THE MULTIFRACTAL ANALYSIS OF BIRKHOFF AVERAGES AND LARGE DEVIATIONS

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ABSTRACT. For one-sided subshifts of finite type, we describe the fine structure of the exceptional set in the Birkhoff ergodic theorem for Hölder continuous functions. We show an intimate connection with large deviation theory for Birkhoff averages, and we provide several applications to probability and number theory, including a problem popularized by Billingsly.

We study the decomposition of the phase space into level sets of the Birkhoff average. We show that there are typically uncountably many dense level sets and that each level set carries an auxiliary equilibrium measure, with constant pointwise dimension (a type of self-similarity). These equilibrium measures, each supported on a measure zero set, are key to our analysis.

Let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be an topologically mixing one-sided subshift of finite type [Wal], and $\phi \in C(\Sigma_A^+, \mathbb{R})$ a continuous function. Denote by $\overline{\phi}(x)$ the Birkhoff average of ϕ along the orbit of the point x, i.e.,

$$\overline{\phi}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\sigma^k x),$$

if the limit exists. The limit function is clearly $\sigma-$invariant and can be identified with the conditional expectation at x of the function ϕ with respect to the sigma-algebra of σ-invariant functions.

For any ergodic probability measure μ, it follows from the Birkhoff ergodic theorem that for μ-almost all x

$$\overline{\phi}(x) \equiv \widehat{\phi} \equiv \int_{\Sigma_A^+} \phi \, d\mu.$$

However, the Birkhoff ergodic theorem provides no structural information about the exceptional set of measure zero.

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In probability theory and much of analysis, sets of measure zero have been considered negligible, since experts believed these sets carry no essential information about the measure. It was well understood that this exceptional set of measure zero supports other ergodic invariant measures, but they seem to have been of no use to study the initial measure. Recent work in the multifractal analysis of dynamical systems has changed this point of view. Out of the large collection of invariant measures supported on this exceptional measure zero set, the multifractal analysis provides a mechanism to select a special one parameter family of equilibrium measures and relate them to a moment generating function associated with the initial measure. We show that this determines the large deviation theory of Birkhoff averages. We make this precise in Section III.

Our multifractal analysis of the exceptional set for Birkhoff averages begins with some natural questions, such as whether $\overline{\phi}$ attains any other values, and if so, what is the range of values, and what is the *dimension* and *topological structure* of the level sets? Also, are there points for which $\overline{\phi}(x)$ does not exist and if so, what is the *dimension* and *topological structure* of this set?

Another way of asking these questions is to describe the fine structure (in the sense of topology and dimension) of the decomposition of the phase space into level sets of the Birkhoff average $\overline{\phi}$, where the decomposition is given by:

$$\Sigma_A^+ = \{x \colon \overline{\phi}(x) = \widehat{\phi}\} \bigcup \bigcup_{\alpha \neq \widehat{\phi}} B_\alpha \bigcup \{x \colon \overline{\phi}(x) \text{ does not exist}\},$$

where the level set $B_\alpha \equiv \{x \colon \overline{\phi}(x) = \alpha\}$.

Our study of this decomposition is important for several reasons. One *practical* reason, which was pointed out by Michael Fisher, is related to calculations of such averages on a computer. While computing Birkhoff averages, one finds that not only does the set of points which have a different Birkhoff limit has measure zero, but points sufficiently close to these exceptional ones also tend to have a different Birkhoff limit, and thus one should understand the *structure* and *distribution* of the exceptional points when doing calculations and/or simulations.

In this note we provide a complete description of this decomposition for an important class of ergodic invariant measures, which are equilibrium measures (Gibbs measures) for Hölder continuous potentials. If ϕ is a Hölder continuous potential (function), we denote by μ_ϕ the unique equilibrium measure for ϕ. We have included a short appendix which contains the definition and some important properties of equilibrium measures. See [Kel, Rue, PP] for excellent expositions.

Equilibrium measures are dynamical systems analogs of Gibbs canonical ensembles in equilibrium statistical physics, and comprise a large class of physically and dynamically interesting measures. The Hölder continuous potential in our case is the analog of the system Hamiltonian in statistical physics. A (mixing) Markov mixing measure is an equilibrium measure, the measure of maximal entropy is an equilibrium measure, and for smooth hyperbolic attractors the natural measure (known also as SBR measure) is an equilibrium measure.

