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Scaled Entropy for Dynamical Systems

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Abstract In order to characterize the complexity of a system with zero entropy we introduce the notions of scaled topological and metric entropies. We allow asymptotic rates of the general form $e^{\alpha a(n)}$ determined by an arbitrary monotonically increasing "scaling" sequence $a(n)$. This covers the standard case of exponential scale corresponding to $a(n) = n$ as well as the cases of zero and infinite entropy. We describe some basic properties of the scaled entropy including the inverse variational principle for the scaled metric entropy. Furthermore, we present some examples from symbolic and smooth dynamics that illustrate that systems with zero entropy may still exhibit various levels of complexity.

1 Introduction

Metric and topological entropies are among the main invariants of dynamics. While the former measures the average amount of information and complexity in the system, the latter characterizes the exponential growth rate of the number of periodic points. Furthermore, if the topological entropy is positive, then due to the variational principle, there exists an invariant measure whose metric entropy is positive too. By the Margulis–Ruelle inequality one concludes that some values of the Lyapunov exponents must be positive thus indicating on the presence of a certain level of chaotic behavior in the system. However, if entropy is zero, little if any meaningful information about the system can be recovered.

The main point of this paper is to observe that the value of the entropy (both metric and topological) depends on the scale in which it is computed and thus the choice of the scale can be crucial. In fact, a system may have its own "internal" scale, which should be used in

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computing entropy as well as some other "scale sensitive" characteristics of the dynamics such as Lyapunov exponents, dimension of invariant sets and measures, etc. The classical notion of entropy is based on the standard exponential scale and if it happens to be the "internal" scale of the system, then one obtains the "correct" value of the entropy. This is the case when the computed value of the entropy is positive and finite (see Theorems 2.8, 2.10, 3.10 and 3.11). Otherwise, one may switch to a different scale (e.g., the polynomial scale) in which the scaled entropy may be positive and/or finite. This would allow one to recover some information and evaluate the level of complexity in the system.

We stress however, that there are systems, which inherently have zero entropy—just think of the identity map (or more generally, of an isometry) whose entropy is zero regardless of the scale. We also emphasize that computing entropy in the standard exponential scale may be substantially simpler than determining the "internal" scale of the system and computing the entropy in this scale. It is worth mentioning that some principle results about entropy such as the Shannon–McMillan–Breiman theorem and related Brin–Katok theorem for (local) metric entropy that hold true for the standard scaled sequence may fail for other scaled sequences (see Example 4.7). Describing those classes of zero entropy systems (with respect to the standard scale) for which these results hold for other scaled sequences is an interesting open problem in the area.

Perhaps, the best way to see how entropy depends on the scale is to define entropy via the Charathéodory-like construction described in [18]. It reveals the "dimension" nature of entropy thus allowing different scales. More precisely, we introduce the notions of *scaled* topological and metric entropies by allowing asymptotic rates of the form $e^{\alpha a(n)}$ where $\alpha > 0$ is a parameter and $a(n)$ is a *scaling sequence*. The classical case corresponds to $a(n) = n$ but some other particular scaling sequences have been used. In particular, the polynomial scale $a(n) = n^s$ was used in [3,5–8,11,14] to compute the entropy dimension in some examples which follow Milnor's idea [17] and the logarithmic scale $a(n) = \log n$ was used in [12,13] to compute the topological entropy in some cases. We point out that our approach to the notion of scaled entropy is quite general and include all the above cases.

In Sect. 2 we introduce the notion of scaled topological entropy and study its basic properties. We also describe how the entropy depends on the scale and in particular, how to determine the "internal" scale of the system. In Sect. 3 we introduce the notion of scaled metric entropy and discuss its basic properties. Finally, in Sect. 4 we present some examples that illustrate the importance of scaling in computing the entropy as well as some new phenomena associated with scaled entropies. The last Sect. 5 contains proofs all the main results of the paper.

2 Scaled Topological Entropy

In this section we introduce the definitions of *scaled topological entropy*, *lower and upper scaled topological entropies* on an arbitrary subset, and study the basic properties of these new defined entropies. From Sects. 2.1 to 2.4, we assume that X is a compact (Hausdorff) topological space and $T\colon X \to X$ is a continuous transformation.

2.1 Definition of Scaled Topological Entropy

We follow the approach described in [18]. Let $\mathcal{U}$ be an open cover of X. Denote by $\mathcal{W}_m(\mathcal{U})$ the set of strings $\mathbf{U} = (U_{i_0}, U_{i_1}, \ldots, U_{i_{m-1}})$ of length $m(\mathbf{U}) = m$ with $U_{i_j} \in \mathcal{U}$ and by

$$\mathcal{W}(\mathcal{U}) = \bigcup_{m\geq 1} \mathcal{W}_m(\mathcal{U}).$$

For $\mathbf{U} \in \mathcal{W}_m(\mathcal{U})$ define

$$X(\mathbf{U}) = U_{i_0} \cap T^{-1}U_{i_1} \cap \cdots \cap T^{-(m-1)}U_{i_{m-1}}.$$

Let $Z \subseteq X$ be a subset of X, which need not be compact or T-invariant. We say that a collection of strings $\Gamma \subseteq \mathcal{W}(\mathcal{U})$ covers Z if $Z \subseteq \bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U})$.

We call a sequence of positive numbers $\mathbf{a} = \{a(n)\}_{n\geq 1}$ a *scaled sequence* if it is monotonically increasing to infinity.

Given a subset $Z \subset X$, $\alpha \in \mathbb{R}$ and a scaled sequence $\mathbf{a} = \{a(n)\}_{n\geq 1}$, let

$$M(Z, \alpha, N, \mathcal{U}, \mathbf{a}) = \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp\bigl(-\alpha a(m(\mathbf{U}))\bigr), \tag{2.1}$$

where the infimum is taken over all covers Γ of Z with $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \Gamma$. It is easy to see that $M(Z, \alpha, N, \mathcal{U}, \mathbf{a})$ is monotone in N and we let $m(Z, \alpha, \mathcal{U}, \mathbf{a}) = \lim_{N\to\infty} M(Z, \alpha, N, \mathcal{U}, \mathbf{a})$. It is easy to show that there is a jump-up value

$$\begin{aligned} E_Z(T, \mathcal{U}, \mathbf{a}) &= \inf\{\alpha : m(Z, \alpha, \mathcal{U}, \mathbf{a}) = 0\} \\ &= \sup\{\alpha : m(Z, \alpha, \mathcal{U}, \mathbf{a}) = +\infty\}. \end{aligned}$$

Definition 2.1 We call the quantity

$$E_Z(T, \mathbf{a}) = \sup\{E_Z(T, \mathcal{U}, \mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\} \tag{2.2}$$

the *scaled topological entropy* of T on the set Z (with respect to the sequence $\mathbf{a} = \{a(n)\}$).

Given a scaled sequence $\mathbf{a} = \{a(n)\}_{n\geq 1}$, $\alpha \in \mathbb{R}$ and $Z \subset X$, define

$$R(Z, \alpha, N, \mathcal{U}, \mathbf{a}) = \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp\bigl(-\alpha a(N)\bigr), \tag{2.3}$$

where the infimum is taken over all covers Γ of Z with $m(\mathbf{U}) = N$ for all $\mathbf{U} \in \Gamma$. We set

$$\begin{aligned} \underline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) &= \liminf_{N\to\infty} R(Z, \alpha, N, \mathcal{U}, \mathbf{a}), \\ \overline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) &= \limsup_{N\to\infty} R(Z, \alpha, N, \mathcal{U}, \mathbf{a}) \end{aligned}$$

and define the jump-up points of $\underline{r}(Z, \alpha, \mathcal{U}, \mathbf{a})$ and $\overline{r}(Z, \alpha, \mathcal{U}, \mathbf{a})$ as

$$\begin{aligned} \underline{E}_Z(T, \mathcal{U}, \mathbf{a}) &= \inf\{\alpha : \underline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) = 0\} = \sup\{\alpha : \underline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) = +\infty\}, \\ \overline{E}_Z(T, \mathcal{U}, \mathbf{a}) &= \inf\{\alpha : \overline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) = 0\} = \sup\{\alpha : \overline{r}(Z, \alpha, \mathcal{U}, \mathbf{a}) = +\infty\} \end{aligned}$$

respectively.

Definition 2.2 We call the quantities

$$\begin{aligned} \underline{E}_Z(T, \mathbf{a}) &= \sup\{\underline{E}_Z(T, \mathcal{U}, \mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\}, \\ \overline{E}_Z(T, \mathbf{a}) &= \sup\{\overline{E}_Z(T, \mathcal{U}, \mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\} \end{aligned}$$

the *lower* and *upper scaled topological entropies* of T on the set Z.

Our definition of the scaled topological entropy and the lower and upper scaled topological entropies follows the generalized Carathéodory construction described in [18] and is associated with the Carathéodory structure in X given as follows: for a finite open cover $\mathcal{U}$ of X and a scaled sequence $\mathbf{a} = \{a(n)\}_{n\geq 1}$ consider the functions $\xi, \eta, \psi : \mathcal{W}(\mathcal{U}) \to \mathbb{R}^+$ such that

$$\xi(\mathbf{U}) = 1, \quad \eta(\mathbf{U}) = \exp\big(-a(m(\mathbf{U}))\big), \quad \psi(\mathbf{U}) = m(\mathbf{U})^{-1}.$$

Then for every subset $Z \subset X$ we have that $E_Z(T, \mathcal{U}, \mathbf{a})$, $\underline{E}_Z(T, \mathcal{U}, \mathbf{a})$ and $\overline{E}_Z(T, \mathcal{U}, \mathbf{a})$ are respectively the Carathéodory dimension and lower and upper Carathéodory capacities of the set Z (see [18] for details and notations).

Remark 2.1 It follows from (2.1) and (2.3) that the quantities $E_Z(T, \mathcal{U}, \mathbf{a})$, $\underline{E}_Z(T, \mathcal{U}, \mathbf{a})$ and $\overline{E}_Z(T, \mathcal{U}, \mathbf{a})$ are always non-negative and hence, in the definitions of $M(Z, \alpha, N, \mathcal{U}, \mathbf{a})$ and $R(Z, \alpha, N, \mathcal{U}, \mathbf{a})$ one can replace $\alpha \in \mathbb{R}$ with $\alpha \geq 0$.

2.2 Properties of Scaled Topological Entropy

For any subset $Z \subset X$ and any open cover $\mathcal{U}$ of X, let $\aleph(\mathcal{U}, Z)$ denote the number of sets in a finite subcover of $\mathcal{U}$ with the smallest cardinality. We have the following equivalent definition of the lower and upper scaled topological entropy (the proof is similar to the proof of Theorem 2.2 in [18]).

Proposition 2.1 *For each scaled sequence* $\mathbf{a} = \{a(n)\}$ *we have*

$$\underline{E}_Z(T, \mathcal{U}, \mathbf{a}) = \liminf_{N\to\infty} \frac{1}{a(N)} \log \aleph\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{U}, Z\Big),$$

$$\overline{E}_Z(T, \mathcal{U}, \mathbf{a}) = \limsup_{N\to\infty} \frac{1}{a(N)} \log \aleph\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{U}, Z\Big).$$

Given two open covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{U}$ is finer than $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is an element $V \in \mathcal{V}$ such that $U \subset V$. We denote such an element by $\mathcal{U} \succeq \mathcal{V}$. Set

$$\mathcal{U} \vee \mathcal{V} := \Big\{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V}\Big\}, \quad T^{-1}\mathcal{U} := \{T^{-1}U : U \in \mathcal{U}\}.$$

In what follows we use the notation $\mathcal{E}$ for either E or $\underline{E}$ or $\overline{E}$. The following propositions describe some basic properties of scaled topological and lower and upper scaled topological entropies.

Proposition 2.2 *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be two open covers of* X *and* $Z \subset X$. *If* $\mathbf{a} = \{a(n)\}$ *is a scaled sequence, then the following properties hold:*

(1) *If* $\mathcal{U} \preceq \mathcal{V}$, *then* $\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq \mathcal{E}_Z(T, \mathcal{V}, \mathbf{a})$;
(2) *For any* $l \geq 1$, *we have*

$$\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq \mathcal{E}_Z\Big(T, \bigvee_{i=0}^{l-1} T^{-i}\mathcal{U}, \mathbf{a}\Big). \tag{2.4}$$

Furthermore, if $\mathbf{a} = \{a(n)\}$ *satisfies*

$$\lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 1, \tag{2.5}$$

then

$$\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) = \mathcal{E}_Z\Big(T, \bigvee_{i=0}^{l-1} T^{-i}\mathcal{U}, \mathbf{a}\Big); \tag{2.6}$$

(3) $E_Z(T, \mathcal{U}, \mathbf{a}) \leq \underline{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq \overline{E}_Z(T, \mathcal{U}, \mathbf{a})$ *and* $E_Z(T, \mathbf{a}) \leq \underline{E}_Z(T, \mathbf{a}) \leq \overline{E}_Z(T, \mathbf{a})$.

The following proposition shows that the scaled topological entropy as well as lower and upper scaled topological entropies are invariant under a topological conjugacy. Its proof is similar to the proofs of Theorems 1.3 and 2.5 in [18].

Proposition 2.3 *Let* $T_i : X_i \to X_i$, $i = 1, 2$ *be two continuous transformations of compact metric spaces and let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence. If there exists a continuous surjection* $\pi : X_1 \to X_2$ *such that* $\pi \circ T_1 = T_2 \circ \pi$, *then for each* $Z \subset X_1$ *and each open cover* $\mathcal{U}$ *of* X_2

$$\mathcal{E}_Z(T_1, \pi^{-1}\mathcal{U}, \mathbf{a}) = \mathcal{E}_{\pi(Z)}(T_2, \mathcal{U}, \mathbf{a}).$$

In particular, $\mathcal{E}_Z(T_1, \mathbf{a}) \geq \mathcal{E}_{\pi(Z)}(T_2, \mathbf{a})$ *for each* $Z \subset X_1$. *Furthermore, if the map* π *is a homeomorphism, then* $\mathcal{E}_Z(T_1, \mathbf{a}) = \mathcal{E}_{\pi(Z)}(T_2, \mathbf{a})$ *for each* $Z \subset X_1$.

