Appendix B. An Example of a Smooth Hyperbolic Measure with Countably Many Ergodic Components

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B.1. Introduction. We construct an example of a diffeomorphism with nonzero Lyapunov exponents with respect to a smooth invariant measure which has countably many ergodic components. More precisely we will prove the following result.

THEOREM B.1. *There exists a C^∞ diffeomorphism f of the three dimensional torus $\mathbb{T}^3$ such that*

1. *f preserves the Riemannian volume μ on $\mathbb{T}^3$;*
2. *μ is a hyperbolic measure;*
3. *f has countably many ergodic components which are open* (mod 0).

B.2. Construction of the Diffeomorphism f. Let $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a linear hyperbolic automorphism. Passing if necessary to a power of A we may assume that A has at least two fixed points p and p'. Consider the map $F = A \times \mathrm{Id}$ of the three dimensional torus $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$. We will perturb F to obtain the desired map f.

Consider a countable collection of intervals $\{I_n\}_{n=1}^{\infty}$ on the circle $\mathbb{S}^1$, where

$$I_{2n} = [(n+2)^{-1}, (n+1)^{-1}], \quad I_{2n-1} = [1-(n+1)^{-1}, 1-(n+2)^{-1}].$$

Clearly, $\bigcup_{n=1}^{\infty} I_n = (0,1)$ and $\operatorname{int} I_n$ are pairwise disjoint.

By Proposition B.2 below, for each n one can construct a C^∞ volume preserving ergodic diffeomorphism $f_n\colon \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [0,1]$ which satisfies:

1. $\|F - f_n\|_{C^n} \le e^{-n^2}$;
2. for all $0 \le m < \infty$, $D^m f_n | \mathbb{T}^2 \times \{z\} = D^m F | \mathbb{T}^2 \times \{z\}$ for $z = 0$ or 1;
3. f_n has nonzero Lyapunov exponents μ-almost everywhere.

Let $L_n\colon I_n \to [0,1]$ be the affine map and $\pi_n = (\mathrm{Id}, L_n)\colon \mathbb{T}^2 \times I_n \to \mathbb{T}^2 \times [0,1]$. We define the map f by setting $f|\mathbb{T}^2 \times I_n = \pi_n^{-1} \circ f_n \circ \pi_n$ for all n and $f|\mathbb{T}^2 \times \{0\} = F|\mathbb{T}^2 \times \{0\}$. Note that for every $n > 0$ and $0 \le m \le n$

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[1] *Key words and phrases.* Hyperbolic measure, Lyapunov exponents, stable ergodicity, accessibility.

D. Dolgopyat was partially supported by the Sloan Foundation. H. Hu was partially supported by the National Science Foundation grant #DMS-9970646. Ya. Pesin was partially supported by the National Science Foundation grant #DMS-9704564 and by the NATO grant CRG970161. D. Dolgopyat and Ya. Pesin wish to thank the Isaac Newton Institute for Mathematical Sciences (Cambridge, UK) for hospitality and support during their stay there in June-July 2000. H. Hu wishes to thank the Centre de Physique Théorique (Marseille-Luminy, France) for hospitality and support during his stay there in May-July 2000.

we have

$$\begin{aligned}\|D^m F|\mathbb{T}^2 \times I_n - \pi_n^{-1} \circ D^m f_n \circ \pi_n\|_{C^n} &\leq \|\pi_n^{-1} \circ (D^m F - D^m f_n) \circ \pi_n\|_{C^n} \\ &\leq e^{-n^2} \cdot (n+1)^n \to 0\end{aligned}$$

as $n \to \infty$. It follows that f is C^∞ on M and has the required properties.

B.3. Main Proposition. The goal of this section is to prove the following statement. Set $I = [0,1]$.

