Journal of Statistical Physics, Vol. 122, No. 6, March 2006 (© 2006)
DOI: 10.1007/s10955-005-9005-7

Phase Transitions for Uniformly Expanding Maps

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Received August 22, 2005; accepted November 7, 2005
Published Online: March 15, 2006

Given a uniformly expanding map of two intervals we describe a large class of potentials admitting unique equilibrium measures. This class includes all Hölder continuous potentials but goes far beyond them. We also construct a family of continuous but not Hölder continuous potentials for which we observe phase transitions. This provides a version of the example in (9) for uniformly expanding maps.

KEY WORDS: Expanding maps; thermodynamical formalism; equilibrium measures; phase transitions.

MSC Classification: 37D25; 37D35; 37E05; 37E10.

1. INTRODUCTION

We consider a uniformly piecewise expanding map f of the interval $I = [0, 1]$, i.e., we assume that $I = I^{(1)} \cup I^{(2)} \cup \cdots \cup I^{(p)}$ and $int I^{(k)} \cap int I^{(j)} = \emptyset$ for $1 \le k, j \le p, j \neq k$ and that $f : I^{(j)} \to I, j = 1, \ldots, p$ is a C^1 diffeomorphism with $|f'| > 1$. Given a potential function $\varphi : I \to \mathbb{R}$, a Borel invariant measure μ_φ is called an *equilibrium measure* if

$$h_{\mu\varphi}(f) + \int_1 \varphi \, d\mu_\varphi = \sup\{h_\mu(f) + \int_1 \varphi \, d\mu\} ,$$

where the supremum is taken over all Borel invariant probability measures. The classical thermodynamical formalism states that if φ is (piecewise) Hölder continuous, then f admits a unique equilibrium measure. The proof is based on representing the system as a Bernoulli shift on p symbol alphabet.

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The goal of this paper is to show that the class of potentials, admitting the unique equilibrium measure, is substantially larger than the class of Hölder continuous potentials. We restrict ourselves to the case of uniformly piecewise expanding maps of two intervals although some results can easily be extended to any finite number of intervals. Our approach is based on constructing a countable Markov partition for f and representing it as the "renewal shift" —a subshift of a special type on a countable set of states. We describe a broad class of potentials which admit unique equilibrium measures and show that this class includes some continuous but not Hölder continuous potentials. Moreover, building on results of Sarig,[9] we construct a family of continuous (but not Hölder continuous) potentials φ_c exhibiting phase transitions: there exists a critical value $c_0 > 0$ such that for every $0 < c < c_0$ there is a unique equilibrium measure for φ_c which is supported on (0, 1] and for $c < c_0$ the equilbrium measure is the Dirac measure at 0. This phase transition phenomenon can be characterized in terms of the behavior of the pressure function (see Sec. 4).

2. THERMODYNAMICS OF COUNTABLE MARKOV SUBSHIFTS

We briefly describe some results on the thermodynamics of countable Markov subshifts (see refs. 1, 2, 5, 8, 10, 12, 13). Let $S = \{1, 2, \dots, \}$ be an infinite alphabet and Σ_A the *Markov shift* with the set of states S and a transition matrix $A = (a_{ij}), i, j \in \mathbb{N}$. This means that Σ_A is the set of all one-sided infinite sequences $\omega \in S^{\mathbb{N}}$ which are admissible by A, i.e., $a_{\omega_i \omega_{i+1}} = 1$ for all $i \in \mathbb{Z}$. Assume that (Σ_A, σ) is topologically mixing. We write $[a_1, \dots, a_n]$ for the cylinder set $\{(x_k) : x_i = a_i, 1 \le i \le n\}$.

Given a continuous function $\phi : \Sigma_A \to \mathbb{R}$, called *potential*, we define the *n—variation* of ϕ to be

$$V_n(\phi) = \sup_{x,y \in [a_1,\dots,a_n]} |\phi(x) - \phi(y)| \ .$$

We say that the potential ϕ has *summable variations* if

$$\sum_{n \ge 2} V_n(\phi) < +\infty \ . \tag{1}$$

Furthermore, we say that ϕ is *weakly Hölder continuous* if there exists $A > 0$ and $0 < \gamma < 1$ such that for all $n \ge 1$,

$$V_n(\phi) \le A\gamma^n \ . \tag{2}$$

If ϕ is weakly Hölder continuous then it has summable variations.

