **148**, AMS, Providence, Rhode Island, 2013.

[3] N. Corabel, E. Barkai, *Pesin-type identity for intermittent dynamics with zero Lyapunov exponent*, Physical Review Letters, 102 (2009) 050601.

[4] L. M. Gaggero-Sager, E. R. Pujals and O. Sotolongo-Costa, *Infinite ergodic theory and non-extensive entropies*, Braz. J. Phys., 41 (2011), 297–303.

[5] G. Halász, *Remarks on the remainder in Birkhoff's ergodic theorem*, Acta Math. Acad. Sci. Hungar. 28 (1976), no. 3-4, 389-395.

[6] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, 54, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1995.

[7] Y. Pesin, *Dimension theory in dynamical systems*, Contemporary Views and Applications, University of Chicago Press, Chicago, 1997.

[8] Y. Pesin, Y. Zhao, *Scaled entropy for dynamical systems*, J. Stat. Phys., **158**, (2015), 447-475.

[9] M. Rabinovich, A. Volkovskii, P. Lecanda, R. Huerta, H. Abarbanel, *Dynamical encoding by networks of competing neuron groups: winnerless competition*, Physical review letters **87**, 2001.

[10] M. Rabinovich, A. Simmons, P. Varona, *Dynamical bridge between brain and mind. Trends in cognitive sciences*, **19** (2015) 453461.

[11] M. Rabinovich, I. Tristan, P. Varona, *Hierarchical nonlinear dynamics of human attention*, Neuroscience and Biobehavioral Reviews **55** (2015) 1835.

[12] A. Zelerowicz, *PSU preprint*.

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