PROPOSITION B.2. *For any $k \geq 2$ and $\delta > 0$, there exists a map g of the three dimensional manifold $M = \mathbb{T}^2 \times I$ such that:*

1. *g is a C^∞ volume preserving diffeomorphism of M;*
2. *$\|F - g\|_{C^k} \leq \delta$;*
3. *for all $0 \leq m < \infty$, $D^m g|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ and 1;*
4. *g is ergodic with respect to the Riemannian volume and has nonzero Lyapunov exponents almost everywhere.*

Before giving the formal proof let us outline the main idea. The result will be achieved in two steps. First applying an argument of [**SW**] we construct a perturbation map which has nonzero *average* central exponent $\int_M \chi^c(x)\, d\mu(x) \neq 0$, where $\chi^c(x)$ denotes the Lyapunov exponent of x along the neutral subspace $E^c(x)$. We then further perturb this diffeomorphism modifying an approach in [**NT**] to ensure that it has the accessibility property and therefore, is ergodic (see Section B.4 for details).

We believe that this approach works in a more general setting. Namely, we conjecture that the following statement holds.

CONJECTURE. *Consider a one parameter family g_ε with $g_0 = F$. Then for sufficiently small ε, g_ε satisfies the conditions of Proposition B.2 except for a positive codimension submanifold in the space of one parameter families.*

PROOF OF PROPOSITION B.2. Consider the linear hyperbolic map A of the torus $\mathbb{T}^2$. We may assume that its eigenvalues are η and η^{-1}, where $\eta > 1$. Let p and p' be fixed points of A. Choose a number $\varepsilon_0 > 0$ such that $d(p,p') \geq 3\varepsilon_0$. Consider the local stable and unstable one-dimensional manifolds for A at points p and p' of "size" ε_0 and denote them respectively by $V^s(p)$, $V^u(p)$, $V^s(p')$, and $V^u(p')$.

Let us choose the smallest positive number n_1 such that the intersection $A^{-n_1}(V^s(p')) \cap V^u(p) \cap B(p,\varepsilon_0)$ consists of a single point which we denote by q_1 (here $B(p,\varepsilon_0)$ is the ball in $\mathbb{T}^2$ of radius ε_0 centered at p). Similarly, we choose the smallest positive number n_2 such that the intersection $A^{n_2}(V^u(p')) \cap V^s(p) \cap B(p,\varepsilon_0)$ consists of a single point which we denote by q_2.

