POINTWISE HYPERBOLICITY IMPLIES UNIFORM HYPERBOLICITY

BORIS HASSELBLATT, YAKOV PESIN, AND JÖRG SCHMELING

ABSTRACT. We provide a general mechanism for obtaining uniform information from pointwise data. A sample result is that if a diffeomorphism of a compact Riemannian manifold has pointwise expanding and contracting continuous invariant cone families, then the diffeomorphism is an Anosov diffeomorphism, *i.e.,* the entire manifold is uniformly hyperbolic.

1. INTRODUCTION

We present a novel combination of ideas (from descriptive set theory and hyperbolic dynamical systems) that provides a way of obtaining uniform information from nonuniform assumptions.

To give a flavor of the immediate application to hyperbolic dynamics (mainly Theorem 4), consider a diffeomorphism f of a compact smooth Riemannian manifold M. In the hyperbolic theory one studies the exponential growth rates of the size of vectors under repeated application of the differential Df and often obtains estimates on a subset $X \subset M$ to the effect that

$$\|D_x f^n(v)\| \geq A(x)\lambda^n(x)\|v\| \text{ for every } n \in \mathbb{N} \text{ and } x \in X,$$

whenever v belongs to a certain subspace E_x of T_xM. Here $A\colon X \to \mathbb{R}^+$ is a Borel function and $\lambda\colon X \to (1,\infty)$ is an f-invariant Borel function. Note that this is equivalent to

$$\varliminf_{n\to\infty} \frac{1}{n} \min_{v\in E_x, \|v\|=1} \log\|D_x f^n(v)\| > 0$$

for all $x \in X$. This condition can also be characterized by the existence of an invariant cone family C such that vectors in $C(x)$ expand in the same way as above. In general, C, E, A and λ are not continuous. Remarkably, we can nevertheless show that if X is compact and A and $\lambda - 1$ are positive and C is continuous (*i.e.,* the system is pointwise hyperbolic in the cones),

---

Date: September 25, 2013.

Boris Hasselblatt was supported in part by a Grant-in-Aid and a Summer Faculty Research Fellowship from the Arts and Sciences Committee on Faculty Research Awards at Tufts University.

Yakov Pesin was supported in part by NSF grant DMS 1101165.

then there is a positive lower bound for both A and $\lambda - 1$ (*i.e.,* the system is *uniformly* expanding in the E-direction). We do allow degenerate cones, so as a special case this includes a theorem about continuous E implying uniform expansion.

In [3], Cao proved that the requirement that X is compact can be weakened to the assumption that X is a set of *total probability*, meaning its complement has zero measure with respect to any invariant probability measure[1]. We stress that our approach is quite different. Subadditivity of the expansion is a crucial ingredient in Cao's arguments, which makes it necessary for him to assume the existence of an invariant subbundle on which the expansion assumption is imposed (and which enables him to improve on [1][2]). By contrast, we only need to assume that there is an invariant expanding cone field; indeed, we do not even assume strict invariance. Our approach is in a manner analogous to the way the definition of hyperbolicity (by Anosov and in terms of invariant subbundles) was followed by the characterization of hyperbolicity in terms of invariant cone fields (by Alekseev).

In [7], Mañé proved a statement similar to ours that differs in two ways. On one hand, he does not require the expansion to be exponential (just unboundedness of orbits of the differential). On the other hand, his method requires information about behavior of the system in the transverse direction, which we do not. A result for continuous-time systems that is analogous to Mañé's was proved earlier by Sacker and Sell [8].

Remark 1 (Pujals)**.** A simple example may illuminate our assumptions and conclusions. A suitably-constructed derived-from-Anosov map [6, Section 17.2] will satisfy the assumptions of Theorem 4, and the conclusion produces a 1-dimensional expanding subbundle on the 2-torus. With respect to our hypotheses this illustrates that we obtain a meaningful conclusion without information about a complementary direction and without needing to construct an invariant 1-dimensional bundle a priori. This example also illustrates that the subbundle we obtain does not represent the unstable direction everywhere (for instance, not at the repelling point, whose unstable manifold is 2-dimensional), but does so in some places. It can be viewed as a fast-stable subbundle of uniform dimension in a context where the dimension of the unstable subbundle varies.

[1] Specifically: **Theorem B.** Let f be a C^1 diffeomorphism with a positively invariant set Λ for which the tangent bundle has a continuous invariant splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$. If f has positive Lyapunov exponents in the E^{cu} direction and negative Lyapunov exponents in the E^{cs} direction on a set of total probability, then Λ is a hyperbolic set.

[2] The pertinent result from [1] is: **Theorem A.** Let $f\colon M \to M$ be a C^1 local diffeomorphism defined in a compact Riemannian manifold. If f is nonuniformly expanding on a set of total probability, then f is uniformly expanding.