Our results on the distribution of Birkhoff averages is a rather straightforward consequence of our Multifractal Analysis (MFA) for equilibrium measures for some low-

dimensional hyperbolic dynamical systems. We have effected a MFA for pointwise dimension and Lyapunov exponents, and in this note we link the Birkhoff average with the pointwise dimension and then quote results from previous work. The relevant references are [PW1, PW2, Wei, Sc, BS], but we have tried to make this presentation almost self-contained. We study the exceptional set by decomposing $\cup_\alpha B_\alpha$ into uncountably many disjoint sets where each element of the decomposition carries an auxiliary equilibrium measure, and exploit properties of these auxiliary equilibrium measures. Most of our tools are from symbolic dynamics and thermodynamic formalism.

To state our main theorem, we need to define the auxiliary function **Birkhoff spectrum for ϕ**

$$b_\phi(\alpha) = \dim_H B_\alpha,$$

where $\dim_H F$ denotes the Hausdorff dimension of the set F.

Let us now state our main theorem. Let μ_{MAX} denotes the measure of maximal entropy (see Appendix).

Theorem 1. *Let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be a topologically mixing one-sided subshift of finite type, $\phi \in C^\gamma(\Sigma_A^+, \mathbb{R})$ a Hölder continuous function, and μ_ϕ the corresponding equilibrium measure.*

(1) *If $\mu_\phi \neq \mu_{\mathrm{MAX}}$, then the function $b_\phi(\alpha)$ is real analytic and strictly convex on an open interval (a, b). It immediately follows that $\overline{\phi}$ attains an interval of values.*

(2) *For $a \leq \alpha \leq b$, each of the level sets B_α is an uncountable dense subset of Σ_A^+. The values a and b are expressible via thermodynamic formalism (see Proof).*

(3) *The interval $[a, b]$ is maximal in the sense that $\overline{\phi}$ does not attain any value outside this interval.*

(4) *If $\mu_\phi \neq \mu_{\mathrm{MAX}}$, the set of points for which $\overline{\phi}(x)$ does not exist has maximal Hausdorff dimension, i.e., the Hausdorff dimension equals the Hausdorff dimension of Σ_A^+.*

Comments.
(i) This theorem illustrates what is sometimes called the *multifractal miracle* – even though the decomposition of the phase space into level sets of $\overline{\phi}$ is intricate and extremely complicated, the function b_ϕ that encodes this decomposition is smooth and convex.

(ii) A priori, one may consider the box dimension instead of the Hausdorff dimension in this definition of b_ϕ. However, since the level sets B_α are dense, it follows from a well known property of the box dimension[1] that the box dimension of these level sets B_α are equal to the box dimension of Σ_A^+.

We now define the notion of local dimension and relate it to Birkhoff averages. For a probability measure μ on Σ_A^+, we define the **pointwise dimension at x**, denoted $d_\mu(x)$, by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

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[1] The box dimension of a set coincides with the box dimension of the closure of the set [Pes].

provided the limit exists. Here $B(x, r)$ is the ball of radius r around the point x measured using the metric ρ_λ (see Appendix). From the definition we see that if $d_\mu(x) = \alpha$ then for balls of sufficiently small radius r the measure μ scales as $\mu(B(x, r)) \asymp r^\alpha$.

The *classical* multifractal analysis is a description of the fine-scale geometry of the decomposition of Σ_A^+ whose constituent components are the level sets

$$K_\alpha = \{x \in X \mid d_\mu(x) = \alpha\},$$

for $\alpha \in \mathbb{R}$. The **dimension spectrum of μ_ϕ** is defined by

$$f_\mu(\alpha) = \dim_H K_\alpha,$$

where $\dim_H K_\alpha$ denotes the Hausdorff dimension of the level set K_α. The dimension spectrum is one of the main objects of study of the usual MFA.

The following proposition relates the Birkhoff average to the pointwise dimension for a special equilibrium measure.

Proposition 1. *Let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be a topologically mixing one-sided subshift of finite type, $\phi \in C^\gamma(\Sigma_A^+, \mathbb{R})$ a Hölder continuous function, and μ_ϕ the corresponding equilibrium measure. Then*

$$\overline{\phi}(x) = P(\phi) - \log \lambda \cdot d_{\mu_\phi}(x), \tag{1}$$

where $P(\phi)$ denotes the thermodynamic pressure of ϕ (see Appendix).