We conclude this section by presenting some more basic properties of the scaled topological entropy as well as lower and upper scaled topological entropies.

Proposition 2.4 *The following statements hold:*

(1) *if* $Z_1 \subset Z_2$, *then* $\mathcal{E}_{Z_1}(T, \mathcal{U}, \mathbf{a}) \leq \mathcal{E}_{Z_2}(T, \mathcal{U}, \mathbf{a})$ *and hence,* $\mathcal{E}_{Z_1}(T, \mathbf{a}) \leq \mathcal{E}_{Z_2}(T, \mathbf{a})$*;*
(2) *if* $Z_i \subset X$, $i \geq 1$ *and* $Z = \bigcup_{i\geq 1} Z_i$, *then* $E_Z(T, \mathbf{a}) = \sup_{i\geq 1} E_{Z_i}(T, \mathbf{a})$, $\underline{E}_Z(T, \mathbf{a}) \geq \sup_{i\geq 1} \underline{E}_{Z_i}(T, \mathbf{a})$ *and* $\overline{E}_Z(T, \mathbf{a}) \geq \sup_{i\geq 1} \overline{E}_{Z_i}(T, \mathbf{a})$*;*
(3) $\mathcal{E}_Z(T, T^{-1}\mathcal{U}, \mathbf{a}) = \mathcal{E}_{T(Z)}(T, \mathcal{U}, \mathbf{a})$ *and* $E_Z(T, \mathbf{a}) = E_{T(Z)}(T, \mathbf{a})$*;*
(4) *if* $\mathbf{a} = \{a(n)\}$ *satisfies* (2.5), *then* $\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) = \mathcal{E}_{T(Z)}(T, \mathcal{U}, \mathbf{a})$ *and* $\mathcal{E}_Z(T, \mathbf{a}) = \mathcal{E}_{T(Z)}(T, \mathbf{a})$.

The following result is an immediate corollary of Statements (3) and (4) of Proposition 2.4.

Corollary 2.5 *For each scaled sequence* $\mathbf{a} = \{a(n)\}$ *satisfying* (2.5) *and each open cover* $\mathcal{U}$ *of* X, *we have*

$$\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) = \mathcal{E}_{T(Z)}(T, \mathcal{U}, \mathbf{a}) = \mathcal{E}_Z(T, T^{-1}\mathcal{U}, \mathbf{a}).$$

2.3 Relations Between Scaled Topological and Lower and Upper Topological Entropies

A sequence of positive numbers $\mathbf{a} = \{a(n)\}$ is said to be *sub-additive* if $a(n + m) \leq a(n) + a(m)$ for all $n, m \in \mathbb{N}$.

Theorem 2.6 *For any compact invariant set* $Z \subset X$ *and any open cover* $\mathcal{U}$ *of* X, *if* $\mathbf{a} = \{a(n)\}$ *is a sub-additive scaled sequence, then*

$$E_Z(T, \mathcal{U}, \mathbf{a}) = \underline{E}_Z(T, \mathcal{U}, \mathbf{a}) = \overline{E}_Z(T, \mathcal{U}, \mathbf{a})$$

and hence,

$$E_Z(T, \mathbf{a}) = \underline{E}_Z(T, \mathbf{a}) = \overline{E}_Z(T, \mathbf{a}).$$

Remark 2.2 While for a general scaled sequence $\mathbf{a} = \{a(n)\}$ Theorem 2.6 may not be true (see Example 4.5 in Sect. 4), sub-additivity assumption is not necessary to ensure the coincidence of the lower and upper scaled topological entropies (see Example 4.4 in Sect. 4 for details).

2.4 Equivalent Scaled Sequences

Let Σ denote the set of all scaled sequences. We call two scaled sequences $\mathbf{a}, \mathbf{b} \in \Sigma$ *equivalent* and we write $\mathbf{a} \sim \mathbf{b}$ if the following condition holds

$$0 < \liminf_{n\to\infty} \frac{b(n)}{a(n)} \le \limsup_{n\to\infty} \frac{b(n)}{a(n)} < \infty.$$

Obviously, $\sim$ defines an equivalence relation on Σ. Given a scaled sequence $\mathbf{a}$, we denote its equivalence class by $[\mathbf{a}] := \{\mathbf{b} \in \Sigma : \mathbf{b} \sim \mathbf{a}\}$ and we let $\mathcal{A} := \Sigma/\sim$. Given two equivalence classes $[\mathbf{a}], [\mathbf{b}] \in \mathcal{A}$, we say that $[\mathbf{a}] \preccurlyeq [\mathbf{b}]$ if for each $\mathbf{a} = \{a(n)\} \in [\mathbf{a}]$ and $\mathbf{b} = \{b(n)\} \in [\mathbf{b}]$ the following holds

$$\limsup_{n\to\infty} \frac{a(n)}{b(n)} = 0.$$

The following result is immediate.

Proposition 2.7 *Let* $\mathbf{a} = \{a(n)\}$ *and* $\mathbf{b} = \{b(n)\}$ *be two scaled sequences. For every* $Z \subset X$ *and every open cover* $\mathcal{U}$ *of* X*, the following properties hold:*

(1) *If* $a(n) \le b(n)$ *for all sufficiently large* n*, then* $\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \ge \mathcal{E}_Z(T, \mathcal{U}, \mathbf{b})$ *and* $\mathcal{E}_Z(T, \mathbf{a}) \ge \mathcal{E}_Z(T, \mathbf{b})$*;*
(2) *For each* $K > 0$ *we have that*

$$K\mathcal{E}_Z(T, \mathcal{U}, K\mathbf{a}) = \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}), \quad K\mathcal{E}_Z(T, K\mathbf{a}) = \mathcal{E}_Z(T, \mathbf{a}),$$

where $K\mathbf{a} = \{Ka(n)\}$*;*
(3) *If there exists a constant* C *such that* $\frac{1}{C}b(n) \le a(n) \le Cb(n)$ *for all sufficiently large* n*, then*

$$\frac{1}{C}\mathcal{E}_Z(T, \mathcal{U}, \mathbf{b}) \le \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \le C\mathcal{E}_Z(T, \mathcal{U}, \mathbf{b})$$

and

$$\frac{1}{C}\mathcal{E}_Z(T, \mathbf{b}) \le \mathcal{E}_Z(T, \mathbf{a}) \le C\mathcal{E}_Z(T, \mathbf{b}).$$

By Statement (3) of Proposition 2.7, for each equivalence class $[\mathbf{a}] \in \mathcal{A}$ and for each $\mathbf{a_1}, \mathbf{a_2} \in [\mathbf{a}]$ we have that $\mathcal{E}_Z(T, \mathbf{a_1}) = \mathcal{E}_Z(T, \mathbf{a_2}) = 0$, or $\mathcal{E}_Z(T, \mathbf{a_1}) = \mathcal{E}_Z(T, \mathbf{a_2}) = \infty$ or both $\mathcal{E}_Z(T, \mathbf{a_1})$ and $\mathcal{E}_Z(T, \mathbf{a_2})$ are positive and finite. In the first two cases, we write $\mathcal{E}_Z(T, [\mathbf{a}]) = 0$ and $\mathcal{E}_Z(T, [\mathbf{a}]) = \infty$ respectively and in the third case, we say that $\mathcal{E}_Z(T, [\mathbf{a}])$ is positive and finite. In this sense, entropy depends not on the scaled sequence but on its class of equivalence. This means that we may have a sequence that does not satisfy our conditions such as (2.5), supper-additivity and sub-additivity, but there may exist an equivalent sequence that does satisfy these conditions.

By Statement (1) of Proposition 2.7, we have $\mathcal{E}_Z(T, [\mathbf{a}]) \le \mathcal{E}_Z(T, [\mathbf{b}])$ whenever $[\mathbf{a}] \succcurlyeq [\mathbf{b}]$.

Theorem 2.8 *If there is* $[\mathbf{a}] \in \mathcal{A}$ *such that* $E_Z(T, [\mathbf{a}])$ *is positive and finite, then*

$$E_Z(T, [\mathbf{b}]) = \begin{cases} 0, & \textit{if } [\mathbf{a}] \preccurlyeq [\mathbf{b}], \\ \infty, & \textit{if } [\mathbf{b}] \preccurlyeq [\mathbf{a}]. \end{cases}$$

In particular, there may exist at most one element in $(\mathcal{A}, \preccurlyeq)$ *such that the corresponding scaled topological entropy is positive and finite. Similar results hold for lower and upper scaled topological entropy.*

Note that there are maps with zero scaled topological entropy with respect to any scaled sequence, see Example 4.6 in Sect. 4. The element in $(\mathcal{A}, \preccurlyeq)$ such that the scaled topological entropy and lower and upper scaled topological entropy is finite and positive may not be the same, see Example 4.5.

2.5 The Case of Compact Metric Spaces

In this and the next subsection, we consider the case when X is a compact metric space with metric d and $T : X \to X$ a continuous transformation.

Let $\mathcal{U}$ be an open cover of X and $|\mathcal{U}| = \max\{\mathrm{diam}(U) : U \in \mathcal{U}\}$ the diameter of the cover $\mathcal{U}$. The following result shows that the supremum in the definition of (lower and upper) scaled topological entropy can be replaced by the limit as the diameter of the cover goes to zero. We continue to use $\mathcal{E}$ for either E or $\underline{E}$ or $\overline{E}$.

Proposition 2.9 *Let $T : X \to X$ be a continuous transformation on a compact metric space X and $\mathcal{U}$ an open cover of X. Then the limit $\lim_{|\mathcal{U}|\to 0} \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a})$ exists and is equal to $\mathcal{E}_Z(T, \mathbf{a})$.*

An open cover $\mathcal{U}$ of X is said to be *generating*, if $\lim_{n\to\infty} |\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}| = 0$. By Proposition 2.2 (2) and Proposition 2.9, if $\mathcal{U}$ is a generating open cover of X and $\mathbf{a} = \{a(n)\}$ a scaled sequence satisfying (2.5), then $\mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) = \mathcal{E}_Z(T, \mathbf{a})$.

Remark 2.3 In this setting, we describe another but equivalent definition of scaled topological entropy. Given $\epsilon > 0$, $n \in \mathbb{N}$ and $x \in X$, denote by $B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\}$ the *Bowen's ball* of radius ϵ centered at x of length n, where $d_n(x, y) := \max\{d(T^i(x), T^i(y)) \colon 0 \le i < n\}$. Given a scaled sequence $\mathbf{a} = \{a(n)\}$, for each subset $Z \subset X$ and each $\alpha, N > 0$ set

$$M(Z, \alpha, N, \delta, \mathbf{a}) = \inf \Big\{ \sum_i \exp\big(-\alpha a(n_i)\big) : \bigcup_i B_{n_i}(x_i, \delta) \supset Z,\ x_i \in X \text{ and } n_i \ge N \text{ for all } i \Big\}.$$

Since $M(Z, \alpha, N, \delta, \mathbf{a})$ is monotonically increasing with N,

$$m(Z, \alpha, \delta, \mathbf{a}) = \lim_{N\to\infty} M(Z, \alpha, N, \delta, \mathbf{a}).$$

We denote the jump-up point of $m(Z, \alpha, \delta, \mathbf{a})$ by

$$E_Z(T, \delta, \mathbf{a}) = \inf\{\alpha : m(Z, \alpha, \delta, \mathbf{a}) = 0\} = \sup\{\alpha : m(Z, \alpha, \delta, \mathbf{a}) = +\infty\}.$$

Let $\mathcal{U}$ be an open cover of X and $\delta(\mathcal{U})$ its Lebesgue number. Clearly, for every $x \in X$ with $x \in X(\mathbf{U})$ for some string $\mathbf{U}$ we have that $X(\mathbf{U}) \subset B_{m(\mathbf{U})}(x, |\mathcal{U}|)$. On the other hand, for each Bowen's all $B_n(x, \delta(\mathcal{U}))$ of radius $\delta(\mathcal{U})$ centered at $x \in X$ and length n, there is a string $\mathbf{U}$ of length n such that $B_n(x, \delta(\mathcal{U})) \subset X(\mathbf{U})$. This implies that

$$E_Z(T, \mathbf{a}) = \lim_{\delta\to 0} E_Z(T, \delta, \mathbf{a}).$$

Similarly, using Bowen's balls, one can define lower and upper scaled topological entropy.

2.6 Relations Between Scaled Topological Entropies and Box Dimension

In this section we describe how to construct scaled sequences that are naturally associated with dynamical systems and in which the corresponding scaled entropies on a subset are closely related to its box dimension.

Recall that given a subset $Z \subset X$ of a compact metric space X, the *lower and upper box dimension* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\delta \to 0} \frac{\log N_d(Z, \delta)}{-\log \delta}, \quad \overline{\dim}_B Z = \limsup_{\delta \to 0} \frac{\log N_d(Z, \delta)}{-\log \delta},$$

where $N_d(Z, \delta)$ denotes the smallest number of balls of radius δ (in d-metric) needed to cover the set Z. For each $n \geq 1$ we set

$$b_n := \min_{x,y \in X} \frac{d_n(x, y)}{d(x, y)}, \quad c_n := \max_{x,y \in X} \frac{d_n(x, y)}{d(x, y)}. \tag{2.7}$$

Lemma 2.1 *For every $r > 0$ and $x \in X$,*

$$B\left(x, \frac{r}{c_n}\right) \subset B_n(x, r) \subset B\left(x, \frac{r}{b_n}\right)$$

where $B(x, r) := \{y \in X : d(x, y) < r\}$ denotes the ball of radius $r > 0$ centered at x.