Set

$$Z_n(\phi, a) = \sum_{\sigma^n(x)=x} \exp \phi_n(x) 1_{[a]}(x) \ ,$$

where $\phi_n(x) = \sum_{i=0}^{n-1} \phi(\sigma^i(x))$. Define the *Gurevich pressure* by

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a) \ .$$

If the potential ϕ has summable variations then the limit exists and does not depend on the choice of the symbol a.

A Borel σ-invariant probability measure μ_ϕ on Σ_A is called an *equilibrium measure* for ϕ if

$$h_{\mu_\phi}(\sigma) + \int_{\Sigma_A} \phi \, d\mu_\phi = \sup\{h_\mu(\sigma) + \int_{\Sigma_A} \phi \, d\mu\},$$

where the supremum is taken over all Borel σ-invariant probability measures on Σ_A for which $\int_{\Sigma_A} \phi \, d\mu_\phi > -\infty$.

We consider the induced map on the cylinder $[a]$. For $x \in [a]$, define

$$r_a(x) = \inf\{n \geq 1; \sigma^n(x) \in [a]\}.$$

It is the first return time of x to $[a]$. We have

$$[a] = \bigcup_{n\geq 0, \xi_i \neq a} [a, \underline{\xi}, a] \cup N,$$

where $\underline{\xi} = \xi_1, \ldots, \xi_{n-1}$, $r_a(x) = n$ for $x \in [a, \underline{\xi}, a]$, and N is a null set for any σ-invariant probability measure. Let $\overline{\sigma} = \sigma^{r_a(x)} : [a] \to [a]$ be the *induced map*. The map $\overline{\sigma}|[a]$ is topologically conjugate to a countable Bernoulli shift with symbols corresponding to $[a, \underline{\xi}, a]$.

Given a potential ϕ, we define the *induced potential* by $\overline{\phi}(x) = \phi_{r_a(x)}(x)$. In order to describe thermodynamics of σ with respect to the potential ϕ we introduce the following crucial property.

Define the *discriminant* of ϕ by

$$\Delta[\phi] = \sup_{p\in\mathbb{R}}\{P_G(\overline{\phi + p}); P_G(\overline{\phi + p}) < +\infty\}.$$

Proposition 2.1. (see [9]). $\Delta[\phi] \geq 0$ *if and only if there exists* p *such that* $P_G(\overline{\phi + p}) = 0$.

Set

$$Z_n^*(\phi, a) = \sum_{\sigma^n(x)=x, r_a(x)=n} e^{\phi_n(x)} 1_{[a](x)}.$$

Proposition 2.2. (see [9]). *We have*

$$\left|\Delta[\phi] - \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, a)|_{\xi=rad.}\right| \leq \sum_{n\geq 2} V_n(\phi),$$

where ξ *= rad. is the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a).$$

Furthermore, if we define $p^*(\phi) = \sup\{p : P_G(\overline{\phi + p}) < \infty\}$*, then* $\log \xi = -p^*(\phi)$*.*

Set $\lambda = \exp P_G(\phi)$. Following [9], we call the potential ϕ *recurrent* if the sum

$$\sum_{n\geq 1} \lambda^{-n} Z_n(\phi, a)$$

diverges. Otherwise we call ϕ *transient*. We call ϕ *positive recurrent* if it is recurrent and

$$\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) < +\infty$$

and *null recurrent* if

$$\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) = \infty\ .$$

By (9), we have that ϕ is recurrent if and only if $\Delta[\phi] \geq 0$. ϕ is positive recurrent when $\Delta[\phi] > 0$ and is either positive recurrent or null recurrent if $\Delta[\phi] = 0$.

Proposition 2.3. (see refs. 9, 11). *Assume that the potential* ϕ *has summable variations (see (1)* $P_G(\phi) < +\infty$ *and* $\sup \phi < \infty$*. Then the variational principle holds:*

$$P_G(\phi) = \sup\{h_\mu(\sigma) + \int_{\Sigma_A} \phi\, d\mu : \mu \circ \sigma^{-1} = \mu,\ \int_{\Sigma_A} \phi\, d\mu > -\infty\}.$$

If ϕ *is positive recurrent then* $P_G(\phi) = -p(\phi)$*, where* $p(\phi)$ *is the solution of the equation* $P_G(\overline{\phi + P}) = 0$*; if* ϕ *is transient then* $P_G(\phi) = -p^*(\phi)$*, where*

$$p^*(\phi) = \sup\{p : P_G(\overline{\phi + P}) < +\infty\}.$$

The following proposition illustrates that if ϕ is transient, there is no equilibrium measure for ϕ.

