

ON THE WORK OF SARIG ON COUNTABLE MARKOV CHAINS AND THERMODYNAMIC FORMALISM

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ABSTRACT. The paper is a non-technical survey and is aimed to illustrate Sarig's profound contributions to statistical physics and in particular, thermodynamic formalism for countable Markov shifts. I will discuss some of Sarig's work on characterization of existence of Gibbs measures, existence and uniqueness of equilibrium states as well as phase transitions for Markov shifts on a countable set of states.

1. INTRODUCTION

Omri Sarig, the winner of the fourth Brin Prize in Dynamical System, has made many fundamental contributions to the theory of dynamical systems in general and specifically to thermodynamics of countable Markov chains. In this paper I will describe some of Sarig's results on characterization of existence of Gibbs measures, on existence and uniqueness of equilibrium states as well as on presence of phase transitions for countable Markov chains. It should be stressed that the results on Gibbs measures for Markov chains on finite or countable set of states serve as a ground to study existence, uniqueness and ergodic properties of equilibrium measures for smooth hyperbolic dynamical systems and that uniformly hyperbolic systems are modeled by subshifts of finite type while nonuniformly hyperbolic systems are modeled by countable Markov chains. Symbolic representation of hyperbolic dynamical systems can be obtained using Markov partitions with a finite or respectively countable collection of partition elements. Constructions of such partitions for uniformly hyperbolic systems is due to Sinai [29] and Bowen [3]. Recently Sarig has constructed countable Markov partitions for nonuniformly hyperbolic surface diffeomorphisms (see [27] and also the paper by Ledrappier [15] in this volume).

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2. GIBBS DISTRIBUTIONS

Thermodynamic formalism, i.e., the formalism of equilibrium statistical physics, originated in the work of Boltzman and Gibbs and was later adapted to the theory of dynamical systems in the classical works of Sinai [28], Ruelle [19, 20] and Bowen [3]. In this paper we shall only discuss thermodynamic formalism for symbolic dynamical systems starting with subshifts of finite type and moving then to countable state Markov chains. It is the latter where Sarig has obtained many principle results that essentially shaped up the area.

2.1. The Gibbs distribution. In statistical mechanics the study of thermodynamics deals with systems which appear to human perception as being "static" despite the motion of the particles which the systems are built of. These systems can be described simply by a set of macroscopically observable variables and are thought of as statistical ensembles that depend on a few observable parameters, and which are in statistical equilibrium. We consider systems (canonical ensembles) with a finite (but sufficiently large) number of particles for which the energy is not known exactly and in place of energy, the temperature is specified. We begin with a simple example that will help us reveal some principle components of thermodynamic formalism.

Consider a physical system \mathcal{A} of finite particles. Each particle is characterized by its position and velocity and we call a given collection of such positions and velocities over all particles a *state*. There are physical systems for which the set of all states is a finite set $X = \{1, \dots, N\}$. We denote by E_i the energy of the state i . We further assume that this mechanical system is in thermal equilibrium with a reservoir, i.e., the particles interact with a *heat bath* \mathcal{B} so that

- (1) \mathcal{A} and \mathcal{B} can exchange energy, but not particles;
- (2) \mathcal{B} is at equilibrium and has temperature T ;
- (3) \mathcal{B} is much larger than \mathcal{A} , so that its contact with \mathcal{A} does not affect its equilibrium state.

Since the energy of the system is not fixed every state can be realized with a probability p_i given by the *Gibbs distribution*

$$p_i = \frac{1}{Z(\beta)} e^{-\beta E_i}, \text{ where } Z(\beta) = \sum_{i=1}^N e^{-\beta E_i},$$

$\beta = \frac{1}{\kappa T}$ is called the *inverse temperature* and κ the *Boltzman's constant*.

It is not difficult to show (see [13]) that the Gibbs distribution maximizes the quantity $H - \beta E = H - \frac{1}{\kappa T} E$, where

$$H = - \sum_{i=1}^N p_i \log p_i$$

is the *entropy* of the Gibbs distribution,

$$E = \sum_{i=1}^N (\beta E_i) p_i = \int_X \varphi d(p_1, \dots, p_n)$$

is the *average energy*, and $\varphi(i) = \beta E_i$ is the *potential*, which in our case is the function that depends on a given coordinate i only.