There are two principal ingredients to our method, and the combination of these is new. One ingredient is the application of ideas in descriptive set theory to the exhaustion of a compact space by compact proper subsets. The other is a careful analysis of the consequences for *smooth* dynamical systems. Descriptive set theory has been used for the study of topological dynamical systems, but we have not seen it applied to smooth dynamics. We believe that the analysis from descriptive set theory is of interest beyond the theory of dynamical systems, and we keep it in a separate section (Section 3) in a form that is ready to "plug in."

The applications presented here are meant to illustrate the practicality of our method, but they are not new, and stronger results have been known, a few of them for some time. A selection of pertinent references is [1,3–5,7].

2. Statement of results

In this section we state results that illustrate our method, beginning in a somewhat generic context and then stating dynamical consequences.

Let (V,π) be a continuous finite-dimensional normed linear bundle over a compact metric space X, $f\colon X\to X$ continuous and f_* a linear extension, *i.e.,* $\pi\circ f_* = f\circ\pi$.

Definition 2. In a linear space L, a cone $C_{E,\theta}$ of angle $\theta\ge 0$ around a subspace E is defined as the set of vectors $v\in L$ such that $\measuredangle(v,E)\le\theta$. The distance between cones C_{E_i,θ_i} is given by $\max(\measuredangle(E_1,E_2),|\theta_1-\theta_2|)$. C_{E_1,θ_1} and C_{E_2,θ_2} are said to be *transverse* if $\measuredangle(E_1,E_2)>\theta_1+\theta_2$. For any cone C in a normed linear space we write $C^1:=\{v\in C\mid \|v\|=1\}$.

A family of cones $C_x\subset V_x$ is said to be *continuous* if the defining subspaces and angles are continuous in x and f*-invariant* if $f_*(C_x)\subseteq C_{f(x)}$.

We call attention to the fact that it is convenient, albeit not essential, for us to consider "circular" cones defined by a direction and an angle. Note also that an f-invariant subbundle of V can be viewed as an invariant cone family by taking $\theta=0$.

Consider a continuous function $a\colon V\to\mathbb{R}^+$. It is said to be *homogeneous* if $a(\alpha v)=|\alpha|a(v)$ for $v\in V$, $\alpha\in\mathbb{R}$. Let

$$\varphi_n(x):=\frac{1}{n}\min_{v\in C_x^1}\log a(f_*^n v).$$

In our results the case of a being a norm is of obvious interest, but we retain this generality to emphasize that we mainly use homogeneity of a. However, we will need to make a nondegeneracy assumption on a that is automatic in the case of norms (and ensures that (5) on page 10 gives a positive number.) Since such an assumption can take various forms, we describe these before stating the results themselves. The most straightforward condition is

$$\min_{z\in X,\ v\in C_z^1} a(v) > 0, \tag{1}$$

where C_x is the cone family in the statement of Theorem 3. This in turn can be seen to be a consequence of the following assumption:

$$\text{there is an } M\in\mathbb{R} \text{ such that if } v\in C_x^1, \text{ then } a(f_* v)\le Ma(v) \tag{2}$$

combined with the assumption $\varphi > 0$ of the theorem. If a is a norm and f_* is the action of the differential of a diffeomorphism, then one can take M to be the maximum of the usual norm of the differential.

A different assumption that serves equally well in the proof is that

$$\min_{z\in X,\ v\in C_z^1} a(f_* v) > 0. \tag{3}$$

At face value, this assumption is slightly weaker than (1), but its main interest lies in the observation that positivity of φ and continuity can be combined to observe that there is an iterate of f for which this condition holds. This means that without any of the preceding assumptions one obtains the conclusion of Theorem 3 for an iterate of f.

Theorem 3. *Assume that a is homogeneous and that there is a continuous invariant cone family $\{C_x\}_{x\in X}$ on X such that (1) or (2) or (3) holds. If $\varphi(x) := \underline{\lim}_{n\to\infty}\varphi_n(x) > 0$ for all $x\in X$, then there exist $\chi > 0$ and $N\in\mathbb{N}$ such that $\varphi_n \ge \chi$ for all $n\ge N$. In particular, there is a $C>0$ such that $a(f_*^n v)\ge C\cdot e^{\chi n}$ for all $n\in\mathbb{N}$ and $v\in C_x^1$.*

Applying Theorem 3 with the function $a(x,v) := \|v\|$ we obtain

Theorem 4. *Let f be a C^1 diffeomorphism of a compact smooth manifold M with a compact invariant set K on which there is a continuous invariant cone family $C_x\subset T_xM$ and*

$$\underline{\lim}_{n\to\infty}\frac{1}{n}\min_{v\in C_x^1}\log(\|D_xf^n(v)\|) > 0 \tag{4}$$

for all $x\in K$. Then there exist $c,\chi>0$ such that

$$\|D_xf^n(v)\|\ge c\cdot e^{\chi n} \text{ for every } x\in K,\ v\in C_x^1 \text{ and } n\in\mathbb{N}.$$

Furthermore, for every $x\in K$ there is a subspace $E_x\subset T_xM$ such that $Df(E_x) = E_{f(x)}$ and $\|D_xf^n(v)\|\ge c\cdot e^{\chi n}$ for every $v\in E_x$, $\|v\|=1$ and $n\in\mathbb{N}$.