Proposition 2.4. (see ref. 2). *Assume that ϕ has summable variations, finite Gurevich pressure and sup $\phi < \infty$. If ϕ admits an equilibrium measure then it is positive recurrent.*

In general, positive recurrence does not imply existence of equilibrium measure. We therefore, consider a particular case of the *renewal shift* (Σ_A, σ) where the entries of the transition matrix A are 1 at $(0, n)$ and $(n, n-1); n \geq 1$ and 0 otherwise.

To study existence and uniqueness of equilibrium measures for the renewal shift one can use results on thermodynamical formalism for general subshifts of countable type (see for example, 1, 2, 5, 8, 12, 13) and reexamine the conditions in this particular case. We however, exploit another approach which is based on inducing schemes, see.[7] Consider the inducing of the renewal shift on the symbol 0. The induced partition is formed by the cylinders $J_n = [0, n-1, \ldots, 1, 0]$ for $n \geq 1$. The collection of intervals $\{J_n\}_{n\geq 1}$ and the inducing time $\tau(J_n) = r_{[0]}(x) = n, x \in J_n$ form an *inducing scheme* (see refs. 6, 7) with the inducing domain [0] and the induced map $\overline{\sigma}(x) = \sigma(x)^n, x \in J_n$. Following refs. 6, 7, we write

$$W = \bigcap_{n\geq 1} \overline{\sigma}^{-n}\left(\bigcup_{n\geq 1} J_n\right), \quad X = \bigcup_{n\geq 1}\bigcup_{k=0}^{n-1} \sigma^k(W \cap J_n).$$

In our case W : [0] and $X = \sum_A$ given a continuous function $\phi : X \to \mathbb{R}$, define the *induced function* $\overline{\phi} : \bigcup_{n\geq 1} J_n \to \mathbb{R}$ by

$$\overline{\phi}(x) = \sum_{k=0}^{n-1} \phi(\sigma^k(x)) \quad \text{for } x \in J_n.$$

Denote by $\mathcal{M}(\sigma)$ the set of all σ-invariant Borel ergodic probability measures μ on X and by

$$s_\phi = \sup(h_\mu(\sigma) + \int_X \phi \, d\mu),$$

where the supremum is taken over all $\mu \in \mathcal{M}(\sigma)$ with $\int_X \phi d\mu > -\infty$. The following result can be derived from (7).

Proposition 2.5. *Assume that the induced potential function $\overline{\phi}$ has summable variations, $-\infty < s_\phi < +\infty$, and $P_G(\overline{\phi - s_\phi}) = 0$. Assume also that sup $(\overline{\phi - s_\phi}) < \infty$ and that*

$$\sum_{n=1}^{\infty} n \sup_{x\in J_n} \exp(\overline{\phi - s_\phi}) < \infty. \tag{3}$$

Then there is a unique equilibrium measure $\mu_\phi \in \mathcal{M}(\sigma)$, i.e., a measure for which

$$s_\phi = h_{\mu_\phi}(\sigma) + \int_1 \phi \, d\mu_\phi = \sup\{h_\mu(\sigma) + \int_1 \phi \, d\mu\} \; .$$

Combining Propositions 2.3 and 2.5, we establish a condition, which guarantees existence and uniqueness of equilibrium measures for the renewal shift.

Theorem 2.6. *Let (Σ_A, σ) be the renewal shift. Assume that:*

1. *ϕ has summable variations;*
2. *sup $\phi < \infty$;*
3. *ϕ is positive recurrent.*

Then there exists a unique equilbrium measure.

Proof. We show that the conditions of Proposition 2.5 are satisfied. Since ϕ has summable variations, so does $\overline{\phi}$. Due to the special structure of the renewal shift, the sum in

$$Z_n(\phi, 0) = \sum_{\sigma^n(x)=x} e^{\phi_n(x)} 1_{[0]}(x)$$

has at most 2^n terms. Hence,

$$Z_n(\phi, 0) \leq 2^n e^{(\sup \phi)n} .$$

This implies that

$$P_G(\phi) \leq \log 2 + \sup \phi < \infty \; .$$

By Proposition 2.3,

$$P_G(\phi) = \left\{ h_\mu(\sigma) + \int_{\Sigma_A} \phi \, d\mu : \mu \circ \sigma^{-1} = \mu, \int_{\Sigma_A} \phi \, d\mu > -\infty \right\}.$$

Since the inducing time in our case is the first return time, we conclude that the right-hand side of the last equation is s_ϕ and hence, $s_\phi = P_G(\phi) < \infty$. Taking the Dirac measure over a periodic orbit, we know that there is at least one invariant measure with $\int \phi \, d\mu > -\infty$. Hence, $s_\phi > -\infty$.