In other words, the Gibbs distribution minimizes the quantity $E - \kappa T H$ known as the *free energy* of the system. The importance of this statement can not be underestimated: it means that one of the main principles of thermodynamics that *nature maximizes entropy* is applicable when energy is fixed and should otherwise be replaced with the principle that *nature minimizes the free energy*.

Consider now a one-dimensional lattice and assume that to each integer one associate a physical system with a finite set $X = \{1, \dots, m\}$ of states. A *configuration* of our infinite systems is a point

$$\omega = (\omega_n) \in \Sigma_N = X^{\mathbb{Z}}.$$

We assume that the set X is endowed with the discrete topology and the the set Σ_n with the direct product topology.

Consider all possible finite configurations $(\omega_{-n}, \dots, \omega_0, \dots, \omega_n)$ called *cylinders*, (there is $N = m^{2n+1}$ possible configurations) and assume that each configuration has energy

$$E_n = E(\omega_{-n}, \dots, \omega_n) = \sum_{j=-n}^n \varphi_0(\omega_j) + \sum_{-n \leq j < k \leq n} \varphi_1(k - j, \omega_k, \omega_j),$$

where φ_0 and φ_1 are continuous functions of their coordinates and that the function φ_1 satisfies

$$\|\varphi_1\|_j = \sup_{k_1, k_2} |\varphi_1(j, k_1, k_2)| < \infty \text{ and } \sum_{j=1}^{\infty} \|\varphi_1\|_j < \infty.$$

Arguing as above we obtain Gibbs distributions with probabilities μ_n that are proportional to $e^{-\beta E_n}$. Assume now that for every configuration $(\omega_{-n}, \dots, \omega_n)$ there exists the limit

$$\mu(\omega_{-n}, \dots, \omega_n) = \lim_{k \rightarrow \infty} \sum \mu_k \{(\omega'_{-k}, \dots, \omega'_k) : \omega'_j = \omega_j \text{ for every } |j| \leq n\},$$

where the sum is taken over all $(\omega'_{-k}, \dots, \omega'_k)$ for which $\omega'_j = \omega_j$ for every $|j| \leq n$. The measure μ is called the *Gibbs distribution on Σ_n* and is an invariant measure for the (left) shift σ on Σ_n .

3. SUBSHIFTS OF FINITE TYPE

We shall now describe a substantial generalization of the above example, which is due to Bowen [3], Parry [17] and Walters [33] (see also [34]). It is designed to describe systems that are modeled by subshifts of finite type.

3.1. Gibbs measures for subshifts of finite type. Let (Σ_A^+, σ) be a (one-sided) *subshift of finite type*.¹ Here $A = (a_{ij})$ is a transition matrix ($a_{ij} = 0$ or 1, no zero columns or rows),

$$\Sigma_A^+ = \{x = (x_n) : a_{x_n x_{n+1}} = 1 \text{ for all } n \geq 0\}$$

and σ is the (left) shift. We view Σ_A^+ as a metric space with the metric $d(x, y)$ given by: for any $x = (x_n)$ and $y = (y_n)$,

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

We assume that A is irreducible (i.e., $A^N > 0$ for some $N > 0$ and all $n \geq N$) implying that σ is topologically mixing. Consider a continuous function φ on Σ_A^+ , which we call a *potential*.

Theorem 3.1 (see [3]). *Assume that the potential φ is Hölder continuous. Then there exist a unique σ -invariant Borel probability measure μ on Σ_A^+ and constants $C_1 > 0$, $C_2 > 0$ and P such that for every $x = (x_i) \in \Sigma_A^+$ and $m \geq 0$,*

$$(3.1) \quad C_1 \leq \frac{\mu(\{y = (y_i) : y_i = x_i, i = 0, \dots, m\})}{\exp(-Pm + \sum_{k=0}^{m-1} \varphi(\sigma^k(x)))} \leq C_2.$$

The measure $\mu = \mu_\varphi$ is called a *Gibbs measure* for the potential φ and the constant $P = P(\varphi)$ the *topological pressure* of φ .

3.2. Ruelle's Perron-Frobenius theorem. The proof of this theorem is based on Ruelle's version of the classical Perron–Frobenius theorem for matrices. Given a continuous potential φ on Σ_A^+ , define a linear operator $\mathcal{L} = \mathcal{L}_\varphi$ on the space $C(\Sigma_A^+)$ by

$$(3.2) \quad (\mathcal{L}_\varphi f)(x) = \sum_{\sigma(y)=x} e^{\varphi(y)} f(y).$$

¹In the literature the pair (Σ_A^+, σ) is also called *atopological Markov chain*.