Theorem 5. *Let $K\subset M$ be a compact f-invariant set that admits two continuous transverse cone families C_x and D_x on $TM\restriction_K$ such that for all $x\in K$ each $v\in C_x$ has positive Lyapunov exponent, and each $v\in D_x$ has* *negative Lyapunov exponent. Then K is a* uniformly *hyperbolic set for f. In particular, if $K = M$, then f is an Anosov diffeomorphism.*

Remark 6. This result in proved in [7], and a continuous-time version is due to Sacker and Sell [8].

Proof. Theorem 4 provides two continuous cone fields with uniform 1-step estimates of contraction and expansion, respectively, for an iterate. This implies hyperbolicity by the Alekseev cone criterion: An invariant set X for a diffeomorphism f is uniformly hyperbolic if and only if it supports continuous cone families C and D that are strictly invariant for f and f^{-1}, respectively, and such that vectors in C_x are expanded and vectors in D_x are contracted by factors that are bounded away from 1 [6]. □

The continuity assumption in Theorem 5 is essential as demonstrated by an example with a homoclinic tangency [5]. In such examples consideration of the images of the tangency points shows that the invariant subbundles cannot be uniformly continuous and are hence discontinuous.

This leads to a natural question: Under what conditions is it possible to prove our main result when the cone family is only assumed to be a Baire family, i.e., a pointwise limit of continuous cone families? (More properly, this should be called a cone family in the first Baire class.)

3. Transfinite hierarchy of set filtrations

This section presents the core of our method, which is a set-theoretic construction that consists of a detailed study of representations of a compact space as a nested union of compact sets. This could easily be presented in even greater generality, but we instead carry it out in a context that is sufficient for our purposes.

What we do in this section is not difficult, but it might nevertheless help to motivate the idea. It is modeled on the proof that a positive continuous function φ on a compact space has a positive minimum: The open cover by sets $\varphi^{-1}((1/n,\infty))$ has a finite subcover. If one wanted to extend this proof to Baire functions (under suitable additional conditions, of course) one might try to cover the space with the interiors of the sets $\varphi^{-1}([1/n,\infty))$. If this does not succeed, one could pass to the set that remains after deleting all these interiors, with the subspace topology, and then repeat the process. The object of this section is to describe a transfinite process of this sort and to show how this provides information of the desired kind. (The main item is Proposition 13(1).) In particular, it provides a handle for showing that the process does indeed terminate in one step which, in the example of a positive Baire function, would then establish that the minimum is positive. Our applications rest on using specific information to show that the process terminates immediately.

3.1. **Filtrations.** Let (X,d) be a compact separable metric space.

Definition 7. A *set filtration* or simply *filtration* of X is a collection of compact sets $X_n \subset X$ such that $\bigcup_{n\in\mathbb{N}} X_n = X$ and $X_n \subseteq X_{n+1}$ for $n \in \mathbb{N}$ with $X_n \subsetneqq X_{n+1}$ if $X_{n+1} \neq X$. We say that X is *uniform* with respect to this filtration if $X = X_n$ for some $n \in \mathbb{N}$.

Lemma 8. *If X is compact, $\{X_n\}_{n\in\mathbb{N}}$ a filtration, then $X = \operatorname{Cl} \bigcup_{n\in\mathbb{N}} \operatorname{Int}(X_n)$, where* Cl *denotes closure.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$. Let B be the closed $1/k$-ball around x. To produce an $x_k \in B \cap \bigcup_{n\in\mathbb{N}} \operatorname{Int}(X_n)$ note that

$$B = B \cap X = B \cap \bigcup_{n\in\mathbb{N}} X_n = \bigcup_{n\in\mathbb{N}} B \cap X_n$$

is a complete metric space and hence not a countable union of sets of first category. Thus, there exists an $N \in \mathbb{N}$ such that $X_N \cap B$ is of second category and hence not nowhere dense. This means that

$$\varnothing \neq \operatorname{Int}_B(\operatorname{Cl} X_N) = \operatorname{Int}_B(X_N) \subset B \cap \operatorname{Int}(X_N),$$

where Int_B denotes the interior in the subspace topology of B. This means that there is an $x_k \in B \cap \bigcup_{n\in\mathbb{N}} \operatorname{Int}(X_n)$. □

The set $\Gamma := X \smallsetminus \bigcup_{n\in\mathbb{N}} \operatorname{Int}(X_n)$ is clearly compact.