Given a sufficiently small number $\varepsilon \in (0,\varepsilon_0)$,

$$\varepsilon \leq \frac{1}{2}\min\{d(p,q_1), d(p,q_2)\},$$

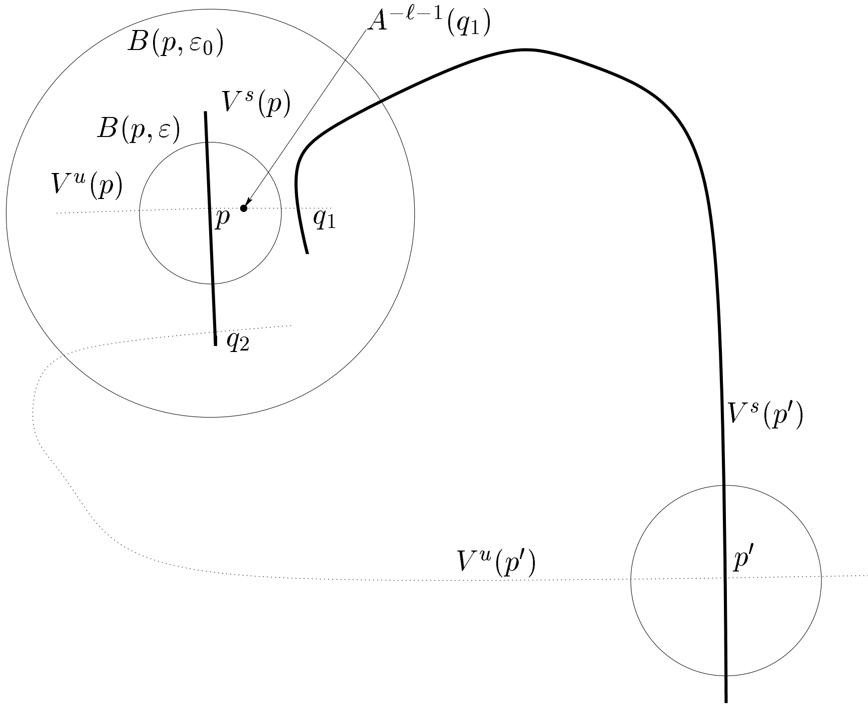

FIGURE B.1

there is $\ell \geq 2$ such that (see Figure B.1)

$$A^{-\ell}(q_1) \notin B(p,\varepsilon), \quad A^{-\ell-1}(q_1) \in B(p,\varepsilon). \tag{B.1}$$

We now choose $\varepsilon' \in (0,\varepsilon)$ such that $A^{-\ell-1}(q_1) \in B(p,\varepsilon')$.

Finally, we assume ε to be so small that for some $q \in \mathbb{T}^2$ we have

$$B(p,\varepsilon) \cap (A^{-n_1}(V^s(p')) \cup A^{n_2}(V^u(p'))) = \varnothing,$$

$$A^i(B(q,\varepsilon)) \cap B(q,\varepsilon) = \varnothing, \quad A^i(B(q,\varepsilon)) \cap B(p,\varepsilon) = \varnothing$$

for $i = 1, \ldots, N$, where $N > 0$ will be determined later, and $\varepsilon = \varepsilon(N)$.

Set $\Omega_1 = B(p,\varepsilon_0) \times I$ and $\Omega_2 = B^{uc}(\bar{q},\varepsilon_0) \times B^s(\bar{q},\varepsilon_0)$, where $\bar{q} = (q, 1/2)$ and $B^{uc}(\bar{q},\varepsilon_0) \subset V^u(q) \times I$ and $B^s(\bar{q},\varepsilon_0) \subset V^s(q)$ are balls of radius ε_0 about $\bar{q}$.

After this preliminary consideration we describe the construction of the map g.

Consider the coordinate system in Ω_1 originated at $(p,0) \in M$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively for the map F. If a point $w = (x,y,z) \in \Omega_1$ and $F(w) \in \Omega_1$ then $F(w) = (\eta x, \eta^{-1} y, z)$.

Choose a C^∞ function $\xi\colon I \to \mathbb{R}^+$ satisfying:

1. $\xi(z) > 0$ on $(0,1)$;
2. $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k$;

3. $\|\xi\|_{C^k} \le \delta$.

We also choose two C^∞ functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ which are defined on the interval $(-\varepsilon_0, \varepsilon_0)$ and satisfy

4. $\varphi(x) = \varphi_0$ if $x \in (-\varepsilon', \varepsilon')$ and $\psi(y) = \psi_0$ if $y \in (-\varepsilon', \varepsilon')$, where φ_0 and ψ_0 are positive constants;
5. $\varphi(x) = 0$ if $|x| \ge \varepsilon$; $\psi(y) \ge 0$ for any y and $\psi(y) = 0$ if $|y| \ge \varepsilon$;
6. $\|\varphi\|_{C^k} \le \delta$, $\|\psi\|_{C^k} \le \delta$;
7. $\int_0^{\pm\varepsilon} \varphi(s)\, ds = 0$.

We now define the vector field X on Ω_1 by

$$X(x,y,z) = \left(-\psi(y)\xi'(z) \int_0^x \varphi(s) ds, 0, \psi(y)\xi(z)\varphi(x) \right).$$

It is easy to check that X is a divergence free vector field supported on $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times I$.

We define the map h_t on Ω_1 to be the time t map of the flow generated by X and we set $h_t = \mathrm{Id}$ on the complement of Ω_1. It is easy to see that h_t is a C^∞ volume preserving diffeomorphism of M which preserves the y coordinate (the stable direction for the map F).