It is called the *Ruelle operator* and it provides a great tool in constructing and studying Gibbs measures. Given a continuous potential φ on Σ_A^+ , let

$$(3.3) \quad \Phi_n(x) = \sum_{k=0}^{n-1} \varphi(\sigma^k(x))$$

be the n -th ergodic sum of φ . Note that for all $n > 0$

$$(\mathcal{L}_\varphi^n f)(x) = \sum_{\sigma^n(y)=x} e^{\Phi_n(y)} f(y).$$

Theorem 3.2 (see [3]). *Let φ be a Hölder continuous potential on Σ_A^+ . Then there exist $\lambda > 0$, a continuous positive function h and a Borel measure ν such that*

1. $\mathcal{L}_\varphi h = \lambda h$ and $\int_{\Sigma_A^+} h d\nu = 1$ (i.e., h is a normalized eigenfunction for the Ruelle operator);
2. $\mathcal{L}_\varphi^* \nu = \lambda \nu$;
3. for every $f \in C(\Sigma_A^+)$

$$(3.4) \quad \lambda^{-n} \mathcal{L}_\varphi^n(f)(x) \rightarrow h(x) \int f d\nu \text{ as } n \rightarrow \infty$$

uniformly in x .

4. the measure $\mu_\varphi = h d\nu$ is a σ -invariant Gibbs measure for φ , which is ergodic (in fact, its natural extension is Bernoulli)².

One can show (see [3]) that the rate of convergence in (3.4) is exponential. This implies that the measure μ_φ has exponential decay of correlations with respect to the class of Hölder continuous *test functions* on Σ_A^+ . Recall that a continuous transformation T has *exponential decay of correlations* with respect to an invariant Borel probability measure μ and a class \mathcal{H} of test functions if there exists $0 < \theta < 1$ such that, for any $h_1, h_2 \in \mathcal{H}$,

$$\left| \int h_1(T^n(x))h_2(x) d\mu - \int h_1(x) d\mu \int h_2(x) d\mu \right| \leq K\theta^n,$$

for some $K = K(h_1, h_2) > 0$.

One can further show (see [3] and [8]) that the measure μ_φ in Theorem 3.2 satisfies the Central Limit Theorem (CLT). Recall that a continuous transformation T satisfies the *Central Limit Theorem* (CLT)

²A Bernoulli automorphism (T, ν) is an invertible (mod 0) measure preserving transformation, which is metrically isomorphic to the Bernoulli shift associated to some Lebesgue space (X, μ)

for test functions from a class \mathcal{H} if for any $h \in \mathcal{H}$, which is not a coboundary (i.e., $h \neq g \circ T - g$ for any g), there exists $\gamma > 0$ such that

$$\mu \left\{ \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} (h(T^i(x)) - \int h d\mu) < t \right\} \rightarrow \frac{1}{\gamma\sqrt{2\pi}} \int_{-\infty}^t e^{-\tau^2/2\gamma^2} d\tau.$$

3.3. Conformal measures. The measure ν in Theorem 3.2 has the important property of being conformal. Recall that given a potential φ on Σ_A^+ , a Borel probability measure μ on Σ_A^+ (which is not necessarily invariant under the shift) is called *conformal* (with respect to φ) if for some constant λ and almost every $x \in \Sigma_A^+$

$$\text{Jac}(\mu)(x) := \frac{d\mu}{d\mu \circ \sigma}(x) = \lambda^{-1} \exp \varphi(x).$$

One can show that in our case the relation $\mathcal{L}_\varphi^* \nu = \lambda \nu$ is equivalent to the fact that ν is a conformal measure for φ .

3.4. The topological pressure and the variational principle. We defined above the topological pressure of the potential φ as a constant in (3.1). Its existence is guaranteed by Theorem 3.1. We shall now give another equivalent definition of the topological pressure.

Denote by

$$Z_m(\varphi) = \sum_{[x_0 x_1 \dots x_{m-1}]} \exp \left(\sup_{x \in [x_0 x_1 \dots x_{m-1}]} \Phi_m(x) \right),$$

where $[x_0 x_1 \dots x_{m-1}]$ is a cylinder and $\Phi_m(x)$ is given by (3.3).