Proof For every $y \in B(x, \frac{r}{c_n})$

$$d_n(x, y) \leq d(x, y) \max_{x,y \in X} \frac{d_n(x, y)}{d(x, y)} < \frac{r}{c_n} c_n = r$$

and the first inclusion follows. On the other hand, for each $y \in B_n(x, r)$

$$d(x, y) \leq d_n(x, y) \max_{x,y \in X} \frac{d(x, y)}{d_n(x, y)} < r \frac{1}{b_n}$$

and the second inclusion follows.

Consider the sequences $\{\log b_n\}$ and $\{\log c_n\}$. If T is Lipschitz, then these sequences satisfy (2.5).

Theorem 2.10 *Let $T : X \to X$ be a continuous transformation of a compact metric space X. Assume that the sequence $\{b_n\}$ is scaled and that $\{\log b_n\}$ and $\{\log c_n\}$ satisfy (2.5). Then for each $Z \subset X$,*

$$\overline{E}_Z(T, \{\log c_n\}) \leq \overline{\dim}_B Z, \quad \underline{E}_Z(T, \{\log b_n\}) \geq \underline{\dim}_B Z.$$

As a manifestation of this theorem we obtain the following:

(1) If $\overline{\dim}_B Z < \infty$, then by Proposition 2.2(4) and Theorem 2.10, the quantities $E_Z(T, \{\log c_n\})$, $\underline{E}_Z(T, \{\log c_n\})$ and $\overline{E}_Z(T, \{\log c_n\})$ are all finite but they may be zero. On the other hand, if $\underline{\dim}_B Z > 0$, then by Theorem 2.10, both $\underline{E}_Z(T, \{\log b_n\})$ and $\overline{E}_Z(T, \{\log b_n\})$ are positive but they may be infinite.

(2) Assume that $E_Z(T, \{\log c_n\}) > 0$ and $\underline{E}_Z(T, \{\log b_n\}) < \infty$. Then by Propositions 2.2 and 2.7,

$$\infty > \underline{E}_Z(T, \{\log b_n\}) \geq E_Z(T, \{\log b_n\}) \geq E_Z(T, \{\log c_n\}) > 0.$$

Using Theorem 2.8 we conclude that the sequence $[\{\log c_n\}]$ and $[\{\log b_n\}]$ are equivalent and that $[\{\log c_n\}]$ is the only equivalence class with positive and finite scaled topological entropy. This is the "built-in" scaling for the map T.

(3) If a subset $Z \subset X$ is such that $\overline{\dim}_{\mathrm{B}} Z = 0$, then Theorem 2.10 yields that $\overline{E}_Z\Big(T, \{\log c_n\}\Big) = 0$ and hence,

$$E_Z\Big(T, \{\log c_n\}\Big) = \underline{E}_Z\Big(T, \{\log c_n\}\Big) = \overline{E}_Z\Big(T, \{\log c_n\}\Big) = 0.$$

3 Scaled Metric Entropy

In this section we introduce different types of scaled metric entropy and we study their properties and relations between them.

3.1 Definition of Scaled Metric Entropy

Let T be a continuous map of a compact (Hausdorff) space X. Denote by $\mathcal{M}_T$ and $\mathcal{E}_T$ the set of all (respectively, ergodic) T-invariant Borel probability measures on X. We follow the approach in [18] and introduce the notion of scaled metric entropy using the *inverse variational principle*. Given a T-invariant measure μ and a scaled sequence $\mathbf{a} = \{a(n)\}$, let

$$\begin{aligned} E_\mu(T, \mathcal{U}, \mathbf{a}) &= \inf\Big\{E_Z\Big(T, \mathcal{U}, \mathbf{a}\Big) \colon \mu(Z) = 1\Big\} \\ &= \lim_{\delta\to 0} \inf\Big\{E_Z\Big(T, \mathcal{U}, \mathbf{a}\Big) \colon \mu(Z) \geq 1-\delta\Big\} \end{aligned}$$

and then let

$$E_\mu\Big(T, \mathbf{a}\Big) = \sup\Big\{E_\mu\Big(T, \mathcal{U}, \mathbf{a}\Big) \mid \mathcal{U} \text{ is a finite open cover of } X\Big\}.$$

We call the quantity $E_\mu(T, \mathbf{a})$ the *scaled metric entropy* of T with respect to μ (and the scaled sequence $\mathbf{a}$). Let further

$$\begin{aligned} \underline{E}_\mu(T, \mathcal{U}, \mathbf{a}) &= \lim_{\delta\to 0} \inf\Big\{\underline{E}_Z(T, \mathcal{U}, \mathbf{a}) \colon \mu(Z) \geq 1-\delta\Big\}, \\ \overline{E}_\mu(T, \mathcal{U}, \mathbf{a}) &= \lim_{\delta\to 0} \inf\Big\{\overline{E}_Z(T, \mathcal{U}, \mathbf{a}) \colon \mu(Z) \geq 1-\delta\Big\}. \end{aligned}$$

We call the quantities

$$\begin{aligned} \underline{E}_\mu(T, \mathbf{a}) &= \sup\Big\{\underline{E}_\mu(T, \mathcal{U}, \mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\Big\}, \\ \overline{E}_\mu(T, \mathbf{a}) &= \sup\Big\{\overline{E}_\mu(T, \mathcal{U}, \mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\Big\} \end{aligned}$$

respectively the *lower* and *upper scaled metric entropy of* T with respect to μ (and the scaled sequence $\mathbf{a}$).

By Statement (3) of Proposition 2.2,

$$E_\mu(T, \mathcal{U}, \mathbf{a}) \leq \underline{E}_\mu(T, \mathcal{U}, \mathbf{a}) \leq \overline{E}_\mu(T, \mathcal{U}, \mathbf{a})$$

and

$$E_\mu(T, \mathbf{a}) \leq \underline{E}_\mu(T, \mathbf{a}) \leq \overline{E}_\mu(T, \mathbf{a}).$$

3.2 Properties of the Scaled Metric Entropy

We describe some basic properties of scaled metric entropy. We will use the notation $\mathcal{E}_\mu$ for either E_μ or $\underline{E}_\mu$ or $\overline{E}_\mu$.

The following proposition is a direct consequence of the definition of scaled metric entropy, Proposition 2.2 and Corollary 2.5.

Proposition 3.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of X and $\mu \in \mathcal{M}_T$. If $\mathbf{a} = \{a(n)\}$ is a scaled sequence, then the following properties hold:*

(1) *If $\mathcal{U} \preccurlyeq \mathcal{V}$, then $\mathcal{E}_\mu(T, \mathcal{U}, \mathbf{a}) \le \mathcal{E}_\mu(T, \mathcal{V}, \mathbf{a})$;*
(2) *If $\mathbf{a} = \{a(n)\}$ satisfies (2.5), then $\mathcal{E}_\mu\Big(T, \mathcal{U}, \mathbf{a}\Big) = \mathcal{E}_\mu\Big(T, \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, \mathbf{a}\Big)$ and $\mathcal{E}_\mu\Big(T, \mathcal{U}, \mathbf{a}\Big) = \mathcal{E}_\mu\Big(T, T^{-1}\mathcal{U}, \mathbf{a}\Big)$.*

The following proposition shows that the scaled metric entropy as well as the lower and upper scaled metric entropies are invariant under a topological conjugacy.

Proposition 3.2 *Let $T_i : X_i \to X_i$, $i = 1, 2$ be two continuous transformations of compact (Hausdorff) spaces and let $\mathbf{a} = \{a(n)\}$ be a scaled sequence. If there exists a homeomorphism $\pi : X_1 \to X_2$ such that $\pi \circ T_1 = T_2 \circ \pi$, then for each $\mu \in \mathcal{M}_{T_1}$*

$$\mathcal{E}_\mu(T_1, \mathbf{a}) = \mathcal{E}_{\pi_*\mu}(T_2, \mathbf{a})$$

where $\pi_*\mu = \mu \circ \pi^{-1}$.

3.3 The Case of Compact Metric Spaces

In the rest of this section, let $T : X \to X$ be a continuous map of a compact metric space X with metric d. In this subsection, we introduce the scaled local metric entropy following the approach of Brin and Katok.

Lemma 3.1 *Let $\mathbf{a} = \{a(n)\}$ be a scaled sequence satisfying (2.5). For any $\epsilon > 0$ and any ergodic measure μ the following two limits*

$$\begin{aligned} \underline{h}_\mu(T, \mathbf{a}, x, \epsilon) &:= \liminf_{n\to\infty} -\frac{1}{a(n)} \log \mu(B_n(x, \epsilon)), \\ \overline{h}_\mu(T, \mathbf{a}, x, \epsilon) &:= \limsup_{n\to\infty} -\frac{1}{a(n)} \log \mu(B_n(x, \epsilon)) \end{aligned} \tag{3.1}$$

are constant almost everywhere.

Proof Given a positive number ϵ, note that $B_{n+1}(x, \epsilon) \subset T^{-1}B_n(Tx, \epsilon)$ for each $n \in \mathbb{N}$ and each $x \in X$. Therefore,

$$\mu(B_{n+1}(x, \epsilon)) \le \mu(T^{-1}B_n(Tx, \epsilon)) = \mu(B_n(Tx, \epsilon)).$$

By (2.5), we have that $\underline{h}_\mu(T, \mathbf{a}, x, \epsilon) \ge \underline{h}_\mu(T, \mathbf{a}, Tx, \epsilon)$ and ergodicity of μ implies that $\underline{h}_\mu(T, \mathbf{a}, x, \epsilon)$ is constant almost everywhere. Similar argument yields the conclusion about the second limit. □

Remark 3.1 Since $\underline{h}_\mu(T, \mathbf{a}, x, \epsilon)$ and $\overline{h}_\mu(T, \mathbf{a}, x, \epsilon)$ are constant almost everywhere, we denote their common values by $\underline{h}_\mu(T, \mathbf{a}, \epsilon)$ and $\overline{h}_\mu(T, \mathbf{a}, \epsilon)$ respectively. Observe that $\underline{h}_\mu(T, \mathbf{a}, \epsilon)$ and $\overline{h}_\mu(T, \mathbf{a}, \epsilon)$ are increasing as ϵ goes to zero and we let

$$\underline{h}_\mu(T, \mathbf{a}) := \lim_{\epsilon\to 0} \underline{h}_\mu(T, \mathbf{a}, \epsilon) \text{ and } \overline{h}_\mu(T, \mathbf{a}) := \lim_{\epsilon\to 0} \overline{h}_\mu(T, \mathbf{a}, \epsilon).$$

These are "scaled" versions of the quantities introduced by Brin and Katok in [2].

For a general scaled sequences $\underline{h}_\mu(T, \mathbf{a})$ may not be equal to $\overline{h}_\mu(T, \mathbf{a})$, see Example 4.7 in the next section.

By Proposition 2.2, in the case of compact metric spaces we have that

$$E_\mu(T, \mathbf{a}) = \lim_{|\mathcal{U}|\to 0} E_\mu(T, \mathcal{U}, \mathbf{a}).$$

The same conclusion holds for $\underline{E}_\mu$ and $\overline{E}_\mu$.

3.4 Relations Between Different Scaled Metric Entropies

In this subsection we study the relations between various versions of scaled metric entropies.

Theorem 3.3 *Let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence satisfying* (2.5). *For any* T*-invariant ergodic measure* μ *we have*

$$\underline{h}_\mu(T, \mathbf{a}) \le E_\mu(T, \mathbf{a}) \le \underline{E}_\mu(T, \mathbf{a}) \le \overline{E}_\mu(T, \mathbf{a}) \le \overline{h}_\mu(T, \mathbf{a}).$$

If a scaled sequence $\mathbf{a} = \{a(n)\}$ is such that the relations

$$\underline{h}_\mu(T, \mathbf{a}) = \overline{h}_\mu(T, \mathbf{a}) := h \tag{3.2}$$

hold for an ergodic measure μ, then by Theorem 3.3, we obtain that

$$E_\mu(T, \mathbf{a}) = \underline{E}_\mu(T, \mathbf{a}) = \overline{E}_\mu(T, \mathbf{a}) = h.$$

This is, for example, the case when $a(n) = n$.

Given a number $h \ge 0$ and an ergodic measure μ, define

$$K_h = \left\{ x \in X\colon \lim_{\epsilon\to 0} \underline{h}_\mu(T, \mathbf{a}, x, \epsilon) = \lim_{\epsilon\to 0} \overline{h}_\mu(T, \mathbf{a}, x, \epsilon) = h \right\}. \tag{3.3}$$

Theorem 3.4 *Let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence satisfying* (2.5). *Assume that* $\mu(K_h) = 1$ *for some* $h \ge 0$. *Then*

$$E_{K_h}(T, \mathbf{a}) = \underline{E}_{K_h}(T, \mathbf{a}) = \overline{E}_{K_h}(T, \mathbf{a}) = h.$$

The following result is a direct consequence of Theorems 3.3 and 3.4.

Corollary 3.5 *Let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence satisfying* (2.5). *Assume that* (3.2) *holds and let* K_h *be the set given by* (3.3). *Then*

$$\begin{aligned} E_\mu(T, \mathbf{a}) &= \underline{E}_\mu(T, \mathbf{a}) = \overline{E}_\mu(T, \mathbf{a}) \\ &= E_{K_h}(T, \mathbf{a}) = \underline{E}_{K_h}(T, \mathbf{a}) = \overline{E}_{K_h}(T, \mathbf{a}) = h. \end{aligned}$$

The next theorem shows that the scaled topological entropy is determined by scaled metric entropy, which extends the result in [15] for scaled entropies.

Theorem 3.6 *Let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence,* μ *a Borel probability measure on* X *and* $L \subset X$ *a Borel subset. Set* $\underline{h}_\mu(T, \mathbf{a}, x) := \lim_{\varepsilon\to 0} \underline{h}_\mu(T, \mathbf{a}, x, \varepsilon)$. *Then for any* $s \ge 0$ *the following properties hold:*

(1) *If* $\underline{h}_\mu(T, \mathbf{a}, x) \le s$ *for all* $x \in L$, *then* $E_L(T, \mathbf{a}) \le s$;
(2) *If* $\underline{h}_\mu(T, \mathbf{a}, x) \ge s$ *for all* $x \in L$ *and* $\mu(L) > 0$, *then* $E_L(T, \mathbf{a}) \ge s$.