Lemma 9. $\Gamma = \{x \in X \mid \exists x_n \to x \colon x_n \notin X_n\}$.

Proof. If $x \in \Gamma = X \smallsetminus \bigcup_{n\in\mathbb{N}} \operatorname{Int}(X_n)$ there exist $y_n \to x$ such that $y_n \notin \operatorname{Int}(X_n)$, and by definition of interior we can find $x_n \notin X_n$ such that $d(x_n, y_n) < 1/n$. Thus $\Gamma \subset \{x \in X \mid \exists x_n \to x \colon x_n \notin X_n\}$. The reverse inclusion is clear because $x_n \notin X_n \Rightarrow x_n \notin \operatorname{Int}(X_n)$. □

3.2. **The hierarchy.** In view of Lemma 8 we wish to exhaust the set X with the interiors of sets X_n from the filtration. This leaves uncovered the compact set Γ, and we now describe how to continue this process recursively in a transfinite way.

Let $X_n^{(0)} := X_n$, $F^{(0)} := X$ and $\Gamma^{(0)} := \Gamma$. Given an ordinal β such that we already have sets $\Gamma^{(\alpha)}$ for all $\alpha < \beta$ we inductively define

- $F^{(\beta)} := \bigcap_{\alpha<\beta} \Gamma^{(\alpha)}$,
- $X_n^{(\beta)} := F^{(\beta)} \cap X_n$,
- $\Gamma^{(\beta)} := \operatorname{Cl}_{F^{(\beta)}} \big(\bigcup_{n\in\mathbb{N}} \operatorname{Int}_{F^{(\beta)}}(X_n^{(\beta)})\big) \smallsetminus \bigcup_{n\in\mathbb{N}} \operatorname{Int}_{F^{(\beta)}}(X_n^{(\beta)}) \subset F^{(\beta)}$,

where $\operatorname{Cl}_{F^{(\beta)}}$ denotes the closure in the subspace topology of $F^{(\beta)}$. Our next lemma implies that taking the ambient closure gives the same set.

Lemma 10 (Compactness)**.** *$\Gamma^{(\beta)}$, $F^{(\beta)}$ and $X_n^{(\beta)}$ are compact.*

Proof. For $\beta = 0$ this is compactness of Γ, X and X_n. We now proceed by induction assuming that $\Gamma^{(\alpha)}$ is compact for all $\alpha < \beta$. Then $F^{(\beta)}$ is compact because it is defined by an intersection of compact sets. Since X_n is compact, this implies compactness of $X_n^{(\beta)}$. Finally, $\Gamma^{(\beta)}$ is a closed subset of $F^{(\beta)}$, hence also compact. □

Proposition 11. *The sets $F^{(\beta)}$, $X_n^{(\beta)}$ and $\Gamma^{(\beta)}$ have the following properties:*

1. *(Nesting) If $\alpha < \beta$, then $F^{(\beta)} \subseteq F^{(\alpha)}$, $X_n^{(\beta)} \subseteq X_n^{(\alpha)}$, and $\Gamma^{(\beta)} \subseteq \Gamma^{(\alpha)}$.*
2. *(Filtration) $\bigcup_{n\in\mathbb{N}} X_n^{(\beta)} = F^{(\beta)}$ and $X_n^{(\beta)} \subseteq X_{n+1}^{(\beta)}$.*
3. *$F^{(\beta)} = \mathrm{Cl}\bigcup_{n\in\mathbb{N}} \mathrm{Int}_{F^{(\beta)}}(X_n^{(\beta)})$. Thus $\Gamma^{(\beta)} = F^{(\beta)} \smallsetminus \bigcup_{n\in\mathbb{N}} \mathrm{Int}_{F^{(\beta)}}(X_n^{(\beta)})$.*
4. *(Stabilization only at $\varnothing$) If $\alpha < \beta$ and $F^{(\alpha)} \neq \varnothing$ then $F^{(\beta)} \subsetneqq F^{(\alpha)}$.*
5. *$F^{(\alpha+1)} = \Gamma^{(\alpha)}$ and hence $X_n^{(\alpha+1)} = \Gamma^{(\alpha)} \cap X_n$.*

Proof. (1) This is clear for $F^{(\beta)}$ from the definition and then immediately follows for $X_n^{(\beta)}$ as well. $\Gamma^{(\beta)} \subseteq F^{(\beta)} = \bigcap_{\tau<\beta} \Gamma^{(\tau)} \subseteq \Gamma^{(\alpha)}$.