Theorem 3.3. *The following limit exists*

$$(3.5) \quad P(\varphi) = \lim_{m \rightarrow \infty} \frac{1}{m} \log Z_m(\varphi).$$

If potential φ is Hölder continuous, then $P(\varphi)$ coincides with the constant P given by (3.1).

The formula (3.5) extends the notion of the topological pressure to continuous potentials.

One of the fundamental results in thermodynamics is the *variational principle* for the topological pressure:

Theorem 3.4. *For every continuous potential φ*

$$(3.6) \quad P(\varphi) = \sup \left\{ h_\mu(f) + \int_{\Sigma_A^+} \varphi d\mu \right\},$$

where the supremum is taken over the set $\mathcal{M}(\Sigma_A^+, f)$ of all σ -invariant Borel probability measures on Σ_A^+ .

Note that one can restrict the supremum in (3.6) to all σ -invariant *ergodic* Borel probability measures on Σ_A^+ . For proofs of Theorems 3.3 and 3.4 we refer the reader to [3] (see also [34]).

Remark. The definition of the topological pressure based on Theorem 3.3 has an advantage over the definition based on formula (3.1): one can replace the whole space Σ_A^+ with a compact subset $K \subset \Sigma_A^+$ which is σ -invariant (i.e., $\sigma^{-1}(K) = K$) and one can replace the requirement that the potential φ is Hölder continuous with a weaker condition that it is just continuous. Furthermore, it is exactly Theorem 3.3 that lays down a way to extend the notion of topological pressure to countable state Markov chains (see Section 4.5), which is the main subject of Sarig's work.

3.5. Equilibrium measures. Given a continuous potential φ , a σ -invariant measure $\mu = \mu_\varphi$ on Σ_A^+ is called an *equilibrium measure* if

$$P(\varphi) = h_{\mu_\varphi} + \int_{\Sigma_A^+} \varphi d\mu_\varphi.$$

Theorem 3.5 (see [3]). *If the potential φ is Hölder continuous, then the Gibbs measure μ_φ in the Ruelle's Perron–Frobenius theorem is the unique equilibrium measure for φ . Moreover, $\log \lambda = P(\varphi)$.*

3.6. Two-sided subshifts. Many results in thermodynamical formalism of one-sided subshifts can be extended to two-sided subshifts (Σ_A, σ) where

$$\Sigma_A = \{x = (x_n) : a_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\}$$

and σ is the left shift. One can view (Σ_A, σ) as at the natural extension of (Σ_A^+, σ) . This is based on results by Sinai [28] and Bowen [3].

Theorem 3.6. *Given a Hölder continuous potential φ on Σ_A there are Hölder continuous functions h on Σ_A and ψ on Σ_A^+ such that for every $x = (x_n) \in \Sigma_A$,*

$$\varphi(x) + h(x) - h(\sigma(x)) = \psi(x_0 x_1 \dots).$$

This equation means that the potentials φ and ψ are *cohomologous*. Due to the variational principle, one can show that two cohomologous potentials have the same set of Gibbs measures.

4. COUNTABLE MARKOV CHAINS

We now move from subshifts of finite type or Markov chains with finitely many states to Markov chains with countably many states or *countable Markov chains* $(X = \Sigma_A^+, \sigma)$ where A is a transition matrix on

a countable set S of states and σ is the left shift. The Borel σ -algebra \mathcal{B} is generated by all cylinders. The main obstacle in constructing equilibrium measures in this case is that the space X is not compact and hence, the space of probability measures on X is not compact either and new methods are needed to overcome this difficulty.

4.1. A bit of a history. I will mention few major developments that preceded Sarig's work. The list below is far from being complete and is only meant to set up the ground for the results described below.

- (1) In 1967-68 Dobrushin, [6], Landford [14] and Ruelle [20] introduced what is now known as *DLR measures*; they characterize Gibbs measures (also called *Gibbsian distributions*) in terms of families of conditional probabilities (see below).
- (2) Gurevic [9, 10] studied the topological entropy (corresponding to the case $\varphi = 0$) and established the variational principle for the topological entropy; later he introduced the notion of topological pressure and obtained the variational principle (see [11] and also [12]). Vere-Jones [30] studied recurrence properties that are central for constructing Gibbs measures. Both Gurevic and Vere-Jones assumed that the potential function depend on finitely many coordinates which allowed them to use some ideas from renewal theory.
- (3) Yuri [35] proved convergence in (3.4) requiring the *finite images property* (see below).
- (4) Aaronson, Denker and Urbanski [2] studied ergodic properties of conformal measures (in particular, they proved that these measures are conservative) and Aaronson and Denker [1] established convergence in (3.4) requiring the *big images property* (which they called the Gibbs-Markov property; see below).