The following result is a direct consequence of Theorem 3.6 and Remark 3.1.

Corollary 3.7 *Let* $\mathbf{a} = \{a(n)\}$ *be a scaled sequence satisfying* (2.5). *For any* T*-invariant ergodic measure* μ*, let* K_h *be the set given by* (3.3)*. Assume that* $\mu(K_h) = 1$ *for some* $h \geq 0$*. Then for each set* Z *of positive* μ*-measure we have*

$$E_{Z\cap K_h}(T, \mathbf{a}) = \underline{h}_\mu(T, \mathbf{a}) = \overline{h}_\mu(T, \mathbf{a}) = h.$$

Proof Consider $L = Z \cap K_h$ and $s = h$ in Theorem 3.6, the desired result immediately follows. □

3.5 Scaled Metric Entropy for Equivalent Scaled Sequences

Following the discussion on scaled topological entropy for equivalent scaled sequence in Sect. 2.4, we introduce a similar notion of equivalence for scaled metric entropy.

The following proposition are direct consequences of Proposition 2.7, and it also holds if X is a compact (Hausdorff) space.

Proposition 3.8 *Let* $\mathbf{a} = \{a(n)\}$ *and* $\mathbf{b} = \{b(n)\}$ *be two scaled sequences. For every* T*-invariant measure* $\mu \in \mathcal{M}_T$*, the following properties hold:*

(1) *If* $a(n) \leq b(n)$ *for all sufficiently large* n*, then* $\mathcal{E}_\mu(T, \mathcal{U}, \mathbf{a}) \geq \mathcal{E}_\mu(T, \mathcal{U}, \mathbf{b})$ *and* $\mathcal{E}_\mu(T, \mathbf{a}) \geq \mathcal{E}_\mu(T, \mathbf{b})$*;*
(2) *For each* $K > 0$*,* $K\mathcal{E}_\mu(T, \mathcal{U}, K\mathbf{a}) = \mathcal{E}_\mu(T, \mathcal{U}, \mathbf{a})$ *and* $K\mathcal{E}_\mu(T, K\mathbf{a}) = \mathcal{E}_\mu(T, \mathbf{a})$*;*
(3) *If there exists a constant* C *such that* $\frac{1}{C}b(n) \leq a(n) \leq Cb(n)$ *for all sufficiently large* n*, then*

$$\frac{1}{C}\mathcal{E}_\mu(T, \mathcal{U}, \mathbf{b}) \leq \mathcal{E}_\mu(T, \mathcal{U}, \mathbf{a}) \leq C\mathcal{E}_\mu(T, \mathcal{U}, \mathbf{b})$$

and

$$\frac{1}{C}\mathcal{E}_\mu(T, \mathbf{b}) \leq \mathcal{E}_\mu(T, \mathbf{a}) \leq C\mathcal{E}_\mu(T, \mathbf{b}).$$

In the following proposition we denote by $\mathcal{H}_\mu$ either $\underline{h}_\mu$ or $\overline{h}_\mu$.

Proposition 3.9 *Let* $\mathbf{a} = \{a(n)\}$ *and* $\mathbf{b} = \{b(n)\}$ *be two scaled sequences. For every* T*-invariant ergodic measure* $\mu \in \mathcal{E}_T$*, the following properties hold:*

(1) *If* $a(n) \leq b(n)$ *for all sufficiently large* n*, then* $\mathcal{H}_\mu(T, \mathbf{a}) \geq \mathcal{H}_\mu(T, \mathbf{b})$*;*
(2) *For each* $K > 0$*,* $K\mathcal{H}_\mu(T, K\mathbf{a}) = \mathcal{H}_\mu(T, \mathbf{a})$*;*
(3) *If there exists a constant* C *such that* $\frac{1}{C}b(n) \leq a(n) \leq Cb(n)$ *for all sufficiently large* n*, then*

$$\frac{1}{C}\mathcal{H}_\mu(T, \mathbf{b}) \leq \mathcal{H}_\mu(T, \mathbf{a}) \leq C\mathcal{H}_\mu(T, \mathbf{b}).$$

By Proposition 3.8(1), we have $\mathcal{E}_\mu(T, [\mathbf{a}]) \leq \mathcal{E}_\mu(T, [\mathbf{b}])$ whenever $[\mathbf{a}] \succcurlyeq [\mathbf{b}]$ and by Proposition 3.8(3), for each equivalence class $[\mathbf{a}] \in \mathcal{A}$ and for each $\mathbf{a}_1, \mathbf{a}_2 \in [\mathbf{a}]$ we have that $\mathcal{E}_\mu(T, \mathbf{a}_1) = \mathcal{E}_\mu(T, \mathbf{a}_2) = 0$ or $\mathcal{E}_\mu(T, \mathbf{a}_1) = \mathcal{E}_\mu(T, \mathbf{a}_2) = \infty$ or both $\mathcal{E}_\mu(T, \mathbf{a}_1)$ and $\mathcal{E}_\mu(T, \mathbf{a}_2)$ are positive and finite.

Respectively, by Proposition 3.9(1), $\mathcal{H}_\mu(T, [\mathbf{a}]) \leq \mathcal{H}_\mu(T, [\mathbf{b}])$ whenever $[\mathbf{a}] \succcurlyeq [\mathbf{b}]$ and by Proposition 3.9(3), for each equivalence class $[\mathbf{a}] \in \mathcal{A}$ and for each $\mathbf{a}_1, \mathbf{a}_2 \in [\mathbf{a}]$ we have that $\mathcal{H}_\mu(T, \mathbf{a}_1) = \mathcal{H}_\mu(T, \mathbf{a}_2) = 0$, or $\mathcal{H}_\mu(T, \mathbf{a}_1) = \mathcal{H}_\mu(T, \mathbf{a}_2) = \infty$ or both $\mathcal{H}_\mu(T, \mathbf{a}_1)$ and $\mathcal{H}_\mu(T, \mathbf{a}_2)$ are positive and finite.

In the first two cases we write $\mathcal{E}_\mu(T, [\mathbf{a}]) = 0$ (respectively, $\mathcal{H}_\mu(T, [\mathbf{a}]) = 0$) and $\mathcal{E}_\mu(T, [\mathbf{a}]) = \infty$ (respectively, $\mathcal{H}_\mu(T, [\mathbf{a}]) = \infty$) and in the third case we say that $\mathcal{E}_\mu(T, [\mathbf{a}])$ (respectively, $\mathcal{H}_\mu(T, [\mathbf{a}])$) is positive and finite.

The first result is clear from the definitions.

Theorem 3.10 *For each T-invariant ergodic measure $\mu \in \mathcal{E}_T$, if there is $[\mathbf{a}] \in \mathcal{A}$ such that $\underline{h}_\mu(T, [\mathbf{a}])$ is positive and finite, then*

$$\underline{h}_\mu(T, [\mathbf{b}]) = \begin{cases} 0, & \text{if } [\mathbf{a}] \preccurlyeq [\mathbf{b}], \\ \infty, & \text{if } [\mathbf{b}] \preccurlyeq [\mathbf{a}]. \end{cases}$$

Similar result holds for $\overline{h}_\mu$.

A similar result also holds for $\mathcal{E}_\mu$.

Theorem 3.11 *For each T-invariant measure $\mu \in \mathcal{E}_T$, if there is $[\mathbf{a}] \in \mathcal{A}$ for which $E_\mu(T, [\mathbf{a}])$ is positive and finite, then*

$$E_\mu(T, [\mathbf{b}]) = \begin{cases} 0, & \text{if } [\mathbf{a}] \preccurlyeq [\mathbf{b}], \\ \infty, & \text{if } [\mathbf{b}] \preccurlyeq [\mathbf{a}]. \end{cases}$$

Similar result holds for $\underline{E}_\mu$ and $\overline{E}_\mu$.

3.6 Relations Between Scaled Metric Entropy and Pointwise Dimension

In this subsection we consider some relations between pointwise dimension and the scaled metric entropy and we describe a result which is similar to Theorem 2.10.

Recall that the *lower and upper pointwise dimensions of the measure* μ at the point $x \in X$ are defined respectively by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}, \quad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Theorem 3.12 *Let $T : X \to X$ be a continuous transformation of a compact metric space X, μ a T-invariant measure, $\{b_n\}$ and $\{c_n\}$ are sequences of numbers defined in (2.7). Assume that the sequence $\{b_n\}$ is scaled and that $\{\log b_n\}$ and $\{\log c_n\}$ satisfy (2.5). Then for each $x \in X$ we have*

(1) $\overline{h}_\mu(T, \{\log c_n\}, x) \le \overline{d}_\mu(x)$;
(2) $\underline{h}_\mu(T, \{\log b_n\}, x) \ge \underline{d}_\mu(x)$,

where $\overline{h}_\mu(T, \mathbf{a}, x) := \lim_{\epsilon \to 0} \overline{h}_\mu(T, \mathbf{a}, x, \epsilon)$ and $\underline{h}_\mu(T, \mathbf{a}, x) := \lim_{\epsilon \to 0} \underline{h}_\mu(T, \mathbf{a}, x, \epsilon)$.

Observe that if the sequences $\{b_n\}_{n \ge 1}$ and $\{c_n\}_{n \ge 1}$ satisfy the conditions of the above theorem, then by Remark 3.1, for μ-almost every $x \in X$,

$$\overline{h}_\mu(T, \{\log c_n\}) \le \overline{d}_\mu(x), \quad \underline{h}_\mu(T, \{\log b_n\}) \ge \underline{d}_\mu(x).$$

4 Examples

Example 4.1 We describe a class of maps T which possess a "built-in" scaled sequence.

Let $T : X \to X$ be an expansive homeomorphism of a compact metric space X (that is there exists $\epsilon > 0$ such that if $d(T^n(x), T^n(y) \le \epsilon$ for any integer n and points $x, y \in X$ then $x = y$). Such a homeomorphism possesses a generating open cover of X that we denote by $\mathcal{V}$. Assume that a subset $Z \subset X$ is such that

(1) $\overline{E}_Z(T, \{n\}) = 0$;
(2) There exists an open cover $\mathcal{U}$ of X such that $\aleph\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, Z\Big) \to \infty$ as $n \to \infty$ (without loss of generality we may assume that the partition $\mathcal{U}$ is finer than $\mathcal{V}$).

By Proposition 2.2, in this case for each scaled sequence $\mathbf{a} = \{a(n)\}$ satisfying (2.5) we have

$$\overline{E}_Z(T, \mathbf{a}) = \limsup_{n\to\infty} \frac{1}{a(n)} \log \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, Z) \tag{4.1}$$

$$\underline{E}_Z(T, \mathbf{a}) = \liminf_{n\to\infty} \frac{1}{a(n)} \log \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, Z). \tag{4.2}$$

In particular, if $a(n) = a(n, \mathcal{U}) = \log \aleph\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, Z\Big)$ increases monotonically and satisfies Condition (2.5), then

$$\overline{E}_Z(T, \mathbf{a}) = \underline{E}_Z(T, \mathbf{a}) = 1.$$

Thus $\mathbf{a}$ is the desired "built-in" scaled sequence. Moreover, if $\mathcal{U}'$ is a cover of X that is finer than $\mathcal{V}$ and if $b(n) = b(n, \mathcal{U}') = \log \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}', Z)$ satisfies (2.5), then

$$\overline{E}_Z(T, \mathbf{b}) = \underline{E}_Z(T, \mathbf{b}) = 1.$$

By Theorem 2.8, for any pair of open covers $\mathcal{U}$ and $\mathcal{U}'$ of X, which are finer than the generating open cover $\mathcal{V}$, if the corresponding sequences $a(n, \mathcal{U})$ and $b(n, \mathcal{U}')$ satisfy (2.5), then $a(n, \mathcal{U})$ and $b(n, \mathcal{U}')$ are equivalent scaled sequences and hence, their scaled entropy is positive and finite.

Example 4.2 We present an example of a map T whose standard topological entropy $E_X(T, \mathbf{b}) = 0$ (where $\mathbf{b} = \{n\}$ is the standard scaled sequence) but its scaled topological entropy $E_X(T, \mathbf{a}) > 0$ for the polynomial scaled sequence $\mathbf{a} = \{a(n) = n^\alpha\}$.

Consider the the full shift T on the one-sided symbolic space $X = \Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$. Given an infinite word $\omega \in X$, let

$$\mathcal{I}_n = \Big\{[x_1x_2\dots x_n]\colon x_1x_2\dots x_n \text{ occurs in } \omega\Big\} \text{ and } Z_n = \bigcup_{I\in\mathcal{I}_n} I,$$

where $[x_1x_2\dots x_n]$ is a cylinder of length n. The set

$$Z = \bigcap_{n=1}^{\infty} Z_n$$

is a closed subset of X and is T-invariant, so that (Z, T) is a subshift.

Let us fix a number $0 < \alpha < 1$. By a result in [4], there is $\omega \in X$ whose complexity function $L_\omega(m)$ satisfies $L_\omega(n) \sim 2^{n^\alpha}$. Recall that $L_\omega(n)$ is the number of finite words of length n that occurs as blocks of consecutive letters in ω.

Let $p(n)$ be the number of n-cylinders in Z. We have that $p(n) \sim 2^{n^\alpha}$ and by Theorem 3.4 in [5],

$$\overline{E}_Z(T, \{n^\alpha\}) = \limsup_{n\to\infty} \frac{1}{n^\alpha} \log p(n) = \log 2.$$

By the choice of α, the standard topological entropy $E_Z(T, \{n\}) = 0$. Since Z is compact and invariant, the sequence $\{n^\alpha\}_{n\geq 1}$ is subadditive, and by Theorem 2.6 we obtain that

$$E_Z(T, \{n^\alpha\}) = \underline{E}_Z(T, \{n^\alpha\}) = \overline{E}_Z(T, \{n^\alpha\}) = \log 2 > 0.$$

Finally, by Theorem 2.8, the class $[\{n^\alpha\}]$ is the only equivalence class whose scaled sequences generate positive and finite scaled topological entropy.