(2) $F^{(\beta)} = F^{(\beta)} \cap X = F^{(\beta)} \cap \bigcup_{n\in\mathbb{N}} X_n = \bigcup_{n\in\mathbb{N}} F^{(\beta)} \cap X_n = \bigcup_{n\in\mathbb{N}} X_n^{(\beta)}$ and $X_n^{(\beta)} = F^{(\beta)} \cap X_n \subset F^{(\beta)} \cap X_{n+1} = X_{n+1}^{(\beta)}$.

(3) By (2) and Lemma 10 we can apply Lemma 8 to $F^{(\beta)} = \bigcup_{n\in\mathbb{N}} X_n^{(\beta)}$.

(4) $\varnothing \neq F^{(\alpha)} = \bigcup_{n\in\mathbb{N}} X_n^{(\alpha)}$ is compact hence complete, so there is an $n_0 \in \mathbb{N}$ such that $X_{n_0}^{(\alpha)}$ is of second category in the induced topology of $F^{(\alpha)}$. Then $\mathrm{Int}\, X_{n_0}^{(\alpha)} \neq \varnothing$ because $X_{n_0}^{(\alpha)}$ is compact and nonempty. It follows that

$$F^{(\beta)} = \bigcap_{\gamma<\beta} \Gamma^{(\gamma)} \subset \bigcap_{\alpha\le\gamma<\beta} \Gamma^{(\gamma)} \subset \Gamma^{(\alpha)} = F^{(\alpha)} \smallsetminus \bigcup_{n\in\mathbb{N}} \mathrm{Int}\, X_n^{(\alpha)} \subset F^{(\alpha)} \smallsetminus \mathrm{Int}\, X_{n_0}^{(\alpha)} \subsetneqq F^{(\alpha)}.$$

(5) The $\Gamma^{(\tau)}$ are nested by (1), so $F^{(\alpha+1)} = \bigcap_{\tau<\alpha+1} \Gamma^{(\tau)} = \bigcap_{\tau\le\alpha} \Gamma^{(\tau)} = \Gamma^{(\alpha)}$. □

3.3. **Termination of the process.**

Proposition 12. *There is a countable ordinal ξ such that $F^{(\xi)} \neq \varnothing = F^{(\xi+1)}$.*

This statement reflects the tacit assumption that $F^{(0)} = X \neq \varnothing$.

Proof. X is second countable, so it has a countable base $\mathcal{U}$. If $F^{(\alpha)} \neq \varnothing$ then $F^{(\alpha)} \smallsetminus F^{(\alpha+1)} \neq \varnothing$ by Proposition 11(4). Since $F^{(\alpha)} \smallsetminus F^{(\alpha+1)}$ is open in the subspace topology of $F^{(\alpha)}$, there is an $O_\alpha \in \mathcal{U}$ such that $O_\alpha \cap F^{(\alpha)} \neq \varnothing$ and $O_\alpha \cap F^{(\alpha+1)} = \varnothing$. These O_α are pairwise distinct, so there are only countably many α for which $F^{(\alpha)} \neq \varnothing$. Thus, $F^{(\alpha_0)} = \varnothing$ for a countable ordinal.

The set $\{\alpha<\omega_1 \mid F^{(\alpha)}=\varnothing\}$, where ω_1 is the first uncountable ordinal, contains α_0 and is hence a nonempty subset of the well-ordered set of countable ordinals. Therefore, it contains a minimal element η.

If η is a limit ordinal, *i.e.,* it is not of the form $\xi+1$ for any ordinal ξ then there is an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of ordinals such that for all $\tau<\eta$ there is an $n\in\mathbb{N}$ for which $\tau<\alpha_n<\eta$. Hence

$$F^{(\eta)}=\bigcap_{\tau<\eta}\Gamma^{(\tau)}=\bigcap_{\tau<\eta}F^{(\tau+1)}=\bigcap_{\tau<\eta}F^{(\tau)}=\bigcap_{n\in\mathbb{N}}F^{(\alpha_n)}\neq\varnothing,$$

since $\varnothing\neq F^{(\alpha_{n+1})}\subset F^{(\alpha_n)}$, a contradiction. So we can write $\eta=\xi+1$. □

Proposition 13. *If ξ is as in Proposition 12,* i.e., $F^{(\xi)}\neq\varnothing=F^{(\xi+1)}$*, then*

(1) *$F^{(\xi)}\subset X_R$ for some $R\in\mathbb{N}$. In particular, if $\xi=0$ then X is uniform with respect to the filtration $(X_n)_{n\in\mathbb{N}}$.*
(2) *If $\tau<\xi$ then $\bigcup_{n=1}^{\infty}\operatorname{Int}(X_n^{(\tau)})\subsetneqq F^{(\tau)}$.*
(3) *If $\xi>0$ then for every $\epsilon>0$ there are $\tau<\xi$ and $N\in\mathbb{N}$ such that*

$$F^{(\tau)}\smallsetminus U_\epsilon(F^{(\xi)})\subset\bigcup_{n=1}^{N}\operatorname{Int}(X_n^{(\tau)})\subset X_N^{(\tau)}\subset X_N,$$

where $U_\epsilon(\cdot)$ denotes ϵ-neighborhood.