4.2. Dobrushin-Lanford-Ruelle (DLR) measures. We begin by describing DLR measures. We think of X as one-dimensional lattice whose points are called *sites*, so that each site n can be in one of countably many states x_n . Given a probability measure μ on X , consider the conditional measures on cylinders $[a_0, \dots, a_{n-1}]$ generated by μ , i.e., the conditional distribution of the configuration of the first n sites (a_0, \dots, a_{n-1}) given that site n is in state x_n , site $(n+1)$ is in state x_{n+1} , etc. More precisely, for almost all $x \in X$,

$$\mu(a_0, \dots, a_{n-1} | x_n, x_{n+1}, \dots)(x) = \mathbb{E}_\mu(1_{[a_0, \dots, a_{n-1}]} | \sigma^{-n} \mathcal{B})(x).$$

Given $\beta > 0$ and a measurable function $U: X \rightarrow \mathbb{R}$, we call a probability measure μ on X a *Dobrushin-Lanford-Ruelle (DLR) measure for*

the potential $\varphi = -\beta U$ if for all $N \geq 1$ and almost every $x \in X$ the conditional measures of μ satisfies the *DLR equation*:

$$\mu(x_0, \dots, x_{n-1} | x_n, x_{n+1}, \dots)(x) = \frac{\exp \Phi_n(x)}{\sum_{\sigma^n(y)=\sigma^n(x)} \exp \Phi_n(y)}.$$

The problem now is to recover μ from its conditional probabilities.

4.3. Conformal measures. In the particular case $\varphi(x) = f(x_0, x_1)$ recovering the measure μ from its conditional probabilities is the well known Kolmogorov's theorem in the theory of classical Markov chains where the stochastic matrix $P = (p_{ij})$ is given by $p_{ij} = \exp(f(i, j))$ if $a_{ij} = 1$ and $p_{ij} = 0$ otherwise.

For general potentials φ DLR measures can be recovered using conformal measures. More precisely, the following statement holds.

Theorem 4.1 (see [1]). *Let φ be a Borel function and μ a non-singular conformal probability measure for φ on a countable Markov chain X . Then μ is a DLR measure for φ .*

Note that this result is quite general as it imposes essentially no restrictions on the potential φ . Indeed, one can obtain much stronger statements assuming certain level of regularity of the potential.

4.4. Regularity requirements: summable variations and local Hölder continuity. Let $\varphi: X \rightarrow \mathbb{R}$ be a potential. We denote by

$$(4.1) \quad \text{var}_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : x_i = y_i, 0 \leq i \leq n-1\}$$

the n -th *variation* of φ . We say that φ has *summable variations* if

$$(4.2) \quad \sum_{n=2}^{\infty} \text{var}_n(\varphi) < \infty$$

and φ is *locally Hölder continuous* if there exist $C > 0$ and $0 < \theta < 1$ such that for all $n \geq 2$

$$(4.3) \quad \text{var}_n(\varphi) \leq C\theta^n.$$

It is easy to see that if the potential is locally Hölder continuous then it has summable variations.

4.5. The Gurevic-Sarig pressure. In what follows we shall always assume that the shift σ is topologically mixing that is given $i, j \in S$ there is $N = N(i, j)$ such that, for any $n \geq N$ there is an admissible word of length n connecting i and j . For $i \in S$ let

$$(4.4) \quad Z_n(\varphi, i) = \sum_{\sigma^n(x)=x, x_0=i} \exp(\Phi_n(x)).$$

The *Gurevic-Sarig pressure* of φ is the number³

$$(4.5) \quad \mathcal{P}(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\varphi, i).$$

This notion is a generalization of the notion of topological entropy $h_G(\sigma)$ for countable Markov chains introduced by Gurevic in [9, 10], so that $\mathcal{P}(0) = h_G(\sigma)$. Existence of the limit in (4.5) and some basic properties of the pressure are described by the following theorem.