Example 4.3 For any fixed number $0 < \alpha < 1$, let (Z, T) be the subshift constructed in Example 4.2. Then the system (Z, T) is uniquely ergodic, i.e., there exists only one T-invariant measure μ supported on Z (see [1, Theorem 3.6] for the proof). Let ξ be the generating partition of (Z, T) induced by the zero coordinate, i.e., $\xi = \{A_0, A_1\}$ with $A_0 = \{x \in Z : x_0 = 0\}$ and $A_1 = \{x \in Z : x_0 = 1\}$, and let $C_{\xi_n}(x)$ denote the element of the refined partition $\bigvee_{i=0}^{n-1} T^{-1}\xi$ which contains the point x. Then, for any $\tau > 0$ we have

$$\lim_{n\to\infty} -\frac{1}{n^\tau} \log \mu(C_{\xi_n}(x)) = 0, \ \mu\text{-a.e. } x \in Z,$$

see Proposition 4.2 in [1] for the proof. This implies that

$$\overline{h}_\mu(T, \{n^\tau\}) = 0$$

for any $\tau > 0$. By Theorem 3.3, we obtain that

$$\begin{aligned}\underline{h}_\mu(T, \{n^\tau\}) &= E_\mu(T, \{n^\tau\}) = \underline{E}_\mu(T, \{n^\tau\}) \\ &= \overline{E}_\mu(T, \{n^\tau\}) = \overline{h}_\mu(T, \{n^\tau\}) = 0.\end{aligned}$$

Therefore,

$$\sup_\nu E_\nu(T, \{n^\alpha\}) = E_\mu(T, \{n^\alpha\}) = 0 < \log 2 = E_Z(T, \{n^\alpha\}).$$

where the supremum is taken over all T-invariant measures (in our case the set of such measures is reduced to the measure μ). This illustrates that the *variational principle for the scaled topological entropy* may fail in general (while it holds for the standard scaled sequence).

Example 4.4 Given a number $s > 1$, there exists $K > 1$ such that $s - 1 - \log(t + K) < 0$ for all $t > 0$. Consider a scaled sequence $\mathbf{a} = \{a(n)\}$ given by $a(n) = \big(\log(n + K)\big)^s$. It is easy to check that the function $a(t) = \big(\log(t + K)\big)^s$ satisfies the following conditions

(1) $\lim_{t\to+\infty} \frac{\log t}{a(t)} = 0$;
(2) a is differentiable, except possibly at 0;
(3) $\lim_{t\to+\infty} a'(t)t^\beta = 0$ for some positive constant $\beta > 0$;
(4) $a'(t)$ is decreasing.

By Theorem 3 in [4], there is a word $\omega \in \Sigma_2$ whose complexity function $L_\omega(n)$ satisfies

$$L_\omega(n) \sim 2^{a(n)}.$$

Let the sets Z_n, $Z \subset \Sigma_2$ be defined as in Example 4.2 and consider a subshift (Z, T). Denote by $p(n)$ the number of n-cylinders in Z. It is easy to see from the construction of Z that $p(n) \sim 2^{a(n)}$. Using similar arguments as in the proof of Theorem 3.4 in [5], one can show that

$$\overline{E}_Z(\sigma, \mathbf{a}) = \limsup_{n\to\infty} \frac{1}{\big(\log(n+K)\big)^s} \log p(n) = \log 2,$$

and

$$\underline{E}_Z(\sigma, \mathbf{a}) = \liminf_{n\to\infty} \frac{1}{\big(\log(n+K)\big)^s} \log p(n) = \log 2.$$

Therefore, for each $s > 1$ one can choose $K > 1$ such that the map T has positive and finite lower and upper scaled topological entropy with respect to the scaled sequence $\{\big(\log(n+K)\big)^s\}$. This is a "built-in" sequence for T which is neither polynomial, nor logarithmic. Moreover, one can see that

(i) $\overline{E}_Z(\sigma, \mathbf{a}) = \underline{E}_Z(\sigma, \mathbf{a}) = \log 2$ despite that fact that the scaled sequence $a(n)$ is neither subadditive nor superadditive (compare with Theorem 2.6);
(ii) there are systems whose scaled topological entropy is zero with respect to the scaled sequence $\{n^\alpha\}$ for any $0 < \alpha < 1$ and it is infinite with respect to the scaled sequence $\{\log n\}$;
(iii) for each $s > 1$ there is a system with positive and finite scaled topological entropy with respect to any scaled sequence in the class $[(\log n)^s]$.

Example 4.5 We describe an example of a smooth dynamical system with zero standard topological entropy and positive scaled topological entropy. It also illustrates that the second statement of Theorem 2.6 may fail to be true when the scaled sequence is not subadditive and that the particular scaled topological entropy with respect to sequence $\{n^s\}$ fails to clarify the complexity of the systems.

Let (M, ω) be a 4-dimensional symplectic manifold and $H\colon M \to \mathbb{R}$ a smooth Hamiltonian function. We denote by X^H the associated vector field and by ϕ_H the associated Hamiltonian flow. We fix a (connected component of) a compact regular energy level $\mathcal{L}$ of H which is an orientable compact connected submanifold of dimension 3.

A first integral $F : M \to \mathbb{R}$ of the vector field X^H is said to be *nondegenerate in the Bott sense* on $\mathcal{L}$ if the critical points of $f := F|_{\mathcal{L}}$ form nondegenerate strict smooth submanifolds of $\mathcal{L}$, that is the Hessian $\partial^2 f$ of f is nondegenerate on the complementary subspaces to these submanifolds. Consider the restrictions of the vector field and the flow to $\mathcal{L}$, which we still denote by X^H and ϕ_H. The triple $(\mathcal{L}, \phi_H, f)$ is called a *nondegenerate Bott system*. It is proved in [9] and [16] that the critical submanifolds for f may only be circles, Lagrangian tori or Klein bottles. A non-degenerate Bott system $(\mathcal{L}, \phi_H, f)$ is said to be *dynamically coherent* if the critical circle for f are either elliptic or hyperbolic periodic orbits.

Let $(\mathcal{L}, \phi_H, f)$ be a nondegenerate dynamically coherent Bott system and $\mathbf{a} = \{a(n) = \log n\}$ a scaled sequence. Denote by ϕ_H^1 the time one map of the Hamiltonian flow ϕ_H. It is shown in [13] that

(1) $E_{\mathcal{L}}(\phi_H^1, \mathbf{a})$ is 0 or 1;
(2) $\overline{E}_{\mathcal{L}}(\phi_H^1, \mathbf{a})$ is 0, 1 or 2; moreover, $\overline{E}_{\mathcal{L}}(\phi_H^1, \mathbf{a}) = 2$ if and only if ϕ_H possesses a hyperbolic orbit.

See [13, Theorem 1] for the proof of Statement (1) and [13, Theorem 2] for the proof of Statement (2).

This implies that

(i) The scaled topological and lower and upper scaled topological entropies are zero with respect to the scaled sequence $\{n^s\}$ for any $0 < s \leq 1$, which means that the particular sequence $\{n^s\}$ fails to describe the complexity of the system $(\mathcal{L}, \phi_H, f)$;
(ii) If ϕ_H possesses a hyperbolic orbit, then $\underline{E}_{\mathcal{L}}(\phi_H^1, \mathbf{a}) < \overline{E}_{\mathcal{L}}(\phi_H^1, \mathbf{a})$. This demonstrates that Statement (2) of Theorem 2.6 may fail if the scaled sequence is not subadditive, and the equivalence class $[\mathbf{a}]$ is the only elements in $(\mathcal{A}, \preccurlyeq)$ (see Sect. 2.4) with positive and finite scaled topological entropy;
(iii) If ϕ_H possesses a hyperbolic orbit, then $\overline{E}_{\mathcal{L}}(\phi_H^1, \mathbf{a}) = 2$. However, $E_{\mathcal{L}}(\phi_H^1, \mathbf{a})$ may be zero. This means that the scaled equivalence class such that the scaled topological entropy and upper scaled topological entropy is positive and finite may be not the same in general.

Example 4.6 Let X be a compact metric space with metric d and T an isometry, that is $d(T(x), T(y)) = d(x, y)$ for any $x, y \in X$. It is easy to see that for any finite open cover $\mathcal{U}$ of X and any $n \in \mathbb{N}$ there is a constant $K > 0$ such that

$$\aleph\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, X\Big) < K.$$

This implies that the system (X, T) has zero scaled topological entropy with respect to any scaled sequence.

Example 4.7 Let T be a rotation by an irrational number θ on $[0, 1)$ and let μ be the Lebesgue measure. Denote by $\mathcal{P}_n$ the partition of $[0, 1)$ generated by the orbit $\{-k\theta\}$, $0 \leq k \leq n$ that is $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$, where the partition $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 1)\}$. Further, for $x \in [0, 1)$, we denote by $\mathcal{P}_n(x)$ the element of the partition $\mathcal{P}_n$ that contains x. For an irrational number $\theta \in (0, 1)$, set

$$\eta := \sup\{t > 0 : \liminf_{j\to\infty} j^t \|j\theta\| = 0\},$$

where $\|j\theta\|$ denotes the distance to the nearest integer. By Theorem 1.1 in [10], for a given irrational number θ, the following properties hold:

(i) For each $x \in [0, 1)$,

$$\frac{1}{\eta} \leq \liminf_{n\to\infty} -\frac{\log \mu(\mathcal{P}_n(x))}{\log n} \leq 1 \text{ and } 1 \leq \limsup_{n\to\infty} -\frac{\log \mu(\mathcal{P}_n(x))}{\log n} \leq \eta;$$

(ii) For almost all $x \in [0, 1)$,

$$\liminf_{n\to\infty} -\frac{\log \mu(\mathcal{P}_n(x))}{\log n} = \frac{1}{\eta} \text{ and } \limsup_{n\to\infty} -\frac{\log \mu(\mathcal{P}_n(x))}{\log n} = 1.$$

Now consider the full shift σ on the one-sided symbolic space Σ_2^+ and the map $\phi : \Sigma_2^+ \to [0, 1)$ given by $\phi(\omega) = \bigcap_{i=0}^{\infty} T^{-i} P_{\omega_i}$. This map determines a symbolic extension of the irrational rotation, where ω_i is the i-th symbol of the word ω and $P_0 = [0, 1-\theta)$, $P_1 = [1-\theta, 1)$. Thus, we obtain an ergodic shift invariant measure m on Σ_2^+ such that $\phi_* m = \mu$. Furthermore, $(\Sigma_2^+, \mathcal{B}_1, m, \sigma)$ is isomorphic via ϕ to $([0, 1), \mathcal{B}_2, \mu, T)$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are Borel sigma algebras on Σ_2^+ and $[0, 1)$ respectively.

Let ξ be the natural partition of (Σ_2^+, σ) into 1-cylinders, i.e., $\xi = \{A_0, A_1\}$ with $A_0 = \{\omega \in \Sigma_2^+ : \omega_0 = 0\}$ and $A_1 = \{\omega \in \Sigma_2^+ : \omega_0 = 1\}$. The isomorphism ϕ maps this partition onto $\mathcal{P}$. We denote by $C_{\xi_n}(\omega)$ the element of the refined partition $\bigvee_{i=0}^{n-1} \sigma^{-1}\xi$ which contains the point ω.

If θ is a Liouville number, then $\eta = \infty$ and by (ii) we obtain that for m-almost every $\omega \in \Sigma_2^+$,

$$\liminf_{n\to\infty} -\frac{\log m(C_{\xi_n}(\omega))}{\log n} = 0 \text{ and } \limsup_{n\to\infty} -\frac{\log m(C_{\xi_n}(\omega))}{\log n} = 1.$$

This means that the scaled version of Brin and Katok's entropy formula fails. On the other hand, since $\eta = 1$ for almost every θ, we conclude that by (i),

$$\lim_{n\to\infty} -\frac{\log m(C_{\xi_n}(\omega))}{\log n} = 1$$

for m-almost every ω. Hence, the scaled version of Brin and Katok's entropy formula holds in this case.

5 Proofs

Proof of Proposition 2.2 (1) Since $\mathcal{U} \preceq \mathcal{V}$, each element $V \in \mathcal{V}$ is contained in some element in $\mathcal{U}$ which we denote by $U(V)$. Therefore, for each string $\mathbf{V} = (V_{i_0}, V_{i_1}, \dots, V_{i_{n-1}}) \in \mathcal{W}_n(\mathcal{V})$ there exists a corresponding string $\mathbf{U}(\mathbf{V}) = (U(V_{i_0}), U(V_{i_1}), \dots, U(V_{i_{n-1}})) \in \mathcal{W}_n(\mathcal{U})$. This yields that

$$\aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, Z) \le \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{V}, Z)$$

and hence,

$$\underline{E}_Z(T,\mathcal{U},\mathbf{a}) \le \underline{E}_Z(T,\mathcal{V},\mathbf{a}) \text{ and } \overline{E}_Z(T,\mathcal{U},\mathbf{a}) \le \overline{E}_Z(T,\mathcal{V},\mathbf{a}).$$

Let $\Gamma \subset \mathcal{W}(\mathcal{V})$ be a collection of strings that covers Z. The corresponding collection of strings $\{\mathbf{U}(\mathbf{V}) : \mathbf{V} \in \Gamma\} \subset \mathcal{W}(\mathcal{U})$ also covers Z. This implies that $M(Z, \alpha, N, \mathcal{U}, \mathbf{a}) \le M(Z, \alpha, N, \mathcal{V}, \mathbf{a})$ for each $\alpha \ge 0$ and $N > 0$ and hence, $m(Z, \alpha, \mathcal{U}, \mathbf{a}) \le m(Z, \alpha, \mathcal{V}, \mathbf{a})$. The first statement follows.