Using the fact that ϕ is positive recurrent we have

$$\sum_{n \geq 1} n\lambda^{-n} Z_n^*(\phi, 0) < +\infty.$$

Since σ is the renewal shift we obtain that $Z_n^*(\phi, 0) = \exp(\overline{\phi}(x))$, where x is the unique periodic orbit of period n in the cylinder $[0, n-1, \ldots, 0]$. For $\lambda =$

$\exp P_G(\phi) = \exp s_\phi$, we have that

$$\left|\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi,0) - \sum_{n=1}^{\infty} n \sup_{x\in J_n} \exp(\overline{\phi - s_\phi})\right| < \sum_{n=2}^{\infty} V_n(\phi).$$

We know that positive recurrence implies Condition (3). Note that Condition (3) in turns, implies that $\sup(\overline{\phi - s_\phi}) < \infty$.

Furthermore, Proposition 2.3 implies that $P_G(\overline{\phi - s_\phi}) = 0$ and the desired result follows. □

3. SYMBOLIC REPRESENTATION OF F VIA THE RENEWAL SHIFT

Recall that f is a uniformly expanding map of the unit interval $I = [0, 1]$. More precisely, we assume that $I = I^{(1)} \cup I^{(2)}$, $int I^{(1)} \cap int I^{(2)} = \emptyset$ and that the map $f : I^{(j)} \to I$, $j = 1, 2$ can be extended to a C^1 diffeomorphism of a small neighborhood of $I^{(j)}$ finally, we assume that $|f'(x)| > \lambda > 1$ for all $x \in int I^{(1)} \cup int I^{(2)}$. Consider the closed intervals $I_0 = I^{(2)}$, $I_n = f^{-1}(I_{n-1}) \cap I^1$, $n \geq 1$. They cover the semi-open interval (0, 1] and form a countable Markov partition for f.

Define the *coding map* h: $\Sigma_A \to I\backslash\{0\}$ by

$$h(i_1, \cdots, i_n, \cdots) = \bigcap_{j=1}^{\infty} f^{-j}(I_{ij}).$$

Here (Σ_A, σ) is the renewal shift.

Denote by Q the set of all end points of the partition intervals and all their preimages. Observe that the only invariant measure μ with $\mu(Q) = 1$ is the Dirac measure at 0, δ_0.

Proposition 3.1. *The map h is onto and is one-to-one and a topological conjugacy between* $(\Sigma_A\backslash h^{-1}(Q), \sigma)$ *and* $(I\backslash Q, f)$.

Given a potential function φ on I, we define $\phi = \varphi \circ h$ to be a potential function on Σ_A. We denote by $\mathcal{H}$ the class of potential functions φ for which the corresponding functions ϕ satisfy all the conditions of Theorem 2.6, i.e., ϕ has summable variations (see (1)), is recurrent and $\sup \phi < \infty$.

The following result is an immediate corollary of Theorem 2.6 and Proposition 3.1.

Proposition 3.2. *If* $\varphi \in \mathcal{H}$ *then*

1. *there exists a unique equilibrium measure* ν_ϕ *for* ϕ;
2. *there exists an equilibrium measure* μ_φ *for* φ *that is either* δ_0 *or* $\nu_\phi \circ h^{-1}$.

The following example illustrates that for some potentials φ the measure μ_φ may not be an equilibrium measure for φ while the Dirac measure δ_0 at 0 may be such a measure.

Example 3.1 Assume that the map f is $C^{1+\epsilon}$, and consider the potential

$$\varphi(x) = \begin{cases} -t \log |f'(x)|, & x \neq 0 \\ f_0, & x = 0 \end{cases}$$

Observe that φ is Hölder continuous on (0, 1). By Theorem 3.3 below, if $f_0 < P_G(\varphi \circ h)$, the measure μ_φ is a unique equilibrium measure for φ (it is the absolutely continuous invariant measure) and if $f_0 > P_G(\varphi \circ h)$, the equilibrium measure is δ_0. Furthermore, if $f_0 = P_G(\varphi \circ h)$ there are two equilibrium measures for φ.