Theorem 4.2 (Sarig, [21, 26]). *Assume that the potential φ has summable variations, i.e., it satisfies (4.2). Then*

1. *The limit in (4.5) exists for all $i \in S$ and is independent of i .*
2. $-\infty < \mathcal{P}(\varphi) \leq \infty$.
3. $\mathcal{P}(\varphi) = \sup\{P(\varphi|K) : K \subset X \text{ compact and } \sigma^{-1}(K) = K\}$, where $P(\varphi|K)$ is the topological pressure of φ on the set K , see Remark 3.4.

4.6. The variational principle for the Gurevic-Sarig pressure.

We now present one of the main results in thermodynamics of countable Markov shifts – the variational principle for the Gurevic-Sarig pressure. This result is a substantial generalization to countable Markov chains of the variational principle for subshifts of finite type (see Theorem 3.4).

Theorem 4.3 (Sarig, [21]). *Assume that φ has summable variations and $\sup \varphi < \infty$. Then*

$$\mathcal{P}(\varphi) = \sup \left\{ h_\mu(\sigma) + \int \varphi d\mu \right\} < \infty,$$

where the supremum is taken over all σ -invariant Borel probability measures on Σ_A^+ such that $-\int \varphi d\mu < \infty$.

Our goal now is to construct an equilibrium measure μ_φ for φ , find conditions under which it is unique and study its ergodic properties. We will achieve this by first constructing a Gibbs measure for φ and then showing that it is an equilibrium measure for φ providing it has finite entropy.

³The notion of the pressure for countable Markov shifts that we discuss in this section was introduced by Gurevic [11] (see also [12]) for potentials of a special kind that only depend on the first coordinate. It was Sarig who extended Gurevic's approach to potentials that may depend on all coordinates (see [21, 26]) and described all the main properties of the pressure (see Theorem 4.2). This is why we propose to call this pressure after both Gurevic and Sarig.

4.7. Gibbs measures for countable Markov chains. The construction of Gibbs measures is based on the study of the Ruelle operator \mathcal{L}_φ and on establishing a generalized version of the Ruelle's Perron-Frobenius theorem. The role of the Ruelle operator in the study of Gibbs measures can be seen from the following result that connects this operator with the Gurevic-Sarig pressure.

We say that a non-zero function f is a test function if it is bounded continuous non-negative and is supported inside a finite union of cylinders.

Theorem 4.4 (Sarig, [21]). *Assume that φ has summable variations. Then for every test function f and all $x \in X$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{L}_\varphi^n f)(x) = \mathcal{P}(\varphi),$$

where \mathcal{L}_φ is the Ruelle operator given by (3.2).

This result implies that if $\mathcal{P}(\varphi)$ is finite, then for every $x \in X$ the asymptotic growth of $\mathcal{L}_\varphi^n f(x)$ is λ^n where $\lambda = \exp \mathcal{P}(\varphi)$.

4.8. Recurrence properties of the potential. We now wish to obtain a more refined information on the asymptotic behavior of $\lambda^{-n} \mathcal{L}_\varphi^n$. To this end given a state $i \in S$, let $Z_n(\varphi, i)$ be given by (4.4) and

$$Z_n^*(\varphi, i) = \sum_{\sigma^n(x)=x, x_0=i} \exp(\Phi_n(x)) 1_{[\varphi_i=n]}(x),$$

where φ_i is the first return time to the cylinder $[i]$.

We say that the potential φ is

- (1) *recurrent* if $\sum \lambda^{-n} Z_n(\varphi, i) = \infty$;
- (2) *positive recurrent* if it is recurrent and $\sum n \lambda^{-n} Z_n^*(\varphi, i) < \infty$;
- (3) *null recurrent* if it is recurrent and $\sum n \lambda^{-n} Z_n^*(\varphi, i) = +\infty$;
- (4) *transient* if $\sum \lambda^{-n} Z_n(\varphi, i) < \infty$.

4.9. Generalized Ruelle's Perron-Frobenius (GRPF) theorem.

Our standing assumption now is that the potential φ has summable variations and finite Gurevic-Sarig pressure, i.e., $\mathcal{P}(\varphi) < \infty$. The following result provides a complete characterization of each of the above types of potentials, i.e., the necessary and sufficient conditions for the potential φ to belong to one of the above classes.