Proof. (1) Proposition 11(3)–(5) give

$$\varnothing=F^{(\xi+1)}=\Gamma^{(\xi)}=F^{(\xi)}\smallsetminus\bigcup_{n\in\mathbb{N}}\operatorname{Int}_{F^{(\xi)}}(X_n^{(\xi)}),$$

so $F^{(\xi)}\subset\bigcup_{n\in\mathbb{N}}\operatorname{Int}_{F^{(\xi)}}(X_n^{(\xi)})$. This open cover has a finite subcover.

(2) If $F^{(\tau)}=\bigcup_{n=1}^{\infty}\operatorname{Int}_{F^{(\tau)}}X_n^{(\tau)}$ then $\varnothing=\Gamma^{(\tau)}=F^{(\tau+1)}$ by Proposition 11(3)–(5), and $\tau\geq\xi$.

(3)

$$\bigcap_{\alpha<\xi}\Gamma^{(\alpha)}=F^{(\xi)}\neq\varnothing.$$

The $\Gamma^{(\alpha)}$ are nested compact sets, so $\inf_{\alpha<\xi}d_H(F^{(\xi)},\Gamma^{(\alpha)})=0$, where d_H is the Hausdorff distance. That is, there is a $\tau<\xi$ such that $\Gamma^{(\alpha)}\subseteq U_\epsilon(F^{(\xi)})$ whenever $\tau\leq\alpha<\xi$. In particular, for $\alpha=\tau$ we have

$$F^{(\tau)}\smallsetminus\bigcup_{n\in\mathbb{N}}\operatorname{Int}(X_n^{(\tau)})=\Gamma^{(\tau)}\subseteq U_\epsilon(F^{(\xi)}),$$

and hence

$$F^{(\tau)}\smallsetminus U_\epsilon(F^{(\xi)})\subseteq\bigcup_{n\in\mathbb{N}}\operatorname{Int}(X_n^{(\tau)}).$$

This is an open cover of a compact set. The claim then follows. □

Remark 14. This transfinite induction does not use the Continuum Hypothesis. It can easily be extended to more general topological spaces.

We close this section with a description of how this method can be used in applications. To that end suppose that $K_1 \subset K_2 \subset X$ are compact, that there is a filtration of X, and that K_1 is uniform. Assume also that $K_2 \smallsetminus O$ is known to be uniform whenever O is open and $K_1 \subset O$. If there is a uniform neighborhood U of K_1 then we can conclude that $K_2 \subset U \cup K_2 \smallsetminus U$ is uniform as well.

In our applications we use this idea to show that X is uniform, and we argue by contradiction. We first establish that $K_1 := F^{(\xi)}$, which is uniform by Proposition 13(1), has a uniform ϵ-neighborhood U_ϵ (see Lemma 15). This is the main step in the proof. Now we observe that if $\xi > 0$ in Proposition 12 then we can take $\tau < \xi$ as in Proposition 13(3) and conclude from the above that $K_2 := F^{(\tau)}$ is uniform. Since this implies that $F^{(\tau+1)} = \varnothing$, we conclude that $\tau + 1 > \xi$ after all, a contradiction. Consequently, $\xi = 0$, and $X = F^{(0)}$ is uniform by Proposition 13(1), as claimed.

4. Proof of Theorem 3

With the assumptions of Theorem 3 consider the filtration of X by

$$X_n := \{x \in X \mid \varphi_k(x) \geq 1/n \text{ for } k \geq n\} \subset \{x \in X \mid \varphi(x) \geq 1/n\}.$$

We will show that the number ξ in Proposition 12 is equal to 0, which by Proposition 13(1) implies that X is uniform with respect to $\{X_n\}_{n \in \mathbb{N}}$.