Consider now the coordinate system in Ω_2 originated at $(q, 1/2)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. We then switch to the cylindrical coordinate system (r, θ, y), where $x = r\cos\theta$, $y = y$, and $z = r\sin\theta$.

Consider a C^∞ function $\rho\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^+$ satisfying:

8. $\rho(r) > 0$ if $0.2\varepsilon' \le r \le 0.9\varepsilon$ and $\rho(r) = 0$ if $r \le 0.1\varepsilon'$ or $r \ge \varepsilon$;
9. $\|\rho\|_{C^k} \le \delta$.

We define now the map $\widetilde{h}_\tau$ on Ω_2 by

$$\widetilde{h}_\tau(r, \theta, y) = (r, \theta + \tau\psi(y)\rho(r), y). \tag{B.2}$$

and we set $\widetilde{h}_\tau = \mathrm{Id}$ on $M \setminus \Omega_2$. It is easy to see that for every τ the map $\widetilde{h}_\tau$ is a C^∞ volume preserving diffeomorphism of M.

Let us set $g = g_{t\tau} = h_t \circ F \circ \widetilde{h}_\tau$. For all sufficiently small $t > 0$ and τ, the map $g_{t\tau}$ is C^k close to F and hence, is a partially hyperbolic (in the narrow sense) C^∞ diffeomorphism of M. It preserves the Riemannian volume in M and is ergodic by Proposition B.3. It remains to show that $g_{t\tau}$ has nonzero Lyapunov exponents almost everywhere.

Denote by $E^s_{t\tau}(w)$, $E^u_{t\tau}(w)$, and $E^c_{t\tau}(w)$ the stable, unstable, and neutral subspaces at a point $w \in M$ for the map $g_{t\tau}$. It suffices to show that for almost everywhere point $w \in M$ and every vector $v \in E^c_\tau(w)$, the Lyapunov exponent $\chi(w, v) \ne 0$.

Set $\kappa_{t\tau}(w) = Dg_{t\tau}|E^u_{t\tau}(w)$, $w \in M$. By Proposition B.6, for all sufficiently small $\tau > 0$,

$$\int_M \log \kappa_{0\tau}(w)\, dw < \log \eta.$$

The subspace $E^u_{t\tau}(w)$ depends continuously on t and τ (for a fixed w; for details see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) and hence, so does $\kappa_{t\tau}$. It follows that for all sufficiently small $\tau > 0$, there is $t > 0$ such that

$$\int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Denote by $\chi^s_{t\tau}(w)$, $\chi^u_{t\tau}(w)$, and $\chi^c_{t\tau}(w)$ the Lyapunov exponents of $g_{t\tau}$ at the point $w \in M$ in the stable, unstable, and neutral directions respectively (since these directions are one-dimensional the Lyapunov exponents do not depend on the vector). By the ergodicity of $g_{t\tau}$, we have that for almost every $w \in M$,

$$\chi^u_{t\tau}(w) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \kappa_{t\tau}(g^i_{t\tau}(w)).$$

By the Birkhoff ergodic theorem, we get

$$\chi^u_{t\tau}(w) = \int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Since $E^s_{t\tau}(w) = E^s_{00}(w) = E^s_F(w)$ for every t and τ, we conclude that $\chi^s_{t\tau}(w) = -\log \eta$ for almost every $w \in M$. Since $g_{t\tau}$ is volume preserving,

$$\chi^s_{t\tau}(w) + \chi^u_{t\tau}(w) + \chi^c_{t\tau}(w) = 0$$

for almost every $w \in M$. It follows that $\chi^c_{t\tau}(w) \neq 0$ for almost every $w \in M$ and hence, $g_{t\tau}$ has nonzero Lyapunov exponents almost everywhere. This completes the proof of the proposition. □

B.4. Ergodicity of the Map $g_{t\tau}$.