We show that the class of functions $\mathcal{H}$ is sufficiently large and in particular, includes all the Hölder continuous functions.

Theorem 3.3. *If φ is Hölder continuous on* I *then $\varphi \in \mathcal{H}$.*

Proof. Since φ is bounded and continuous, so is ϕ. Moreover, as the map h is Hölder continuous (while h^{-1} is not), we have that ϕ is Hölder continuous on Σ_A. This implies that ϕ has summable variations (see (1)) and is bounded from above.

To establish the recurrent condition we consider inducing on the cylinder [0] as described in Sec. 2. Note that the only induced cylinder with return time n is $[0, n-1, \dots, 1, 0]$. Fix $x_i \in [0, i-1, \dots, 1, 0]$ and consider the function ψ such that $\psi(x) = \phi(x_i)$ whenever $x \in [0, i-1, \dots, 1, 0]$. For simplicity we write $a_i = \phi(x_i)$. The function ψ has the following properties:

1. $V_n(\psi) = 0$ (this is true since ψ is constant on every cylinder);
2. there exist a, $A > 0$ and $0 < \gamma < 1$ such that $|a_i - a| < A\gamma^i$ (this is true since $h(x_i) \to 0$ in I and ϕ is Hölder continuous);
3. $|\phi(x) - \psi(x)| \leq V_n(\phi)$ for any $x \in [0, n-1, \dots, 1, 0]$.

We claim that

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0)|_{\xi = rad.} = +\infty.$$

To show this observe that for $x \in [0, n-1, \dots, 1, 0]$,

$$Z_n^*(\psi, 0) = \exp\left\{\sum_{i=0}^{n-1} \phi(\sigma^i x)\right\} = \exp\left\{\sum_{i=0}^{n-1} a_i\right\}.$$

As $n \to +\infty$, we have that

$$Z_n^*(\psi, 0) \asymp e^{na} .$$

Hence, for $\xi = e^{-a}$,

$$\begin{aligned}\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) &= \log \sum_{n=1}^{\infty} e^{-an} Z_n^*(\psi, 0) \\ &= \log \sum_{n=1}^{\infty} \exp\left\{\sum_{i=0}^{n-1}(a_i - a)\right\} \\ &\geq \log \sum_{n=1}^{\infty} \exp\left\{-\sum_{i=0}^{n-1} A\gamma^n\right\} \\ &\geq \log \sum_{n=1}^{\infty} \exp\left\{-A\frac{1}{1-\gamma}\right\} = +\infty.\end{aligned}$$

Using Property 3 of the function ψ, we obtain that

$$|Z_n^*(\phi, 0)) - Z_n^*(\psi, 0)| \leq \sum_{k=2}^{n+1} V_n(\phi).$$

Since $\sum_{n\geq 2} V_n(\phi) < +\infty$, we conclude that the power series

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a), \quad \sum_{n=1}^{\infty} z^n Z_n^*(\psi, a)$$

have the same radius of convergence. It follows that

$$\begin{aligned}\Delta[\phi] &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, 0) - \sum_{n\geq 2} V_n(\phi) \\ &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) - 2\sum_{n\geq 2} V_n(\phi) = +\infty.\end{aligned}$$

The desired result follows. □

We now show that the class of function $\mathcal{H}$ admits some non-Hölder continuous potentials which thus have equilibrium measures. Consider the one-parameter family of functions

$$\varphi_c(x) = \begin{cases} -c(1 - \log x)^{-a}, & x \in (0, 1] \\ 0, & x = 0, \end{cases} \tag{4}$$

where $c \in \mathbb{R}$. Observe that φ_c is continuous on [0, 1] but is not Hölder continuous at zero.

Theorem 3.4. *For any $\alpha > 1$ the potential $\phi_c = \varphi_c \circ h \in \mathcal{H}$, i.e., it satisfies all the conditions of Theorem 2.6.*

Proof. Set $\beta = \min |f'|^{-1}$ and $\gamma = \max |f'|^{-1}$. Since f is uniformly expanding $\beta \leq \gamma < 1$. We write $I_0 = [d_0, 1], I_1 = [d_1, d_0], \cdots, I_n = [d_n, d_{n-1}]$, and $I_{i_1,\cdots,i_n} = [i_1 \ldots, i_n]$ as long as $i_1, \ldots, i_n$ is admissible. Note that $f(d_n) = d_{n-1}$ and that

1. $I_n \subset [d_n, 1]$ and $\frac{\beta}{\gamma} \leq \frac{|[0,d_n]|}{|I_n|} \leq \frac{\gamma}{\beta}$;
2. $|I_{i_1}, \ldots, i_n| \leq \gamma^n$;
3. $d_n \geq \beta^n$.