(2) Since $\mathcal{U} \preceq \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$, the inequality (2.4) follows.
Fix $k > l$ and let $\mathcal{V} = \bigvee_{i=0}^{l-1} T^{-i}\mathcal{U}$. For $\mathbf{U} = (U_{i_0}, U_{i_1}, \dots, U_{i_{k-1}}) \in \mathcal{W}_k(\mathcal{U})$ let $\mathbf{V}(\mathbf{U}) = (V_{i_0}, V_{i_1}, \dots, V_{i_{k-l}}) \in \mathcal{W}_{k-l+1}(\mathcal{V})$ be the corresponding string. Clearly, $X(\mathbf{U}) = X(\mathbf{V}(\mathbf{U}))$ and hence,

$$\aleph(\bigvee_{i=0}^{k-l+1} T^{-i}\mathcal{V}, Z) \le \aleph(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{U}, Z).$$

The requirement that $\lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 1$ implies that $\mathcal{E}_Z(T, \mathcal{V}, \mathbf{a}) \le \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a})$, here $\mathcal{E}$ is $\underline{E}$ or $\overline{E}$.
On the other hand, if $\Gamma \subset \bigcup_{j\ge k} \mathcal{W}_j(\mathcal{U})$ covers the set Z, then the collection of strings $\{\mathbf{V}(\mathbf{U}) : \mathbf{U} \in \Gamma\} \in \bigcup_{j\ge k-l+1} \mathcal{W}_j(\mathcal{V})$ also covers Z. Using again the requirement

$\lim_{n\to\infty}\frac{a(n)}{a(n+1)}=1$, for each $\beta>0$ we can find $N'>0$ such that for each $\mathbf{U}$ with $m(\mathbf{U})\geq N'$

$$1-\beta<\frac{a(m(\mathbf{U}))}{a(m(\mathbf{U})-l+1)}=\prod_{i=1}^{l-1}\frac{a(m(\mathbf{U})-l+i+1)}{a(m(\mathbf{U})-l+i)}<1+\beta.$$

Therefore, by Remark 2.1,

$$M(Z,\alpha,N,\mathcal{U},\mathbf{a})\geq M(Z,\alpha(1+\beta),N-l+1,\mathcal{V},\mathbf{a})$$

for all sufficiently large N. Letting $N\to\infty$ on both sides of the above inequality yields that $m(Z,\alpha,\mathcal{U},\mathbf{a})\geq m(Z,\alpha(1+\beta),\mathcal{V},\mathbf{a})$. Therefore,

$$E_Z(T,\mathcal{U},\mathbf{a})(1+\beta)\geq E_Z(T,\mathcal{V},\mathbf{a}).$$

The arbitrariness of β implies the third statement.

(3) The last statement follows immediately from the definitions.

□

Proof of Proposition 2.4 Statements (1) and (2) follow from Theorems 1.1 and 2.1 in [18]. To prove Statement (3) we apply Proposition 2.3 with $X_i=X$, $\pi=T$ and $T_i=T$, $i=1,2$ and obtain that $\mathcal{E}_Z(T,T^{-1}\mathcal{U},\mathbf{a})=\mathcal{E}_{T(Z)}(T,\mathcal{U},\mathbf{a})$ and, consequently

$$\mathcal{E}_Z(T,\mathbf{a})\geq\mathcal{E}_{T(Z)}(T,\mathbf{a}). \tag{5.1}$$

On the other hand, let $\Gamma\subset\mathcal{W}(\mathcal{U})$ be a collection of strings that covers $T(Z)$, i.e.,

$$T(Z)\subset\bigcup_{\mathbf{U}\in\Gamma}\big(U_{i_0}\cap T^{-1}U_{i_1}\cap\cdots\cap T^{-m(\mathbf{U})+1}U_{i_{m(\mathbf{U})-1}}\big).$$

Then

$$Z\subset\bigcup_{\mathbf{U}\in\Gamma}\big(T^{-1}U_{i_0}\cap T^{-2}U_{i_1}\cap\cdots\cap T^{-m(\mathbf{U})}U_{i_{m(\mathbf{U})-1}}\big).$$

Since X is compact, we can choose a finite subcover $\{U_1,\ldots,U_k\}$ of $\mathcal{U}$ that covers Z. Thus

$$Z\subset\bigcup_{\mathbf{U}\in\Gamma}\bigcup_{j=1}^{k}\big(U_j\cap T^{-1}U_{i_0}\cap T^{-2}U_{i_1}\cap\cdots\cap T^{-m(\mathbf{U})}U_{i_{m(\mathbf{U})-1}}\big).$$

This implies that

$$\aleph\Big(\bigvee_{i=0}^{N}T^{-i}\mathcal{U},Z\Big)\leq k\,\aleph\Big(\bigvee_{i=0}^{N-1}T^{-i}\mathcal{U},T(Z)\Big) \tag{5.2}$$

and together with the monotonicity of $a(n)$ yields that

$$M(Z,\alpha,N+1,\mathcal{U},\mathbf{a})\leq kM(T(Z),\alpha,N,\mathcal{U},\mathbf{a}). \tag{5.3}$$

Letting $N\to\infty$ on both sides of (5.3) yields that

$$m(Z,\alpha,\mathcal{U},\mathbf{a})\leq k\,m(T(Z),\alpha,\mathcal{U},\mathbf{a})$$

and hence,

$$E_Z(T,\mathcal{U},\mathbf{a})\leq E_{T(Z)}(T,\mathcal{U},\mathbf{a}). \tag{5.4}$$

This implies that

$$E_Z(T,\mathbf{a}) \le E_{T(Z)}(T,\mathbf{a}). \tag{5.5}$$

By (5.1) and (5.5), we have $E_Z(T,\mathbf{a}) = E_{T(Z)}(T,\mathbf{a})$. To prove Statement (4), by Proposition 2.1, (2.5) and (5.2) we have that

$$\underline{E}_Z(T,\mathcal{U},\mathbf{a}) \le \underline{E}_{T(Z)}(T,\mathcal{U},\mathbf{a}), \quad \overline{E}_Z(T,\mathcal{U},\mathbf{a}) \le \overline{E}_{T(Z)}(T,\mathcal{U},\mathbf{a}). \tag{5.6}$$

Hence,

$$\underline{E}_Z(T,\mathbf{a}) \le \underline{E}_{T(Z)}(T,\mathbf{a}) \text{ and } \overline{E}_Z(T,\mathbf{a}) \le \overline{E}_{T(Z)}(T,\mathbf{a}). \tag{5.7}$$

The above inequalities together with (5.1) yield that

$$\underline{E}_Z(T,\mathbf{a}) = \underline{E}_{T(Z)}(T,\mathbf{a}), \quad \overline{E}_Z(T,\mathbf{a}) = \overline{E}_{T(Z)}(T,\mathbf{a}).$$

To prove the other results, we let $\mathcal{G} \subset \mathcal{W}(\mathcal{U})$ be a collection of strings that covers Z, i.e.,

$$Z \subset \bigcup_{\mathbf{U}\in\mathcal{G}} \left(U_{i_0} \cap T^{-1}U_{i_1} \cap \cdots \cap T^{-m(\mathbf{U})+1}U_{i_{m(\mathbf{U})-1}}\right).$$

Hence,

$$T(Z) \subset \bigcup_{\mathbf{U}\in\mathcal{G}} \left(U_{i_1} \cap \cdots \cap T^{-m(\mathbf{U})+2}U_{i_{m(\mathbf{U})-1}}\right).$$

This implies that

$$\aleph\Big(\bigvee_{i=0}^{N-2} T^{-i}\mathcal{U}, T(Z)\Big) \le \aleph\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{U}, Z\Big). \tag{5.8}$$

Using (2.5), for each $\beta > 0$ there exits $N' > 0$ such that

$$1-\beta < \frac{a(n+1)}{a(n)} < 1+\beta \quad \text{for all } n \ge N'.$$

Hence,

$$M(T(Z), \alpha(1+\beta), N-1, \mathcal{U}, \mathbf{a}) \le M(Z, \alpha, N, \mathcal{U}, \mathbf{a}) \tag{5.9}$$

for all sufficiently large N. Letting $N \to \infty$ in (5.9) yields

$$m(T(Z), \alpha(1+\beta), \mathcal{U}, \mathbf{a}) \le m(Z, \alpha, \mathcal{U}, \mathbf{a})$$

and hence, $E_{T(Z)}(T,\mathcal{U},\mathbf{a}) \le (1+\beta)E_Z(T,\mathcal{U},\mathbf{a})$. The arbitrariness of β implies $E_{T(Z)}(T,\mathcal{U},\mathbf{a}) \le E_Z(T,\mathcal{U},\mathbf{a})$ and hence, $E_{T(Z)}(T,\mathbf{a}) \le E_Z(T,\mathbf{a})$. This together with (5.4) and (5.5) yield that $E_{T(Z)}(T,\mathcal{U},\mathbf{a}) = E_Z(T,\mathcal{U},\mathbf{a})$ and $E_{T(Z)}(T,\mathbf{a}) = E_Z(T,\mathbf{a})$. Finally, by (2.5) and (5.8) we obtain that $\mathcal{E}_{T(Z)}(T,\mathcal{U},\mathbf{a}) \le \mathcal{E}_Z(T,\mathcal{U},\mathbf{a})$ and hence, $\mathcal{E}_{T(Z)}(T,\mathbf{a}) \le \mathcal{E}_Z(T,\mathbf{a})$ (here $\mathcal{E}$ denotes $\underline{E}$ or $\overline{E}$). Combing these two inequalities and (5.6) and (5.7), yield the desired results. This completes the proof of the proposition. □

Proof of Theorem 2.6 Observe that by Proposition 2.2 it suffices to show that $E_Z(T,\mathcal{U},\mathbf{a}) \ge \overline{E}_Z(T,\mathcal{U},\mathbf{a})$. Choose $\alpha > E_Z(T,\mathcal{U},\mathbf{a})$. There are $N > 0$ and $\mathcal{G} \in \bigcup_{m\ge N} \mathcal{W}_m(\mathcal{U})$ such that $\mathcal{G}$ covers Z and

$$A(\mathcal{G}) := \sum_{\mathbf{U}\in\mathcal{G}} \exp\Big(-\alpha a(m(\mathbf{U}))\Big) < 1.$$

Since Z is compact, we can choose $\mathcal{G}$ to be finite and hence, $\mathcal{G} \subset \bigcup_{m=1}^{M} \mathcal{W}_m(\mathcal{U})$ for some $M \geq 1$. Set $\mathcal{G}^n = \{\mathbf{U}_1 \ldots \mathbf{U}_n : \mathbf{U}_i \in \mathcal{G}\}$ and $\Gamma = \bigcup_{n=1}^{\infty} \mathcal{G}^n$. Since Z is invariant, Γ covers Z. It is easy to check that $A(\mathcal{G}^n) \leq A(\mathcal{G})^n$ and hence,

$$A(\Gamma) = \sum_{n=1}^{\infty} A(\mathcal{G}^n) < \infty.$$

Let us fix $K > 0$ and a point $x \in Z$. Since Γ covers Z there is a string $\mathbf{U} \in \Gamma$ such that $x \in X(\mathbf{U})$ and $K \leq m(\mathbf{U}) < K + M$. Denote by $\mathbf{U}^*$ the substring that consists of the first K symbols of the string $\mathbf{U}$. Let Γ_K denote the collection of all substrings $\mathbf{U}^*$ constructed above. It is easy to see that

$$\sharp\Gamma_K \geq \aleph\Big(\bigvee_{i=0}^{K-1} T^{-i}\mathcal{U}, Z\Big).$$

It follows that

$$\exp\big(-\alpha a(K)\big)\aleph\Big(\bigvee_{i=0}^{K-1} T^{-i}\mathcal{U}, Z\Big) \leq \sum_{\mathbf{U}^* \in \Gamma_K} \exp(-\alpha a(K)).$$

Since $\mathbf{a} = \{a(n)\}$ is a subadditive scaled sequence, we find that

$$a(m(\mathbf{U})) \leq a(K) + a(m(\mathbf{U}) - K) \leq a(K) + a(M).$$

Therefore,

$$\sum_{\mathbf{U}^* \in \Gamma_K} \exp(-\alpha a(K)) \leq \exp[\alpha a(M)] A(\Gamma) < \infty.$$

By Proposition 2.1,

$$\overline{E}_Z\Big(T, \mathcal{U}, \mathbf{a}\Big) = \limsup_{K \to \infty} \frac{1}{a(K)} \log \aleph(\bigvee_{i=0}^{K-1} T^{-i}\mathcal{U}, Z) < \alpha.$$

Since α is arbitrary, we conclude that $\overline{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq E_Z(T, \mathcal{U}, \mathbf{a})$. Hence,

$$E_Z(T, \mathcal{U}, \mathbf{a}) = \underline{E}_Z(T, \mathcal{U}, \mathbf{a}) = \overline{E}_Z(T, \mathcal{U}, \mathbf{a})$$

implying that $E_Z(T, \mathbf{a}) = \underline{E}_Z(T, \mathbf{a}) = \overline{E}_Z(T, \mathbf{a})$. □

Proof of Theorem 2.8 We shall prove the result for the scaled topological entropy $E_Z(T, [\mathbf{a}])$; the arguments for the lower and upper scaled topological entropies are similar.