It is obvious from this proposition that results on the level sets of pointwise dimension $d_{\mu_\phi}(x)$ can be translated into results on the level sets for the Birkhoff average $\overline{\phi}(x)$.

Proof. It follows from the definition of equilibrium measure that

$$\mu(C_n(x)) \asymp C \cdot \exp(S_n\phi(x) - nP(\phi)),$$

where $C_n(x)$ denotes the $n-$cylinder that contains the point x and $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(\sigma^k x)$. It immediately follows that

$$\lim_{n \to \infty} \frac{1}{n} \log \mu(C_n(x)) = \overline{\phi}(x) - P(\phi).$$

Since cylinder sets are metric balls, we have that

$$d_{\mu_\phi}(x) = \lim_{n \to \infty} \frac{\log \mu(C_n(x))}{\log |C_n(x)|} = \lim_{n \to \infty} \frac{\log \mu(C_n(x))}{n} \frac{n}{\log |C_n(x)|},$$

where $|C_n(x)|$ denotes the diameter of the cylinder (ball) $C_n(x)$. By definition (of the metric) there exists $B > 0$ such that $|C_n(x)| = B \exp(-n)$, and we immediately obtain that

$$\overline{\phi}(x) = P(\phi) - \log \lambda \cdot d_{\mu_\phi}(x).$$

$\blacksquare$

As an immediate consequence we obtain the following relationship between the functions $b_\phi(\alpha)$ and $f_{\mu_\phi}(\alpha)$.

Corollary 1.

$$b_\phi(\alpha) = f_{\mu_\phi}\left(\frac{P(\phi) - \alpha}{\log \lambda}\right). \tag{2}$$

Remarks and Ideas on Proof of Theorem 1.

Let $\phi \in C^\gamma(\Sigma_A^+, \mathbb{R})$ and let $\mu = \mu_\phi$ be the corresponding equilibrium measure. Define the one parameter family of functions φ_q, $q \in (-\infty, \infty)$ on Σ_A^+ by

$$\varphi_q(x) = -T(q)\log \lambda + q \log \psi(x),$$

where $\log \psi = \phi - P(\phi)$ and $T(q)$ is chosen such that

$$P(\varphi_q) = P(-T(q)\log \lambda + q\log \psi(x)) = 0. \tag{3}$$

One can show that $T(q)$ exists for every $q \in \mathbb{R}$. It is obvious that for all q the functions $\varphi_q \in C^\gamma(\Sigma_A^+, \mathbb{R})$.

The following results used to prove parts (i) and (ii) can easily be extracted from the more sophisticated argument for hyperbolic sets contained in [PW1, PW2].

The pointwise dimension $d_{\mu_\phi}(x)$ exists for μ_ϕ-almost every $x \in \Sigma_A^+$ and

$$d_{\mu_\phi}(x) = \frac{1}{\log \lambda}\int_{\Sigma_A^+} \log \psi \, d\mu_\phi.$$

The function $T(q)$ is real analytic for all $q \in \mathbb{R}$, $T(0) = \dim_H \Sigma_A^+$, $T(1) = 0$, $T'(q) \leq 0$ and $T''(q) \geq 0$ (see Figure 1A). This is a manifestation of analyticity of pressure (see Appendix) and the explicit formulas for its first and second (Frechet) derivatives.

The function $\alpha(q) = -T'(q)$ attains values in the interval $[\alpha_1, \alpha_2]$, where $0 \leq \alpha_1 \leq \alpha_2 < \infty$. The function $f_{\mu_\phi}(\alpha(q)) = T(q) + q\alpha(q)$ (see Figure 1B). This statement uses results about the family $\{\mu_{\phi_q}\}$ of equilibrium measure for $\{\phi_q\}$. The key points in the proof are verifying that $\mu_{\phi_q}(K_{\alpha(q)}) = 1$ and that the pointwise dimension $d_{\mu_{\phi_q}}(x) = T(q) + q\alpha(q)$ for $x \in K_{\alpha(q)}$. Since equilibrium measures are positive on open sets, the first property implies that the level sets K_α and B_α are dense. It is not too hard to see that the interval of definition of b_ϕ is $[a, b]$ where $a = -\lim_{q\to\infty} \alpha(q)$ and $b = -\lim_{q\to-\infty} \alpha(q)$.