We first verify the summable variation condition (see (1)). Choose $x, y \in I_{i_1,\ldots,i_n}$ Let a be an integer such that $\gamma < \beta^{\frac{1}{a}}$. Set $i_1 = m$. If $m \geq \frac{n}{a}$, we have that $I_{i_1,\ldots,i_n} \subset I_m \subset [d_m, 1]$ and hence, $\beta^m \leq x, y \leq \gamma^m$. It follows that

$$\begin{aligned}
|\varphi_c(x) - \varphi_c(y)| &= |\varphi_c{}'(\xi)||x - y| \\
&= |c|\frac{\alpha(1 - \log\xi)^{-\alpha-1}}{|\xi|}|x - y| \\
&\leq |c|\frac{\alpha(1 - \log\xi)^{-\alpha-1}}{|[0, d_m]|}|I_m| \\
&\leq |c|\frac{|I_m|}{|[0, d_m]|}\alpha(1 - m\log\gamma)^{-a-1} \\
&\leq C|c||m|^{-\alpha-1} \geq C|c|\left(\frac{n}{a}\right)^{-\alpha-1}
\end{aligned}$$

If $m < \frac{n}{a}$, we have that

$$\begin{aligned}
|\varphi_c(x) - \varphi_c(y)| &= |\varphi_c{}'(\xi)||x - y| \\
&= |c|\frac{\alpha(1 - \log\xi)^{-\alpha-1}}{|\xi|}|x - y| \\
&\leq |c|\frac{\alpha(1 - \log\xi)^{-\alpha-1}}{|[0, d_m]|}|I_{i_1,\cdots,i_n}| \\
&\leq |c|\frac{|I_m|}{|[0, d_m]|}\frac{|I_{i_1,\cdots,i_n}|}{|I_m|}\alpha(1 - m\log\gamma)^{-\alpha-1} \\
&\leq C'|c|\frac{\gamma^n}{\beta^{\frac{n}{a}}}
\end{aligned}$$

With n sufficient large, we can guarantee that

$$\left(\frac{\gamma}{\beta^{\frac{1}{a}}}\right)^n \leq C'' \left(\frac{n}{a}\right)^{-\alpha-1} .$$

This yields that

$$V_n(\phi_c) \asymp n^{-1-\alpha} ,$$

and hence, ϕ_c has summable variations (see (1)). Clearly, this function is bounded from above. We shall now show that

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi_c, 0)|_{\xi=rad.} = +\infty.$$

In fact,

$$Z_n^*(\phi_c, 0) = \exp \sum_{k=0}^{n-1} \varphi_c(x_k)$$

with $x_k \in I_k$. Taking into account that $\alpha > 1$ we find that

$$\begin{aligned} \exp\left\{\sum_{k=0}^{n-1} \varphi_c(x_k)\right\} &\geq \exp\left\{-\sum_{k=0}^{n-1} |\varphi_c(d_k)|\right\} \\ &\geq \exp\left\{-|c| \sum_{k=0}^{n-1} (1 + k|\log \gamma|)^{-\alpha}\right\} \\ &\geq \exp\left\{-|c|(|\log \gamma|)^{-\alpha} \sum_{k=0}^{n-1} (k+1)^{-\alpha}\right\} > e^{-C}, \end{aligned}$$

where $C > 0$ is a constant. It follows that for $\xi = 1$,

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, 0) = \log \sum_{n=1}^{\infty} e^{-C} = +\infty.$$

We have $\Delta[\phi_c] = +\infty$ and the desired result follows. □

Remark. *The above argument shows indeed, that the function ϕ_c has summable variations and is bounded from above for all $\alpha > 0$.*

4. PHASE TRANSITIONS

Phase transitions for the renewal shift were studied by refs. 3, 4, 9. Let (Σ_A, σ) be the renewal shift, $\phi : \Sigma_A \to \mathbb{R}$ a potential function and $\bar{\phi}(x) = \phi_{r_a(x)}(x)$ corresponding induced potential function.