Suppose there is $[\mathbf{a}] \in \mathcal{A}$ such that $E_Z(T, [\mathbf{a}])$ is positive and finite. Then for each $[\mathbf{b}] \succ [\mathbf{a}]$,

$$\limsup_{n \to \infty} \frac{a_1(n)}{b_1(n)} = 0$$

for arbitrary $\mathbf{a}_1 = \{a_1(n)\} \in [\mathbf{a}]$ and $\mathbf{b}_1 = \{b_1(n)\} \in [\mathbf{b}]$. Let us fix such two scaled sequences $\mathbf{a}_1$ and $\mathbf{b}_1$. Given a small number $\beta > 0$, for all sufficiently large n we have that $a_1(n) < \beta b_1(n)$ and hence, $m(Z, \alpha, \mathcal{U}, \mathbf{a}_1) \geq m(Z, \alpha, \mathcal{U}, \beta\mathbf{b}_1)$. This implies that

$$E_Z(T, \mathbf{a}_1) \geq E_Z(T, \beta\mathbf{b}_1) = \frac{1}{\beta} E_Z(T, \mathbf{b}_1),$$

i.e., $\beta E_Z(T, \mathbf{a}_1) \geq E_Z(T, \mathbf{b}_1)$. Since β is arbitrary, we conclude that $E_Z(T, \mathbf{b}_1) = 0$ and hence, $E_Z(T, [\mathbf{b}]) = 0$.

On the other hand, if $[\mathbf{b}] \preccurlyeq [\mathbf{a}]$ then

$$\limsup_{n\to\infty} \frac{b_2(n)}{a_2(n)} = 0$$

for arbitrary $\mathbf{a}_2 = \{a_2(n)\} \in [\mathbf{a}]$ and $\mathbf{b}_2 = \{b_2(n)\} \in [\mathbf{b}]$. Given a small number $\beta > 0$, for all sufficiently large n we have that $b_2(n) < \beta a_2(n)$ and hence, $m(Z, \alpha, \mathcal{U}, \mathbf{b}_2) \geq m(Z, \alpha, \mathcal{U}, \beta\mathbf{a}_2)$. It follows that $E_Z(T, \mathbf{b}_2) > \frac{1}{\beta} E_Z(T, \mathbf{a}_2)$. Again since β is arbitrary, $E_Z(T, \mathbf{b}_2) = \infty$ implying that $E_Z(T, [\mathbf{b}]) = \infty$. □

Proof of Proposition 2.9 It follows from the definitions that

$$\liminf_{|\mathcal{U}|\to 0} \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq \limsup_{|\mathcal{U}|\to 0} \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a}) \leq \mathcal{E}_Z(T, \mathbf{a}). \tag{5.10}$$

On the other hand, if $\mathcal{V}$ is a finite open cover of X with the Lebesgue number δ and $\mathcal{U}$ an open cover of X with $|\mathcal{U}| < \delta$, then $\mathcal{V} \preceq \mathcal{U}$. By Proposition 2.2 (1), we obtain that $\mathcal{E}_Z(T, \mathcal{V}, \mathbf{a}) \leq \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a})$. This implies that

$$\mathcal{E}_Z(T, \mathbf{a}) \leq \liminf_{|\mathcal{U}|\to 0} \mathcal{E}_Z(T, \mathcal{U}, \mathbf{a})$$

and together with (5.10) complete the proof of the second statement. □

Proof of Theorem 2.10 Let $\overline{\dim}_B Z := \overline{\alpha}$ and $\underline{\dim}_B Z := \underline{\alpha}$. By Lemma 2.1, for all small $r > 0$,

$$N_d(Z, \frac{r}{c_n}) \geq N_{d_n}(Z, r) \geq N_d(Z, \frac{r}{b_n}), \tag{5.11}$$

where $N_{d_n}(Z, r)$ is the minimal number of balls of radius r in the d_n-metric needed to cover the set Z.

Fix a small number $r > 0$. Since the sequence $\{b_n\}$ is scaled, so is the sequence $\{c_n\}$ and hence, $\frac{r}{c_n} \to 0$. Moreover, since the sequence $\{\log c_n\}$ satisfies (2.5), we have that

$$\frac{\log \frac{r}{c_n}}{\log \frac{r}{c_{n+1}}} \to 1$$

as $n \to \infty$.

Claim. *If a sequence of numbers* $\{b_n\}$ *is such that* $b_n \to 0$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \frac{\log b_n}{\log b_{n+1}} = 1,$$

then

$$\underline{\dim}_B Z = \liminf_{n\to\infty} \frac{\log N_d(Z, b_n)}{-\log b_n}, \quad \overline{\dim}_B Z = \limsup_{n\to\infty} \frac{\log N_d(Z, b_n)}{-\log b_n}.$$

□

Proof of Claim For each $r > 0$ there exists a positive integer n such that $b_{n+1} \leq r < b_n$. It follows that

$$N_d(Z, b_n) \leq N_d(Z, r) \leq N_d(Z, b_{n+1}).$$

This yields that

$$\frac{\log N_d(Z, b_n)}{-\log b_{n+1}} \leq \frac{\log N_d(Z, r)}{-\log r} \leq \frac{\log N_d(Z, b_{n+1})}{-\log b_n}.$$

The desired result now immediately follows from the requirement that $\lim_{n\to\infty} \frac{\log b_n}{\log b_{n+1}} = 1$. □

It follows from the claim that

$$\limsup_{n\to\infty} \frac{\log N_d\left(Z, \frac{r}{c_n}\right)}{-\log \frac{r}{c_n}} = \overline{\alpha}.$$

Therefore, for each $\epsilon > 0$ there exists $K > 0$ such that for all $n > K$

$$N_d(Z, \frac{r}{c_n}) < \left(\frac{r}{c_n}\right)^{-(\overline{\alpha}+\epsilon)}.$$

Combining this with the first inequality in (5.11), we find that for all $n > K$,

$$N_{d_n}(Z, r) < \left(\frac{r}{c_n}\right)^{-(\overline{\alpha}+\epsilon)}.$$

It follows that

$$\limsup_{n\to\infty} \frac{\log N_{d_n}(Z, r)}{\log c_n} \leq \overline{\alpha} + \epsilon. \tag{5.12}$$

Since r and ϵ are arbitrary, by Remark 2.3 we conclude that

$$\overline{E}_Z(T, \{\log c_n\}) < \overline{\alpha}.$$

The second inequality can be proven in a similar fashion. First note that

$$\liminf_{n\to\infty} \frac{\log N_d\left(Z, \frac{r}{b_n}\right)}{-\log \frac{r}{b_n}} = \underline{\alpha}.$$

Hence, for each $\epsilon > 0$ there exists $K' > 0$ such that for all $n > K'$

$$N_d\left(Z, \frac{r}{b_n}\right) \geq \left(\frac{r}{b_n}\right)^{-\underline{\alpha}+\epsilon}.$$

Combining this with the second inequality in (5.11), we obtain that for all $n > K'$,

$$N_{d_n}(Z, r) \geq (\frac{r}{b_n})^{-\underline{\alpha}+\epsilon}.$$

It follows that

$$\liminf_{n\to\infty} \frac{\log N_{d_n}(Z, r)}{\log b_n} \geq \underline{\alpha} - \epsilon$$

and since r and ϵ are arbitrary, by remark 2.3 we conclude that $\underline{E}_Z(T, \{\log b_n\}) > \underline{\alpha}$. □

Proof of Proposition 3.2 By Proposition 2.3, we know that

$$\begin{aligned} E_\mu(T_1, \pi^{-1}\mathcal{U}, \mathbf{a}) &= \inf\left\{E_Z(T_1, \pi^{-1}\mathcal{U}, \mathbf{a}) : \mu(Z) = 1\right\} \\ &= \inf\left\{E_{\pi(Z)}(T_2, \mathcal{U}, \mathbf{a}) : \mu(Z) = 1\right\} \end{aligned}$$

$$= \inf\left\{E_Y(T_2, \mathcal{U}, \mathbf{a}) : \pi_*\mu(Y) = 1\right\}$$
$$= E_{\pi_*\mu}(T_2, \mathcal{U}, \mathbf{a})$$

the third equality follows from the fact that π is a homeomorphism. Hence,

$$E_\mu(T_1, \mathbf{a}) \geq E_{\pi_*\mu}(T_2, \mathbf{a}).$$

Since $\pi^{-1} \circ T_2 = T_1 \circ \pi^{-1}$, applying the above inequality, we find that $E_{\pi_*\mu}(T_2, \mathbf{a}) \geq E_{\pi_*^{-1}\pi_*\mu}(T_1, \mathbf{a}) = E_\mu(T_1, \mathbf{a})$ and hence, $E_\mu(T_1, \mathbf{a}) = E_{\pi_*\mu}(T_2, \mathbf{a})$. The other two equalities for $\underline{E}_\mu$ and $\overline{E}_\mu$ can be proved in a similar fashion. □

Proof of Theorem 3.3 By Lemma 3.1, the quantities $\underline{h}_\mu(T, \mathbf{a})$ and $\overline{h}_\mu(T, \mathbf{a})$ are well-defined. Since $E_\mu(T, \mathbf{a}) \leq \underline{E}_\mu(T, \mathbf{a}) \leq \overline{E}_\mu(T, \mathbf{a})$, it suffices to show that $\overline{E}_\mu(T, \mathbf{a}) \leq \overline{h}_\mu(T, \mathbf{a})$ and $E_\mu(T, \mathbf{a}) \geq \underline{h}_\mu(T, \mathbf{a})$.

We first prove that $\overline{E}_\mu(T, \mathbf{a}) \leq \overline{h}_\mu(T, \mathbf{a})$. We may assume that $\overline{h}_\mu(T, \mathbf{a})$ is finite. Set $\overline{h} = \overline{h}_\mu(T, \mathbf{a}) \geq 0$ and choose $\delta > 0$ and a small number $\eta > 0$. Let $\epsilon_\eta > 0$ be such that if $\epsilon \in (0, \epsilon_\eta]$, then for μ-almost every $x \in X$,

$$\limsup_{n\to\infty} -\frac{1}{a(n)} \log \mu(B_n(x, \epsilon)) \leq \overline{h} + \eta/2.$$

This is possible in view of Remark 3.1. It follows that for μ-almost every $x \in X$, there exists a number $N(x) > 0$ such that for any $n \geq N(x)$,

$$\frac{1}{a(n)} \log \mu(B_n(x, \epsilon/2)) + \overline{h} \geq -\eta. \tag{5.13}$$

Given a positive integer N, let $K_N = \{x \in X : N(x) \leq N\}$. We have that $K_N \subset K_{N+1}$, and $\bigcup_{N\geq 0} K_N$ is a set of full μ-measure. Therefore, one can find $N_0 > 0$ for which $\mu(K_{N_0}) > 1 - \delta$.

Fix a number $N > N_0$. Let E be a maximal (n, ϵ)-separated subset of K_N, i.e, E is a maximal subset satisfying that every two distinct points $x, y \in E$ imply that $d_n(x, y) > \epsilon$, then $K_N \subseteq \bigcup_{x\in E} B_n(x, \epsilon)$. Furthermore, the balls $\{B_n(x, \epsilon/2) : x \in E\}$ are pairwise disjoint and by (5.13), the cardinality of E is less than or equal to $\exp\big[a(n)(\overline{h} + \eta)\big]$. Let $\Lambda(Z, n, \epsilon)$ denote the smallest number of Bowen's balls $\{B_n(x, \epsilon)\}$ whose union covers the subset Z. For all sufficiently large n we have

$$\Lambda(K_N, n, \epsilon) \leq \exp\big[a(n)(\overline{h} + \eta)\big].$$

It follows that

$$\overline{E}_{K_N}(T, \epsilon, \mathbf{a}) := \limsup_{n\to\infty} \frac{1}{a(n)} \log \Lambda(K_N, n, \epsilon) \leq \overline{h} + \eta.$$

Since $\mu(K_N) \geq 1 - \delta$, we have

$$\overline{E}_\mu(T, \epsilon, \mathbf{a}) := \lim_{\delta\to 0} \inf\{\overline{E}_Z(T, \epsilon, \mathbf{a}) : \mu(Z) \geq 1 - \delta\} \leq \overline{h} + \eta.$$

Letting $\epsilon \to 0$ in the above inequality and taken into account that η can be arbitrary, we conclude that $\overline{E}_\mu(T, \mathbf{a}) \leq \overline{h}$.

We shall now prove that $E_\mu(T, \mathbf{a}) \geq \underline{h}_\mu(T, \mathbf{a})$. Set $\underline{h} = \underline{h}_\mu(T, \mathbf{a})$, and assume that $\underline{h} > 0$. It suffices to prove that $E_Z(T, \mathbf{a}) \geq \underline{h}$ for any subset $Z \subseteq X$ of full μ-measure. Choose $\eta > 0$ and $\delta \in (0, 1/2)$ and denote $\lambda = \underline{h} - \eta$. Let

$$K = \left\{x \in X : \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B_n(x, \epsilon))}{a(n)} = \underline{h}\right\}$$

and $K' = K \cap Z$. By Lemma 3.1 and Remark 3.1, we have $\mu(K) = 1$ and then $\mu(K') = 1$. There exists a positive number $\epsilon_\eta > 0$ such that for $\epsilon \in (0, \epsilon_\eta]$, one can find a set $K_1 \subset K'$ with $\mu(K_1) > 1 - \delta$ and a number $N_1 > 0$ such that for any $x \in K_1$ and $n \geq N_1$,

$$\mu(B_n(x, 2\epsilon)) \leq \exp\big[-a(n)(\underline{h} - \eta)\big].$$

We may assume further that K_1 is compact since otherwise we can approximate it from within by a compact subset. Take an open cover $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ of K_1 with $n_i \geq N_1$ for all i. Since K_1 is compact, we may assume that the cover is finite and consists of Bowen's balls $B_{n_1}(x_1, \epsilon), \dots, B_{n_l}(x_l, \epsilon)$.