Finally, if $\mu_\phi \neq \mu_{\text{MAX}}$ then the functions $f_{\mu_\phi}(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair (see Appendix). The strict convexity follows from convexity of Pressure along with explicit derivative formulas.

The proof of (3) is due to Schmeling [Sch]. The key ingredient is Simpelaere's variational formula [Sim]. The proof of (4) is due to Bareirra and Schmeling [BS]. See [PPit] (and also [She]) for a precursor result. The idea is to insert the set of points which are typical with respect to a given equilibrium measure into the set of non-typical (non-generic) points by means of a Lipschitz continuous homeomorphism. ∎

Applications and Connections to Probability and Number Theory

The following are applications of Theorem 1, Proposition 1, and Corollary 1.

FIGURE 1A GRAPH OF $T(q)$

FIGURE 1B GRAPH OF $f_{\mu_\phi}(\alpha)$

I: Borel's Law of Large numbers and Birkhoff Averages of Functions Over Markov Chains.

The Birkhoff ergodic theorem is a generalization of Borel's Strong Law of Large Numbers in the special case when the sequence of *random variables* $\{\phi \circ \sigma^n\}$ is IID (independent and identically distributed). Thus, Theorem 1 yields refined new information about this fundamental law of probability theory.

More generally, let $\{X_n\}$ be a (one-sided) irreducible Markov chain on a finite state space which has a stationary distribution π (see [Dur]). Via a well known construction [Pet, p 6] this Markov chain can be identified, via a coding map χ, with an ergodic one-sided subshift of finite type equipped with a Markov measure μ^π. Since Markov measures are equilibrium measures (for Hölder continuous potentials), we can apply Theorem 1 to describe the decomposition of the probability space $(\Omega, \mathcal{F}, \pi)$ into level sets of the

6

Birkhoff averages for a Hölder continuous function. More precisely, one can decompose the probability space

$$\Omega = \tilde{\Omega} \bigcup \left(\bigcup_\alpha \tilde{B}_\alpha \right),$$

where $\tilde{B}_\alpha = \chi(B_\alpha)$ and $\tilde{\Omega}$ is the set of points for which the Birkhoff averages do not converge. Note that $\nu(\Omega) = 0$ for *every* ergodic stationary measure ν while the set $\chi^{-1}(\tilde{\Omega}) \subset \Sigma_A^+$ has full Hausdorff dimension.

II: Fine Distribution in Borel's Normal Number Theorem. Consider the base 2 expansion of a number $x \in [0,1]$, i.e., $x = \sum_{k=0}^\infty a_k(x) 2^{-k}$, where $a_k \in \{0,1\}$. With only countably many exceptions this representation is unique. Borel proved that for (Lebesgue) almost every x the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k(x) = \frac{1}{2}.$$

This is an easy consequence of the Birkhoff ergodic theorem applied to the full one-sided shift on two symbols $\{0,1\}$, the $(1/2, 1/2)$ Bernoulli measure μ, and the (Hölder continuous) characteristic function I_k of the cylinder set $C_k = \{x \in [0,1]: x = kx_2x_3 \dots \}$.

Eggleston [Egg] and Besicovitch [Bes] (see laso [Bil]) found the following remarkable formula for the dimension of the set of points where the above limit is equal to α for $0 \le \alpha \le 1$. We provide an alternate proof of this result using our MFA of the equilibrium measure μ_{I_k} and Corollary 1.

We note that the deep connection between dynamical systems and dimension theory seems to have been first discovered by Billingsley [Bill] while studying this formula. Billingsley interpreted the formula in terms of ergodic theory, and reproved it using tools from ergodic theory.

Theorem 2. *For $0 \le \alpha \le 1$, the Hausdorff dimension*

$$\dim_H \left(x \in [0,1]: \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k(x) = \alpha \right) = H(\alpha),$$