Lemma 15. *There exist $C, \epsilon > 0$ and $\lambda > 1$ such that if $f^n(x) \in U_\epsilon(F^{(\xi)})$ whenever $0 \leq n \leq K$ for some $K \in \mathbb{N}$ then*

$$\min_{v \in C_x^1} a(f_*^n(v)) \geq C\lambda^n \text{ whenever } 0 \leq n \leq K.$$

Proof. By Proposition 13(1) there is an $R \in \mathbb{N}$ (which depends on ξ) such that $F^{(\xi)} \subset X_R$. Thus for all $n \geq R$ and $y \in F^{(\xi)}$ we have

$$\frac{1}{R} \leq \varphi_n(y) = \frac{1}{n} \min_{v \in C_y^1} \log a(f_*^n(v)),$$

hence $\min_{v \in C_y^1} a(f_*^n(v)) \geq e^{n/R}$.

Now take $L \in \mathbb{N}$ such that if $y \in F^{(\xi)}$, then

$$\min_{v \in C_y^1} a(f_*^L(v)) \geq 3 \max_{x \in X, v \in C_x^1} a(v)$$

and write $g = f_*^L$. Note that L depends only on R and hence only on ξ. If $v \in C_y^1$, then

$$a(g^n v) = \frac{a\Big(g\big(\frac{g^{n-1}v}{\|g^{n-1}v\|}\big)\Big)}{a\big(\frac{g^{n-1}v}{\|g^{n-1}v\|}\big)} a(g^{n-1}v) \geq 3a(g^{n-1}v) \geq \dots$$

$$\geq 3^{n-1} a(gv) \geq 3^{n-1}(3 \max_{x\in X, v\in C_x^1} a(v))$$

Thus, for $n \in \mathbb{N}$ and $y \in F^{(\xi)}$ we have

$$\min_{v\in C_y^1} a(g^n(v)) \geq 3^n \max_{x\in X, v\in C_x^1} a(v).$$

If $K \leq L$ the conclusion of Lemma 15 is obtained by taking

$$C \leq \min_{1\leq n\leq L} \min_{v\in C_x^1} a(f_*^n(v))\lambda^{-n}, \tag{5}$$

where $\lambda > 1$ can be chosen arbitrarily, *e.g.*, as below. This is positive by our nondegeneracy assumption (1) or (2) or (3) on a (see page 4). For $K > L$ we continue as follows.

Recall that the choice of L gives

$$\min_{v\in C_y^1} a(g(v)) \geq 3 \max_{x\in X, v\in C_x^1} a(v).$$

For any $x \in U_\epsilon(F^{(\xi)})$ we can choose $y \in F^{(\xi)}$ such that $d(x, y) < \epsilon$. Then

$$\min_{v\in C_x^1} a(g(v)) = \min_{v\in C_y^1} a(g(v)) \frac{\min_{v\in C_x^1} a(g(v))}{\min_{v\in C_y^1} a(g(v))},$$

and by continuity of a and of the cone family C_x we can choose ϵ so small that the last fraction is bounded below by $2/3$. (Thus, ϵ depends on L and R and hence ultimately only on ξ. Note also that this is the only place where we use continuity of a and the cone family C_x instead of mere boundedness conditions.) This gives

$$\min_{v\in C_x^1} a(g(v)) \geq \frac{2}{3} \min_{w\in C_y^1} a(g(w)) \geq 2 \max_{z\in X, w\in C_z^1} a(w)$$

$$\geq 2 \max_{z\in U_\epsilon(F^{(\xi)}), w\in C_z^1} a(w) \geq 2 \max_{v\in C_x^1} a(v), \tag{6}$$

which implies that $a(g(v)) \geq 2a(v)$ whenever $\|v\| = 1$. Thus, for any $n \in \mathbb{N}$ such that $nL \leq K$ we find that

$$\min_{v\in C_x^1} a(g^n(v)) = \min_{v\in C_x^1} a(g(g^{n-1}(v))) \geq 2\min_{v\in C_x^1} a(g^{n-1}(v)) \geq \dots$$
$$\geq 2^{n-1}\min_{v\in C_x^1} a(g(v)) \geq 2^n \max_{z\in X, w\in C_z^1} a(w)$$

by (6).

With $C' := \Big(\min_{z\in X, w\in C_z^1} \frac{a(f_*(w))}{a(w)}\Big)^L$, $C := \frac{C'}{2}\max_{z\in X, w\in C_z^1} a(w)$, $\lambda := 2^{1/L}$, $n = kL + r$, $0 \leq r < L$, and $v \in C_x^1$ we get

$$a(f_*^n(v)) = a(f_*^{kL+r}(v)) = \frac{a(f_*^n(v))}{a(f_*^{n-1}(v))}\cdots\frac{a(f_*^{n-r+1}(v))}{a(f_*^{n-r}(v))}a(g^k(v))$$
$$\geq C'\cdot 2^k \max_{z\in X, w\in C_z^1} a(w) = \frac{C'}{2}(2^{(k+1)/n})^n \max_{z\in X, w\in C_z^1} a(w) \geq C\cdot\lambda^n. \quad \square$$

We now conclude the proof of Theorem 3. Recall that we chose ξ as in Proposition 12 which determines R via Proposition 13(1), and these parameters in turn determine the choice of ϵ in Lemma 15.