Proposition 4.1. *Assume that ϕ satisfies Conditions 1 and 2 of Proposition 2.3 (i.e., φ has summable variations and is bounded from above) and that $\overline{\phi}$ is weakly Hölder continuous (see (2)). Then there exists a critical value $0 < c_0 \leq +\infty$ such that the potential $c\phi$ is positive recurrent for all $0 < c < c_0$ and is transient for $c > c_0$.*

Note that this result does not exclude the case when $c_0 = \infty$.

In fact, if φ is a Hölder continuous function on I then the function $c\varphi$ is Hölder continuous and hence, is positive recurrent for all c (see Theorem 3.4). We provide an example of a one-parameter family of functions for which the phase transition actually occurs.

Theorem 4.2. *For the function $\varphi_c(x)$, given by (4) with $0 < \alpha \leq 1$, the potential function $\phi_c = \varphi_c \circ h$ has summable variations and $\phi_c(x)$ is weakly Hölder continuous. There exists $c_0 > 0$ such that*

1. *ϕ_c is positive recurrent for $c_0 > c > 0$ and there is a unique equilbrium measure for φ_c among all invariant measures. This measure is supported on (0, 1].*
2. *ϕ_c is transient for $c > c_0$ and there is no equilibrium measure for φ_c among the measures supported on (0, 1], the Dirac measure at 0 is the equilibrium measure.*

Proof. The fact that ϕ_c has summable variations and is bounded from above for $0 < \alpha \leq 1$ (and actually for all $\alpha > 0$) was established in the proof for Theorem 3.4 (see Remark in the end of Sec. 3).

We shall show that the induced potential $\overline{\phi}_c(x)$ is weakly Hölder continuous. This will imply that it has summable variations. Choose two points x and y in the interval $[0, n_1, \cdots, 0, n_2, \cdots, 0, \cdots, n_k, \cdots, 0]$. We have

$$|\overline{\phi_c}(x) - \overline{\phi_c}(y)| = |\sum_{i=0}^{n_1}(\phi_c(f^i(x)) - \phi_c(f^i(y)))|$$

$$\leq \sum_{i=0}^{n_1} |c| \frac{\alpha(1 - \log \xi_i)^{-1-\alpha}}{|\xi_i|} |\phi_c(f^i(x)) - \phi_c(f^i(y))|$$

$$\leq \sum_{i=0}^{n_1} |c| \frac{\alpha(1-(n_1-i)\log\gamma)^{-1-\alpha}}{|[0, d_{n_1-i}]|}$$

$$|[n_1 - i, \cdots, 0, n_2, \cdots, 0, \cdots, n_k, \cdots, 0]|$$

$$\leq \sum_{i=0}^{n_1} |c| \frac{|I_{n_1-i}|}{|[0, d_{n_{1-i}}]|} \gamma^{k-1} |\alpha(1-(n_1-i)\log\gamma)^{-1-\alpha}|$$

$$\leq |c| C \sum_{i=0}^{n_1} (n_1 - i)^{-\alpha-1} \gamma^{k-1} \leq A\gamma^{k-1} \ ,$$

where $A = |c|C \sum_{n=0}^{\infty} n^{-\alpha-1}$. Hence, $\phi_c(x)$ is weakly Hölder continuous.

We shall show that for c sufficiently close to zero, the potential ϕ_c is positive recurrent and for c sufficiently large, it is transient. Set

$$F_n(\phi_c) = \exp\left\{ \sum_{k=0}^{n-1} \sup_{x \in I_k} \varphi_c(x) \right\} \ .$$

Observe that

$$\log \sum_{n\geq 1} \xi^n F_n(\phi_c)|_{\xi=rad.} - 2\sum_{n\geq 2} V_n(\phi_c) \leq \Delta[\phi_c]$$

$$\leq \log \sum_{n\geq 1} \xi^n F_n(\phi_c)|_{\xi=rad}.$$

and that

$$F_n(\phi_c) = \exp\left\{ \sum_{k=0}^{n-1} \varphi_c(d_k) \right\},$$

where d_k were introduced in the proof of Theorem 3.4 and are the endpoints of the basic intervals I_n (i.e., $I_n = [d_n, d_n - 1]$).

We consider the two cases.