For each $i = 1, \dots, l$, we choose $y_i \in K_1 \cap B_{n_i}(x_i, \epsilon)$. Hence, $B_{n_i}(x_i, \epsilon) \subset B_{n_i}(y_i, 2\epsilon)$, and $\{B_{n_i}(y_i, 2\epsilon)\}_i$ form an open cover of K_1 as well. Now we have

$$\begin{aligned}\sum_{B_{n_i}(x_i,\epsilon)\in\Gamma} \exp\big(-\lambda a(n_i)\big) &\geq \sum_{i=1}^{l} \exp\big(-\lambda a(n_i)\big) = \sum_{i=1}^{l} \exp\big(-(\underline{h} - \eta) a(n_i)\big) \\ &\geq \sum_{i=1}^{l} \mu(B_{n_i}(y_i, 2\epsilon)) \geq 1 - \delta.\end{aligned}$$

Since the inequality holds for any cover $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ of K_1, we conclude that $M(K_1, \lambda, N, \epsilon, \mathbf{a}) \geq 1 - \delta$. Hence, $m(K_1, \lambda, \epsilon, \mathbf{a}) \geq 1 - \delta$. This implies that

$$E_{K_1}(T, \epsilon, \mathbf{a}) \geq \lambda = \underline{h} - \eta, \text{ and } E_{K_1}(T, \mathbf{a}) \geq \underline{h} - \eta.$$

Using Proposition 2.4 and the fact that η is arbitrary, we find that

$$E_Z(T, \mathbf{a}) \geq E_{K_1}(T, \mathbf{a}) \geq \underline{h}. \tag{5.14}$$

So by definition, $E_\mu(T, \mathbf{a}) \geq \underline{h}$.

In the case when $\underline{h}_\mu(T, \mathbf{a}) = +\infty$, one can slightly modify the argument in the proof of the second inequality to obtain that $E_\mu(T, \mathbf{a}) = +\infty$. This completes the proof of the theorem. □

Proof of Theorem 3.4 Fix $h \geq 0$ and set $K = K_h$. Fix now a small number $\eta > 0$ and choose ϵ_η as in the proof of Theorem 3.3. Since

$$\lim_{\epsilon\to 0} \underline{h}_\mu(T, \mathbf{a}, x, \epsilon) = \lim_{\epsilon\to 0} \overline{h}_\mu(T, \mathbf{a}, x, \epsilon) = h$$

for all $x \in K$ and $\mu(K) = 1$, for some $\epsilon \in (0, \epsilon_\eta]$ and μ-almost every $x \in X$ there exists a number $N(x) > 0$ such that for any $n \geq N(x)$

$$\left| \frac{1}{a(n)} \log \mu(B_n(x, \epsilon/2)) + h \right| \leq \eta.$$

Given a positive integer $N > 0$, set $K_N = \{x \in K : N(x) \leq N\}$. We have $K_N \subset K_{N+1}$ and $\bigcup_{N\geq 0} K_N = K$. Hence, given $\delta > 0$, we can find $N_0 > 0$ for which $\mu(K_{N_0}) > 1 - \delta$.

Fix a number $N \geq N_0$, as in the proof of Theorem 3.3. We have that $\overline{E}_{K_N}(T, \epsilon, \mathbf{a}) \leq h + 2\eta$. Letting $\epsilon \to 0$ and taken into account that η is arbitrary, we obtain that $\overline{E}_{K_N}(T, \mathbf{a}) \leq h$. Letting $N \to \infty$, we conclude that $\overline{E}_K(T, \mathbf{a}) \leq h$.

The inequality $E_K(T, \mathbf{a}) \geq h$ is contained in (5.14) since Z is an arbitrary set of full μ-measure and in our case $\mu(K) = 1$. The desired result follows now from Theorem 3.3 and Proposition 2.2. □

Proof of Theorem 3.6 For a fixed $r > 0$ and $k \in \mathbb{N}$, let

$$L_k = \left\{x \in L : \liminf_{n\to\infty} \frac{-\log \mu(B_n(x,\epsilon))}{a(n)} < s + r \text{ for all } \epsilon \in \left(0, \frac{1}{k}\right)\right\}.$$

Then we have $L = \bigcup_{k=1}^{\infty} L_k$, since $\underline{h}_\mu(T, \mathbf{a}, x) \le s$ for all $x \in L$.

Now fix $k \ge 1$ and $0 < \epsilon < \frac{1}{5k}$. For each $x \in L_k$, there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ (depending on the point x) such that

$$\mu(B_{n_j}(x,\epsilon)) \ge \exp\big(-a(n_j)(s+r)\big) \quad \text{for all } j \ge 1.$$

For any $N \ge 1$, the set L_k is contained in the union of the sets in the family

$$\mathcal{F} = \left\{B_{n_j}(x,\epsilon) : x \in L_k, n_j \ge N\right\}.$$

By Lemma 1 in [15], there exists a subfamily $\mathcal{G} = \left\{B_{n_i}(x_i,\epsilon)\right\}_{i\in I} \subset \mathcal{F}$ of pairwise disjoint balls such that

$$L_k \subset \bigcup_{i\in I} B_{n_i}(x_i, 3\epsilon).$$

The subfamily is at most countable since μ is a probability measure and the elements in $\mathcal{G}$ are pairwise disjoint and have positive μ-measure. Note that

$$\mu(B_{n_i}(x_i,\epsilon)) \ge \exp\big(-a(n_i)(s+r)\big) \text{ for all } i \in I.$$

The disjointness of $\left\{B_{n_i}(x_i,\epsilon)\right\}_{i\in I}$ yields that

$$M(L_k, s+r, N, 3\epsilon, \mathbf{a}) \le \sum_{i\in I} \exp\big(-a(n_i)(s+r)\big) \le \sum_{i\in I} \mu(B_{n_i}(x_i,\epsilon)) \le 1.$$

It follows that

$$m(L_k, s+r, 3\epsilon, \mathbf{a}) = \lim_{N\to\infty} M(L_k, s+r, N, 3\epsilon, \mathbf{a}) \le 1.$$

Hence,

$$E_{L_k}(T, 3\epsilon, \mathbf{a}) \le s + r.$$

Since ϵ can be arbitrary, this implies that

$$E_{L_k}(T, \mathbf{a}) \le s + r \quad \text{for all } k \ge 1.$$

Hence,

$$E_L(T, \mathbf{a}) = E_{\bigcup_{k=1}^{\infty} L_k}(T, \mathbf{a}) = \sup_{k\ge 1} E_{L_k}(T, \mathbf{a}) \le s + r.$$

Since r can be arbitrary, this implies that $E_L(T, \mathbf{a}) \le s$.

Now we prove the second statement. Fix $r > 0$ and for each $k \ge 1$ set

$$L_k = \left\{x \in L : \liminf_{n\to\infty} \frac{-\log \mu(B_n(x,\epsilon))}{a(n)} > s - r \text{ for all } \epsilon \in (0, \frac{1}{k})\right\}.$$

Since $\underline{h}_\mu(T, \mathbf{a}, x) \ge s$ for all $x \in L$, we have that $L_k \subset L_{k+1}$ and $\bigcup_{k=1}^{\infty} L_k = L$. Fix a sufficiently large $k \ge 1$ with $\mu(L_k) > \frac{1}{2}\mu(L) > 0$. For each $N \ge 1$, set

$$L_{k,N} = \left\{x \in L_k : \frac{-\log \mu(B_n(x,\epsilon))}{a(n)} > s - r \text{ for all } n \ge N, \epsilon \in (0, \frac{1}{k})\right\}.$$

It is easy to see that $L_{k,N} \subset L_{k,N+1}$ and $\bigcup_{N=1}^{\infty} L_{k,N} = L_k$. Thus we can pick $N^* \geq 1$ such that $\mu(L_{k,N^*}) > \frac{1}{2}\mu(L_k) > 0$. For simplicity of notation, let $L^* = L_{k,N^*}$ and $\epsilon^* = \frac{1}{k}$. By the choice of L^*, we have that

$$\mu(B_n(x,\epsilon)) \leq \exp\big(-a(n)(s-r)\big) \text{ for all } x \in L^*, 0 < \epsilon < \epsilon^*, n \geq N^*.$$

Fix a sufficiently large $N > N^*$. For each cover $\mathcal{F} = \{B_{n_i}(y_i, \frac{\epsilon}{2})\}_{i\geq 1}$ of L^* with $0 < \epsilon < \epsilon^*$ and $n_i \geq N \geq N^*$ for each $i \geq 1$. Without loss of generality, assume that $L^* \bigcap B_{n_i}(y_i, \frac{\epsilon}{2}) \neq \varnothing$ for all i. Thus, for each $i \geq 1$ pick a point $x_i \in L^* \bigcap B_{n_i}(y_i, \frac{\epsilon}{2})$ so that

$$B_{n_i}(y_i, \frac{\epsilon}{2}) \subset B_{n_i}(x_i, \epsilon).$$

It follows that

$$\sum_{i\geq 1} \exp\big(-a(n_i)(s-r)\big) \geq \sum_{i\geq 1} \mu(B_{n_i}(x_i,\epsilon)) \geq \mu(L^*).$$

Therefore ,

$$M(L^*, s-r, N, \frac{\epsilon}{2}, \mathbf{a}) \geq \mu(L^*) > 0.$$

Consequently

$$m(L^*, s-r, \frac{\epsilon}{2}, \mathbf{a}) = \lim_{N\to\infty} M(L^*, s-r, N, \frac{\epsilon}{2}, \mathbf{a}) \geq \mu(L^*) > 0,$$

which implies that $E_{L^*}(T, \frac{\epsilon}{2}, \mathbf{a}) \geq s-r$. Letting $\epsilon \to 0$, we find that $E_{L^*}(T, \mathbf{a}) \geq s-r$. It follows that

$$E_L(T, \mathbf{a}) \geq E_{L^*}(T, \mathbf{a}) \geq s-r.$$

Since r can be arbitrary, this implies that $E_L(T, \mathbf{a}) \geq s$ completing the proof of the theorem. □

Proof of Proposition 3.11 Suppose there is $[\mathbf{a}] \in \mathcal{A}$ such that $\mathcal{E}_\mu(T, [\mathbf{a}])$ is finite. Then for each $[\mathbf{b}] \succcurlyeq [\mathbf{a}]$,

$$\limsup_{n\to\infty} \frac{a_1(n)}{b_1(n)} = 0$$

for arbitrary $\mathbf{a}_1 = \{a_1(n)\} \in [\mathbf{a}]$ and $\mathbf{b}_1 = \{b_1(n)\} \in [\mathbf{b}]$. Let us fix such two scaled sequences $\mathbf{a}_1$ and $\mathbf{b}_1$. Given a small number $\beta > 0$, for all sufficiently large n we have that $a_1(n) < \beta b_1(n)$. By Proposition 3.8, we have that

$$\mathcal{E}_\mu(T, \mathbf{a}_1) \geq \frac{1}{\beta}\mathcal{E}_\mu(T, \mathbf{b}_1),$$

i.e., $\beta\mathcal{E}_\mu(T, \mathbf{a}_1) \geq \mathcal{E}_\mu(T, \mathbf{b}_1)$. Since β is arbitrary, we conclude that $\mathcal{E}_\mu(T, \mathbf{b}_1) = 0$ and hence, $\mathcal{E}_\mu(T, [\mathbf{b}]) = 0$.

On the other hand, if $[\mathbf{b}] \preccurlyeq [\mathbf{a}]$ then

$$\limsup_{n\to\infty} \frac{b_2(n)}{a_2(n)} = 0$$

for arbitrary $\mathbf{a}_2 = \{a_2(n)\} \in [\mathbf{a}]$ and $\mathbf{b}_2 = \{b_2(n)\} \in [\mathbf{b}]$. Given a small number $\beta > 0$, for all sufficiently large n we have that $b_2(n) < \beta a_2(n)$. It follows that $\mathcal{E}_\mu(T, \mathbf{b}_2) > \frac{1}{\beta}\mathcal{E}_\mu(T, \mathbf{a}_2)$. Again since β is arbitrary, $\mathcal{E}_\mu(T, \mathbf{b}_2) = \infty$ implying that $\mathcal{E}_\mu(T, [\mathbf{b}]) = \infty$. □

Proof of Theorem 3.12 For each $x \in X$, by Lemma 2.1, we have that for all $r > 0$,

$$B\left(x, \frac{r}{c_n}\right) \subset B_n(x, r) \subset B\left(x, \frac{r}{b_n}\right).$$

Fix a small number $r > 0$, since $\frac{r}{c_n} \to 0$ and $\frac{\log \frac{r}{c_n}}{\log \frac{r}{c_{n+1}}} \to 1$ as n approaches infinity, using Young's result [19, Proposition 2.1], we have that

$$\limsup_{n\to\infty} \frac{\log \mu(B(x, \frac{r}{c_n}))}{\log \frac{r}{c_n}} = \overline{d}_\mu(x).$$

Therefore, for each $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$\mu\left(B(x, \frac{r}{c_n})\right) > \left(\frac{r}{c_n}\right)^{(\overline{d}_\mu(x)+\epsilon)}.$$

Hence,

$$\mu(B_n(x, r)) > \left(\frac{r}{c_n}\right)^{(\overline{d}_\mu(x)+\epsilon)} \text{for all } n > N.$$

And this implies that

$$\limsup_{n\to\infty} \frac{\log \mu(B_n(x, r))}{-\log c_n} \leq \overline{d}_\mu(x) + \epsilon.$$

Since r and ϵ can be arbitrary, this imply that

$$\overline{h}_\mu(T, \{\log c_n\}, x) \leq \overline{d}_\mu(x).$$

To prove the second statement, note that

$$\liminf_{n\to\infty} \frac{\log \mu\left(B(x, \frac{r}{b_n})\right)}{\log \frac{r}{b_n}} = \underline{d}_\mu(x).$$

Therefore, for each $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$\mu(B(x, \frac{r}{b_n})) < \left(\frac{r}{b_n}\right)^{(\underline{d}_\mu(x)-\epsilon)}.$$

Hence,

$$\mu(B_n(x, r)) < \left(\frac{r}{b_n}\right)^{(\underline{d}_\mu(x)-\epsilon)} \text{ for all } n > N.$$

Thus, it follows that

$$\liminf_{n\to\infty} \frac{\log \mu(B_n(x, r))}{-\log b_n} \geq \underline{d}_\mu(x) - \epsilon.$$

The arbitrariness of r and ϵ imply that

$$\overline{h}_\mu(T, \{\log b_n\}, x) \geq \underline{d}_\mu(x).$$

This completes the proof of the theorem. □

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