PROPOSITION B.3. *For every sufficiently small $t > 0$ and $\tau \geq 0$ the map $g_{t\tau}$ is ergodic.*

PROOF. Consider a partially hyperbolic (in the narrow sense) diffeomorphism f of a compact Riemannian manifold M preserving the Riemannian volume. Two points x, $y \in M$ are called *accessible (with respect to f)* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either E^u or E^s. The diffeomorphism f satisfies the *essential accessibility property* if almost any two points in M (with respect to the Riemannian volume) are accessible. We will show that the map $g_{t\tau}$ has the essential accessibility property. The ergodicity of the map will then follow from the result by Pugh and Shub (see [**PS**]; see also the paper by Burns, Pugh, Shub, and Wilkinson in this volume).

Given a point $w \in M$, denote by $\mathcal{A}(w)$ the set of points $q \in M$ such that w and q are accessible. Set $I_p = \{p\} \times (0, 1)$.

LEMMA B.4. *For every $z \in (0, 1)$,*

$$\mathcal{A}(p, z) \supset I_p. \tag{B.3}$$

PROOF OF LEMMA B.4. We use the coordinate system (x, y, z) in Ω_1 described above. Since the map h_t preserves the center leaf I_p, we have that

$$h_t(0,0,z) = (h_t^{(1)}(0,0,z), h_t^{(2)}(0,0,z), h_t^{(3)}(0,0,z)) = (0,0,h_t^{(3)}(0,0,z))$$

for $z \in (0,1)$. It suffices to show that for every $z \in (0,1)$,

$$\mathcal{A}(p,z) \supset \{(p,a) : a \in [(h_t^{-\ell})^{(3)}(p,z), z]\}, \tag{B.4}$$

where ℓ is chosen by (B.1). In fact, since accessibility is a transitive relation and $h_t^{-n}(p,z) \to (p,0)$ for any $z \in (0,1)$, (B.4) implies that $\mathcal{A}(p,z) \supset \{(p,a) : a \in (0,z]\}$. Since this holds true for all $z \in (0,1)$ and accessibility is a reflexive relation, we obtain (B.3).

Now we proceed with the proof of (B.4).

Let $q_1 \in V^u_{t\tau}(p)$ and $q_2 \in V^s_{t\tau}(p)$ be two points constructed in Section B.3. The intersection $V^s_{t\tau}(q_1) \cap V^u_{t\tau}(q_2)$ is not empty and consists of a single point q_3. We will prove that for any $z_0 \in (0,1)$, there exist $z_i \in (0,1)$, $i = 1,2,3,4$ such that

$$\begin{aligned} &(q_1, z_1) \in V^u_{t\tau}((p,z_0)), \quad (q_3,z_3) \in V^s_{t\tau}((q_1,z_1)), \\ &(q_2,z_2) \in V^u_{t\tau}((q_3,z_3)), \quad (p,z_4) \in V^s_{t\tau}((q_2,z_2)) \end{aligned}$$

and

$$z_4 \le (h_t^{-\ell})^{(3)}(p,z_0). \tag{B.5}$$

See Figure B.2. This means that $(p,z_4) \in \mathcal{A}(p,z_0)$. By continuity, we conclude that

$$\{(p,a) : a \in [z_4, z_0]\} \subset \mathcal{A}(p,z_0)$$

and (B.4) follows.

Since $g_{t\tau}$ preserves the xz-plane, we have that $V^{uc}_{t\tau}((p,z_0)) = V^{uc}_F((p,z_0))$. Hence, there is a unique $z_1 \in (0,1)$ such that $(q_1,z_1) \in V^u_{t\tau}((p,z_0))$. Notice that

$$g_{t\tau}^{-n}(p,z_0) = (p, h_t^{-n}((p,z_0)), \quad g_{t\tau}^{-n}(q_1,z_1) = (A^{-n}q_1, z_1)$$

for $n \le \ell$. This is true because the points $A^{-n}q_1$, $n = 0, 1, \ldots, \ell$ lie outside the ε-neighborhood of I_p, where the perturbation map $h_t = \mathrm{Id}$. Similarly, since the points $A^{-n}q_1$, $n > \ell$ lie inside the ε'-neighborhood of I_p, and the third component of h_t depends only on the z-coordinate, we have