Theorem 4.5 (Sarig, [21, 22, 25]). *We have that*

1. φ is recurrent if and only if there are $\lambda > 0$, a positive continuous function h and a conservative measure ν (i.e., a measure

that allows no nontrivial wandering sets) which is finite and positive on cylinders such that

$$\mathcal{L}_\varphi h = \lambda h, \quad \mathcal{L}_\varphi^* \nu = \lambda \nu.$$

In this case $\lambda = \exp \mathcal{P}(\varphi)$.

2. φ is positive recurrent if and only if $\int h d\nu = 1$; in this case for every cylinder $[a]$,

$$\lambda^{-n} \mathcal{L}_\varphi^n(1_{[a]})(x) \rightarrow h(x) \frac{\nu([a])}{\int h d\nu} \text{ as } n \rightarrow \infty$$

uniformly in x on compact sets. Furthermore, ν is a conformal measure for φ (and hence, a DLR measure) and $\mu_\varphi = h\nu$ is the unique σ -invariant Gibbs measure for φ .

3. φ is null recurrent if and only if $\int h d\nu = \infty$; in this case for every cylinder $[a]$,

$$\lambda^{-n} \mathcal{L}_\varphi^n(1_{[a]})(x) \rightarrow 0 \text{ as } n \rightarrow \infty$$

uniformly in x on compact sets.

4. φ is transient if and only if there are no conservative measures ν which are finite on cylinders and such that $\mathcal{L}_\varphi^* \nu = \lambda \nu$ for some $\lambda > 0$.

This theorem is a far reaching generalization of Theorem 3.2 and it covers earlier results by Vere-Jones [30, 31], by Aaronson and Denker [1] and by Yuri [35, 36]. In particular, the result by Yuri requires

- *finite images property (FIP)*: the set $\{\sigma([i]) : i \in S\}$ is finite;

and the result by Aaronson and Denker requires the big image property; the latter was extended by Sarig to

- *big images and pre-images property (BIP)*: there exist $i_1, \dots, i_m \in S$ such that for all $j \in S$ there are $1 \leq k, \ell \leq m$ for which $a_{i_k j} a_{j i_\ell} = 1$.

In fact, the BIP property can be used to characterize existence of σ -invariant Gibbs measures.

Theorem 4.6 (Sarig, [25]). *Assume that the potential φ has summable variations. Then φ admits a unique σ -invariant Gibbs measure μ_φ if and only if*

1. X satisfies the BIP property;
2. $\mathcal{P}(\varphi) < \infty$ and $\text{var}_1 \varphi < \infty$ (i.e., $\sum_{n \geq 1} \text{var}_n(\varphi) < \infty$).

In this case φ is positive recurrent and $\mu_\varphi = h\nu$, where ν is the conformal measure for φ in the GRPF Theorem 4.5.

4.10. Existence and uniqueness of equilibrium measures. Let φ be a potential on Σ_A^+ . The following result by Sarig, [25] provides some sufficient conditions on φ that guarantee existence of an equilibrium measure for φ .

Theorem 4.7. *Assume that the potential φ has summable variations and $\mathcal{P}(\varphi) < \infty$. Assume also that φ is positive recurrent and $\sup \varphi < \infty$. If the measure ν in the GRPF theorem is such that the measure $\mu_\varphi = h\nu$ has finite entropy then μ_φ is an equilibrium measure for φ .*

We shall now state a result by Buzzi and Sarig, [4] that guarantees uniqueness of the equilibrium measure for the potential φ . It also shows that the requirement that φ is positive recurrent is necessary for the existence of the equilibrium measure provided the potential is bounded from above.

Theorem 4.8. *Assume that the potential φ has summable variations and $\mathcal{P}(\varphi) < \infty$. Assume also that $\sup \varphi < \infty$. Then φ has at most one equilibrium measure. In addition, if such a measure exists then φ is positive recurrent and this measure coincides with the measure ν in the GRPF theorem and has finite entropy.*

4.11. Ergodic properties.

Theorem 4.9 (Sarig, [21, 25]). *Assume that the potential φ has summable variations and $\mathcal{P}(\varphi) < \infty$. Assume also that $\sup \varphi < \infty$. If $\mu = \mu_\varphi$ is an equilibrium measure for φ then μ is strongly mixing and*

$$h_\mu(\sigma) = \int \log \frac{d\mu}{d\mu \circ \sigma} d\mu \text{ (the entropy formula).}$$

The strong mixing property is a corollary of a general result by Aaronson, Denker and Urbanski that claims that if ν is a non-singular σ -invariant measure, which is finite on cylinders, conservative and whose log of the Jacobian has summable variations, then ν is strongly mixing. One can show that the measure μ in the above theorem is indeed a Bernoulli measure.