where

$$H(\alpha) = \frac{-(1-\alpha)\log(1-\alpha) - \alpha \log \alpha}{\log 2}.$$

Proof. Consider the full shift σ on the space Σ_2^+ endowed with the metric ρ_2 (see Appendix). To effect a MFA, one considers the one parameter family of (locally constant) potentials $\phi_q = -T(q) \log 2 + q(I_k - P(I_k))$, where $T(q)$ is chosen such that $P(\phi_q) = 0$. A simple calculation shows that $P(I_k) = \log(1+e)$ and thus $\phi_q = -T(q) \log 2 + qI_k - \log(1+e)$. Since the functions ϕ_q depend only on the first coordinate, i.e., $\phi_q(\omega_1 \omega_2 \cdots) = \phi_q(\omega_1)$, it follows that $e^{\phi_q(0)} + e^{\phi_q(1)} = e^{P(\phi_q)}$. A simple calculation shows that

$$T(q) = \frac{1}{\log 2} \log \left(\frac{e^q + 1}{(1+e)^q} \right).$$

Differentiating the expression for $T(q)$ yields

$$u \equiv \alpha(q) = -T'(q) = \frac{e^q - \log(1+e) - e^q \log(1+e)}{\log(2) + e^q \log(2)},$$

and one immediately obtains $\alpha(\infty) = (-1 + \log(1+e))/\log 2$ and $\alpha(-\infty) = (\log(1 + e)/\log 2$. It follows that the dimension spectrum $f_{\mu_{I_k}}$ is defined on the interval $[(-1 + \log(1+e))/\log 2, (\log(1+e)/\log 2]$ and by Proposition 1 we see that the Birkhoff spectrum for I_k is defined on the interval $[0, 1]$. We note that α is invertible with inverse

$$q \equiv \alpha^{-1}(u) = \log\left(\frac{-u \log 2 + \log(1+e)}{1 + u \log 2 - \log(1+e)}\right).$$

By the Legendre transform relation we know that $f_{\mu_{I_k}}(\alpha(q)) = T(q) + q\alpha(q)$. It follows that $f_{\mu_{I_k}}(u) = T(\alpha^{-1}(u)) + \alpha^{-1}(u)u$. The explicit expression for $f_{\mu_{I_k}}(u)$ is complicated, but through tedious algebraic manipulation, one obtains that

$$f_{\mu_{I_k}}(u) = H(-u \log 2 + \log(1+e)),$$

where

$$H(u) = \frac{-(1-u)\log(1-u) - u \log u}{\log 2}.$$

By (2), it immediately follows that

$$b_{I_k}(\alpha) = f_{\mu_{I_k}}\left(\frac{\log(1+e) - \alpha}{\log 2}\right)$$

and thus,

$$b_{I_k}(\alpha) = H(\alpha) = \frac{-(1-\alpha)\log(1-\alpha) - \alpha \log \alpha}{\log 2}.$$

∎

The same method yields analogs of Theorem 2 for expansions with respect to an arbitrary base $b \in \{2, 3, \cdots\}$.

III: Connections With Large Deviation Theory for Birkhoff Sums.

In this section we establish an intimate connection between the Birkhoff spectrum for ϕ and (global) large deviations of the Birkhoff sums $S_n \phi$. This is an important application of our multifractal analysis for pointwise dimension.

Consider a subshift of finite type $\sigma \colon \Sigma_A^+ \to \Sigma_A^+$. Let ϕ be a Hölder continuous function on Σ_A^+ and μ the coresponding Gibbs measure. Define N_n to be the total number of cylinder sets $\{C_n^k\}$ at level n, and consider the family of random variables $X_k = \log \mu(C_n^k)$, where the integer k is randomly chosen with uniform distribution from $1, \ldots, N_n$. The moment generating function of X_k is

$$c_n(q) = \exp(qX_k) = (1/N_n) \sum_{C_n^j} \mu(C_n^j)^q.$$

In [PP1, PP2] we show that

$$\lim_{n\to\infty} \frac{\log c_n(q)}{\log n} = R(0) - R(q),$$

where

$$R(q) = \lim_{n\to\infty} \frac{\log \sum_{C_n^j} \mu(C_n^j)^q}{\log n}.$$

We also show that $R(q)$ coincides with $T(q)$, which is defined by (3), with $\lambda = 2$ and $\log \psi = \phi - P(\phi)$. We stress again that as a consequence of our multifractal analysis we obtain that $T(q)$, and thus $R(q)$, is smooth and convex. Combining these results with Ellis' large deviation theorem [E], we immediately obtain the following counting formula for the dimension spectrum:

Theorem 3. *If $\mu \neq \mu_{\max}$, then*

$$f_\nu(\alpha) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log J_n(\alpha, \varepsilon)}{\log n},$$

where $J_n(\alpha, \varepsilon)$ is the number of cylinder sets C_n^j such that $\alpha - \varepsilon < \mu(C_n^j) \leq \alpha + \varepsilon$.