$$g_{t\tau}^{-n}(q_1,z_1) = (A^{-n}q_1, h_t^{-n+\ell}z_1).$$

Since $d(g_{t\tau}^{-n}((p,z_0)), g_{t\tau}^{-n}((q_1,z_1))) \to 0$ as $n \to \infty$, we have

$$d(h_t^{-n}((p,z_0)), h_t^{-n+\ell}((p,z_1))) \to 0$$

as $n \to \infty$. It follows that $z_1 = (h_t^{-\ell})^{(3)}((p,z_0))$.

By the construction of the map h_t (that is $h_t = \mathrm{Id}$ outside Ω_1) the sets $A^{-n_1}V^s_{t\tau}(p')$ and $A^{n_2}V^u_{t\tau}(p')$ are pieces of horizontal lines. This means that $z_2 = z_3 = z_1$.

Since the third component of h_t is nondecreasing from (q_2,z_2) to (p,z_4) along $V^s_{t\tau}(p)$, we conclude that $z_4 \le z_3 = z_1 = (h_t^{-\ell})^3(p,z_0)$ and thus (B.5) holds. $\square$

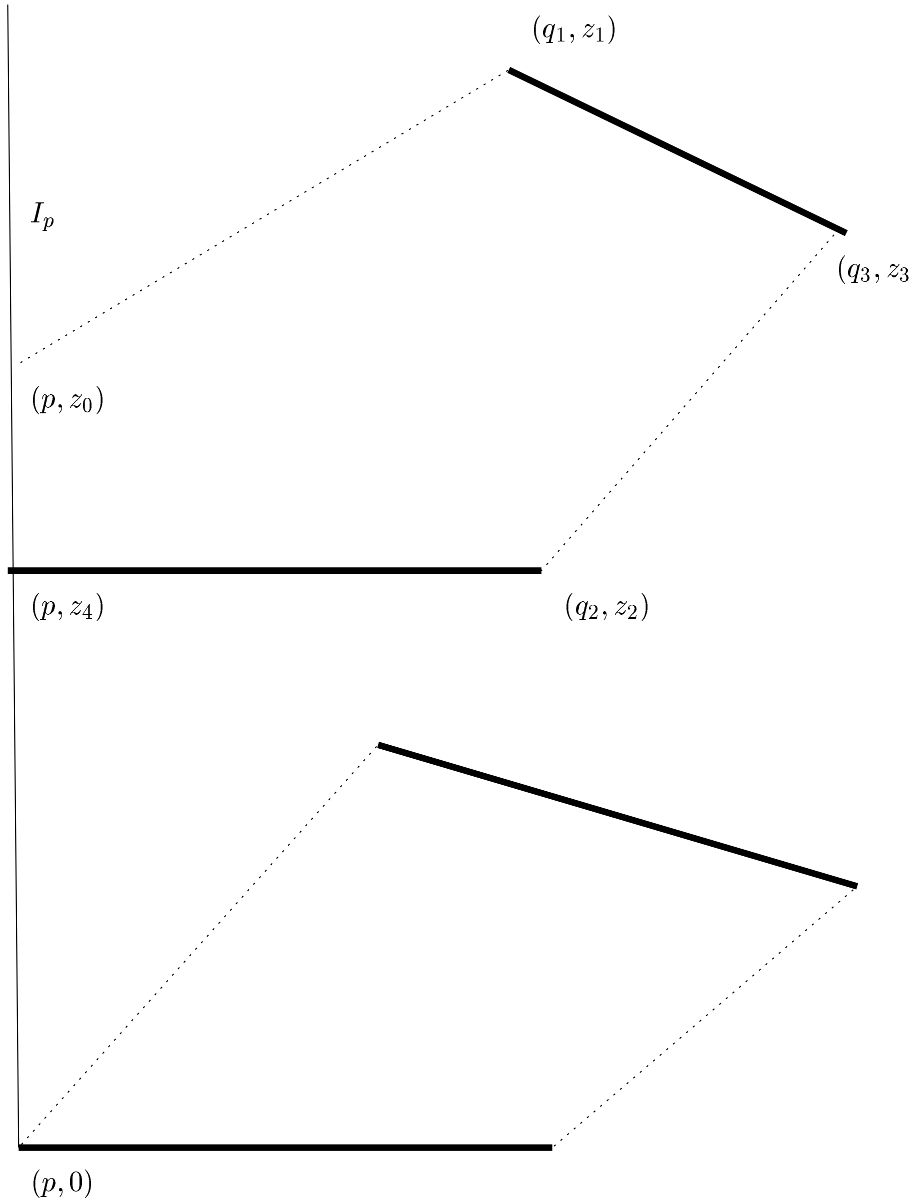

FIGURE B.2. Bold lines are stable manifolds, dotted — unstable ones

The essential accessibility property follows from Lemma B.4 and the following statement.

LEMMA B.5 (see [**NT**]). *Assume that any two points in I_p are accessible. Then the map $g_{t\tau}$ satisfies the essential accessibility property.*

Proof of Lemma B.5. It is easy to see that for any two points $x, y \in M$ which do not lie on the boundary of M one can find points $x', y' \in I_p$ such that the pairs (x, x') and (y, y') are accessible. By Lemma B.4 the points x', y' are accessible. Since accessibility is a transitive relation the result follows. $\square$

This completes the proof of the proposition. $\square$