4.12. Decay of correlations and CLT.

Theorem 4.10 (Sarig, [25]). *Assume that the potential φ is locally Hölder continuous and $\mathcal{P}(\varphi) < \infty$. Assume also that $\sup \varphi < \infty$. Then the equilibrium measure μ_φ for φ has exponential decay of correlations (with respect to the class of Hölder continuous functions on X) and satisfies the Central Limit Theorem.*

The proof of this results is based on the crucial *spectral gap property* (SGP) of the Ruelle operator that claims that in an appropriate (sufficiently “large”) Banach space \mathcal{B} of continuous functions $\mathcal{L}_\varphi = \lambda\mathcal{P} + \mathcal{N}$ where $\lambda = \mathcal{P}(\varphi)$ and

$$\mathcal{P}\mathcal{N} = \mathcal{N}\mathcal{P} = 0, \quad \mathcal{P}^2 = \mathcal{P}, \quad \dim(\text{Im}\mathcal{P}) = 1.$$

Furthermore, the spectral radius of \mathcal{N} is less than λ . The SGP implies the exponential rate of convergence in (3.4) leading to the exponential decay of correlations and the Central Limit Theorem for functions in \mathcal{B} (see [8]).

For subshifts of finite type there is a subspace \mathcal{B} on which the Ruelle operator has the SGP (due to Ruelle and Doeblin-Fortet) but this may not be true for countable Markov chains due to the presence of phase transitions. Indeed, the SGP guarantees that the function $t \rightarrow p(t) = \mathcal{P}(\varphi + t\psi)$ (where φ and ψ are locally Hölder continuous) is real-analytic implying uniqueness of equilibrium measures.⁴ However, for countable Markov chains as t varies the function $\varphi + t\psi$ can change its mode of recurrence (e.g., move from being positive recurrent to null recurrent or to transient) resulting in non-analyticity of the function $p(t)$ and hence, the appearance of phase transitions. Given a subshift of countable type, Cyr and Sarig found a necessary and sufficient condition for the existence of a space \mathcal{B} on which the Ruelle operator has the SGP. Using this condition they showed that absence of phase transitions is open and dense in the space of locally Hölder continuous potentials (see [5]).

4.13. Analyticity of the pressure function. We complete our presentation by describing a result of Sarig that guarantee analyticity of the pressure function (and hence absence of phase transitions). We recall that our standing assumption is that the potential φ has summable variations and finite Gurevich-Sarig pressure. We are interested in the existence of directional derivatives $\frac{d}{dt}|_{t=0}\mathcal{P}(\varphi + t\psi)$ and we restrict ourself to the set of directions $\text{Dir}(\varphi)$ which consists of those ψ for which $\sum_{n=2}^{\infty} V_n(\psi) < \infty$ (recall that $V_n(\psi)$ is the n -th variation of ψ) and there exists $\varepsilon > 0$ such that for any $|t| < \varepsilon$ we have $\mathcal{P}(\varphi + t\psi) < \infty$.

Theorem 4.11 (Sarig,[22]). *If φ is strongly positive recurrent then for every $\psi \in \text{Dir}(\varphi)$ that is Hölder continuous, there exists $\varepsilon > 0$ such that $\varphi + t\psi$ is positive recurrent for all $|t| < \varepsilon$ and such that the function $t \rightarrow \mathcal{P}(\varphi + t\psi)$ is real analytic in $(\varepsilon, \varepsilon)$.*

⁴In statistical physics this situation corresponds to absence of phase transitions: a phase transition occurs if there is more then one equilibrium measure for the given potential φ .

Of particular interest is the case when $\psi = \varphi$ as it appears in the study of the one-parameter family of potentials $\{\beta\varphi\}_{\beta \geq \beta_0}$. One can show that if $\mathcal{P}(\beta_0\varphi) < \infty$, then $\mathcal{P}(\beta\varphi) < \infty$ for all $\beta > \beta_0$ and hence, $\varphi \in \text{Dir}(\beta\varphi)$. This however, may not be true for $\beta = \beta_0$; see an example in [24].

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