For each $x \in \Sigma_A^+$ and $r > 0$, let $m_r(x)$ be the positive integer defined by

$$e^{S_{m_r}\phi(x)} \geq r \quad \text{and} \quad e^{S_{m_r+1}\phi(x)} < r.$$

As an immediate consequence of Theorem 3 and Corollary 1, we obtain a formula for large deviations of the Birkhoff sums $S_n\phi$. In this explicit form, the formula first appeared in the recent thesis of M. Kessebohmer [Kess], where he also proves a stronger *local* version of this formula.

Theorem 4. *For $\alpha(q) = -T'(q)$, where $T(q)$ is defined by (3), one has that for $q > 0$:*

$$b_\phi\left(\frac{P(\phi) - \alpha(q)}{\log 2}\right) - \dim_H(\Sigma_A^+) = \lim_{r\to 0} \frac{\log\left(\mu_{max}\left\{x : \frac{S_{n_r(x)}\phi(x)}{-\log r} \geq -\alpha(q)\right\}\right)}{-\log r}.$$

Appendix: Equilibrium Measures and Thermodynamic Formalism

This Appendix contains some essential definitions and facts from symbolic dynamics and thermodynamic formalism. For details consult [Bow, K, PP, Rue].

Let $\sigma\colon \Sigma_A^+ \to \Sigma_A^+$ be an topologically mixing one-sided subshift of finite type. The space Σ_A^+ has a natural family of metrics defined by

$$\rho_\lambda(x, y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{\lambda^k},$$

where λ is any number satisfying $\lambda > 1$. Let us choose $\lambda = 2$. The set Σ_A^+ is compact with respect to the topology induced by ρ_λ and the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is a homeomorphism.

(1) Let $g \in C(\Sigma_A^+, \mathbb{R})$. We define the **pressure** $P \colon C(\Sigma_A^+, \mathbb{R}) \to \mathbb{R}$ by

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \inf_{x \in C_{i_1 \cdots i_n}} \exp \sum_{j=0}^{n-1} \phi(\sigma^j(x)) \right).$$

(2) The pressure function $P \colon C^\gamma(\Sigma_A^+, \mathbb{R}) \to \mathbb{R}$ is real analytic. Let $\varphi \in C^\gamma(\Sigma_A^+, \mathbb{R})$. The map $\mathbb{R} \to \mathbb{R}$ defined by $t \to P(t\varphi)$ is convex. It is strictly convex unless φ is cohomologous to a constant ($\varphi \sim C$), i.e., there exists $C > 0$ and $g \in C^\gamma(\Sigma_A^+, \mathbb{R})$ such that $\varphi(x) = g(\sigma x) - g(x) + C$.

(3) Let $\varphi \in C(\Sigma_A^+, \mathbb{R})$. A Borel probability measure $\mu = \mu_\varphi$ on Σ_A^+ is called a **equilibrium measure** for the potential φ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leq \frac{\mu\{y \mid y_i = x_i, \ i = 0, \cdots, n-1\}}{\exp(-nP(\varphi) + \sum_{k=0}^{n-1} \varphi(\sigma^k x))} \leq D_2$$

for all $x = (x_1 x_2 \cdots) \in \Sigma_A^+$ and $n \geq 0$. For subshifts of finite type, equilibrium measures exist for any Hölder continuous potential φ, are unique, and coincide with the equilibrium measure for φ. Two equilibrium measure μ_ϕ and μ_ψ coincide if and only if the potentials ϕ and ψ are cohomologous. The measure of maximal entropy μ_{MAX} is the equilibrium measure with constant potential.

Facts About The Legendre Transform

Let f be a C^2 strictly convex map on an interval I, hence, $f''(x) > 0$ for all $x \in I$. The **Legendre transform** of f is the function g of a new variable p defined by

$$g(p) = \max_{x \in I}(p\,x - f(x)).$$

It is easy to show that g is strictly convex and that the Legendre transform is involutive. One can also show that strictly convex functions f and g form a Legendre transform pair if and only if $g(\alpha) = f(q) + q\alpha$, where $\alpha(q) = -f'(q)$ and $q = g'(\alpha)$.

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