Suppose that $\xi > 0$ and choose $\tau < \xi$ and N as in Proposition 13(3).

Consider any $x \in F^{(\tau)}$. If there is a $k_0 \in \mathbb{N}_0$ such that $f^k(x) \in U_\epsilon(F^{(\xi)})$ for $k < k_0$ and $f^{k_0}(x) \notin U_\epsilon(F^{(\xi)})$ then $f^{k_0}(x) \in X_N^{(\tau)}$. Thus for any $v \in C_x^1$ we have

$$a(f_*^n(v)) = a(f_*^{\max(0,n-k_0)}(f_*^{\min(n,k_0)}(v)))$$
$$\geq e^{\max(0,n-k_0)/N} a(f_*^{\min(n,k_0)}(v)) \geq C\lambda^{\min(n,k_0)} e^{\max(0,n-k_0)/N} \geq C\gamma^n$$

for all $n \in \mathbb{N}$, where $\gamma := \min(\lambda, e^{1/N}) > 1$. Note that the same estimate holds if $f^n(x) \in U_\epsilon(F^{(\xi)})$ for all $n \in \mathbb{N}$, so it holds for all $x \in F^{(\tau)}$.

It is easy to check that this implies that $F^{(\tau)} \subset X_{2\max(1,-\log C)/\log\gamma}$. By Proposition 11(3) we conclude that $F^{(\tau+1)} = \varnothing$, and hence $\tau \geq \xi$, which is contrary to our choice of τ.

Remark 16 (Fathi)**.** One might recast the arguments in a slightly different way that does not use as much descriptive set theory; we indicate this in the case where $f: M \to M$ is pointwise hyperbolic. Denote by F the family of closed (hence compact) nonempty f-invariant subsets that are *not* uniformly hyperbolic; it is stable under nested intersections due to the stability of uniformly hyperbolic sets. To see that $F = \varnothing$, note that otherwise by Zorn's Lemma there is an element of F that is minimal with respect to inclusion—and yet, by the arguments using the filtration X_n

and the Baire Category Theorem, contains a proper subset that is not uniformly hyperbolic. A contradiction.

By Brouwer's Reduction Principle one should here even be able to replace Zorn's Lemma by an induction over the integers.

Acknowledgments. This paper was conceived during Research-in-Pairs visits of all three authors at the Mathematisches Forschungsinstitut Oberwolfach July 15–28, 2007 and June 15–28, 2008, and finalized while visiting the Centre Interfacultaire Bernoulli of the École Polytechnique Fédérale de Lausanne March 12-21, 2013. We are grateful for the hospitality and support of both institutions—and for the careful reading by the referee.

References

[1] J. F. Alves, V. Araújo and B. Saussol, On the uniform hyperbolicity of some nonuniformly hyperbolic systems, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1303–1309

[2] Luis Barreira and Yakov Pesin: *Nonuniform Hyperbolicity: Dynamics of Systems with nonzero Lyapunov Exponents*, Cambridge University Press (2007).

[3] Yongluo Cao: *Non-zero Lyapunov exponents and uniform hyperbolicity*, Nonlinearity **16** (2003) 1473–1479

[4] Yongluo Cao, Stefano Luzzatto, Isabel Rios: *A minimum principle for Lyapunov exponents and a higher-dimensional version of a theorem of Mañé*, Qual. Theory Dyn. Syst. **5** (2004), no. 2, 261–273.

[5] Yongluo Cao, Stefano Luzzatto, Isabel Rios: *Some non-hyperbolic systems with strictly non-zero Lyapunov exponents for all invariant measures: horseshoes with internal tangencies*, Discrete Contin. Dyn. Syst. **15** (2006), no. 1, 61–71.

[6] Anatole Katok, Boris Hasselblatt: *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995

[7] Ricardo Mañé: *Quasi-Anosov diffeomorphisms and hyperbolic manifolds*, Transactions of the American Mathematical Society, **229** (1977), 351–370.

[8] Robert Sacker, George Sell: *Existence of dichotomies and invariant splittings for linear differential systems. I.*, J. Differential Equations 15 (1974), 429–458.

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155, USA
E-mail address: bhasselb@tufts.edu

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA
E-mail address: pesin@math.psu.edu

DEPARTMENT OF MATHEMATICS, LUND INSTITUTE OF TECHNOLOGY, LUNDS UNIVERSITET, BOX 118, SE-22100 LUND, SWEDEN
E-mail address: Jorg.Schmeling@math.lth.se