Case I: $\alpha = 1$. We have

$$F_n(\phi_c) = \exp\left\{ \sum_{k=0}^{n-1} \varphi_c(d_k) \right\}$$

$$\geq \exp\left\{ \sum_{k=0}^{n-1} -c(1-\log\gamma^k)^{-\alpha} \right\}$$

$$\geq \exp\left\{ \sum_{k=0}^{n-1} -c|\log\gamma|(k+1)^{-\alpha} \right\}$$

$$\geq \exp\{-cM \log(n+1)\} \geq C(n+1)^{-cM} \ ,$$

where $C > 0$ and $M > 0$ are constants. Similarly, one can show that for some $C' > 0$ and $M' > 0$,

$$F_n(\phi_c) \leq C'(n+1)^{-cM'}.$$

Therefore, the radius of convergence $\xi = 1$ and

$$\sum_{n\geq 1} F_n(\phi_c) \geq \sum_{n\geq 1} C(n+1)^{-cM} = +\infty$$

as long as $-1 < -cM < 0$ (i.e., $0 < c < \frac{1}{M}$). For these values of c we have $\Delta[\phi_c] = +\infty$ and hence, ϕ_c is positive recurrent.

Observe that

$$\log \sum_{n\geq 1} F_n(\phi_c) \leq \log \sum_{n\geq 1} C'(n+1)^{(-cM')}.$$

The expression in the right-hand side of this inequality tends to $-\infty$ as $c \to \infty$. Hence, for some c we have

$$\Delta[\phi_c] \leq \log \sum_{n\geq 1} F_n(\phi_c) < 0 \ .$$

This implies that for c sufficiently large, ϕ is transient.

Case II: $0 < \alpha < 1$. We find that

$$\exp\{-cM'(n+1)^{1-\alpha}\} \leq F_n(\phi_c) \leq \exp\{-cM(n+1)^{1-\alpha}\} \ .$$

The radius of convergence $\xi = 1$ and

$$\sum_{n\geq 1} F_n(\phi_c) \geq \sum_{n\geq 1} \exp\{-cM'(n+1)^{1-\alpha}\}.$$

Therefore,

$$\sum_{n\geq 1} F_n(\phi_c) \geq ke^{-1} \tag{5}$$

as long as $cM' < \frac{1}{(k+1)^{1-\alpha}}$. Observe that $\sum_{n\geq 2} V_n(\phi) \to 0$ as $c \to 0$. In view of (5), this imples that for c sufficiently close to zero,

$$\Delta[\phi_c] \geq \log \sum_{n\geq 1} F_n(\phi_c) - 2\sum_{n\geq 2} V_n(\phi_c) > 0$$

and hence, ϕ_c is positive recurrent.

We also have that

$$\log \sum_{n\geq 1} F_n(\phi_c) \leq \log \sum_{n\geq 1} \exp\{-cM(n+1)^{1-\alpha}\} \to -\infty$$

as $c \to +\infty$, and hence, ϕ_c is transient for sufficiently large c.

Since $\xi = 1$, by Proposition 2.2, $0 = \log \xi = p^*(\phi_c)$. When c is small, $\Delta[\phi_c] > 0$ implies that $p(\phi_c) < p^*(\phi_c) = 0$. Proposition 2.3 give that $P_G(\phi_c) = -p(\phi_c) > 0$. Because ϕ_c is positive recurrent, by Theorem 2.6, there is a unique equilibrium measure for the renewal shift, by toplogical conjugacy, there is a unique equilibrium measure μ_{φ_c} among the measures supported on (0, 1]. Since the only invariant measure supported outside of (0, 1] is the Dirac measure δ_0, and from

$$h_{\delta_0}(f) + \int_{\mathrm{I}} \varphi_c \, d\delta_0 = \varphi_c(0) < P_G(\phi_c)$$

we know that δ_0 is not an equilbrium measure, hence μ_{φ_c} is the equilibrium measure among all invariant measures.

In the case when c is sufficiently large, $\Delta[\phi_c] < 0$ and Proposition 2.3 implies that $P_G(\phi_c) = -p^*(\phi_c) = 0$. In this case ϕ_c is transient, by Proposition 2.4, there is no equilibrium measure supported on (0, 1].

$$h_{\delta_0}(f) + \int_{\mathrm{I}} \varphi_c \, d\delta_0 = 0 < P_G(\phi_c)$$

implies that the Dirac measure is the equilibrium measure. □

ACKNOWLEDGMENTS

We would like to thank Omri Sarig for numerous useful discussions we had with him while working on the paper. The work was partially supported by the National Science Foundation grant #DMS-0503810 and U.S.-Mexico Collaborative Research grant 0104675.

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