B.5. Hyperbolicity of the Map $g_{0\tau}$. In this section we show that for all sufficiently small τ, the map $g_{0\tau}$ has nonzero average Lyapunov exponent in the central direction. Since this map is ergodic this implies that $g_{0\tau}$ has nonzero Lyapunov exponents almost everywhere.

Proposition B.6. *For any sufficiently small $\tau > 0$,*

$$\int_M \log \kappa_{0\tau}(w) dw < \log \eta.$$

Proof. Our approach is an elaboration of an argument in [**SW**].

For any $w \in M$, we introduce the coordinate system in $T_w M$ associated with the splitting $E^u_F(w) \oplus E^s_F(w) \oplus E^c_F(w)$. Given $\tau \geq 0$ and $w \in M$, there exists a unique number $\alpha_\tau(w)$ such that the vector $v_\tau(w) = (1, 0, \alpha_\tau(w))^t$ lies in $E^u_{0\tau}(w)$ (where t denotes the transpose). Since the map $\widetilde{h}_\tau$ preserves the y coordinate, by the definition of the function $\alpha_\tau(w)$, one can write the vector $Dg_{0\tau}(w)v_\tau(w)$ in the form

$$Dg_{0\tau}(w)v_\tau(w) = (\bar{\kappa}_\tau(w), 0, \bar{\kappa}_\tau(w)\alpha_\tau(g_{t0}(w)))^t \tag{B.6}$$

for some $\bar{\kappa}_\tau(w) > 1$. Since the expanding rate of $Dg_{0\tau}(w)$ along its unstable direction is $\kappa_{0\tau}(w)$ we obtain that

$$\kappa_{0\tau}(w) = \bar{\kappa}_\tau(w) \frac{\sqrt{1 + \alpha_\tau(g_{0\tau}(w))^2}}{\sqrt{1 + \alpha_\tau(w)^2}}.$$

Since $E^u_{0\tau}(w)$ is close to $E^u_{00}(w)$ the function $\alpha_\tau(w)$ is uniformly bounded. Using the fact that the map $g_{0\tau}$ preserves the Riemannian volume we find that

$$L_\tau = \int_M \log \kappa_{0\tau}(w)\, dw = \int_M \log \bar{\kappa}_\tau(w)\, dw. \tag{B.7}$$

Consider the map $\widetilde{h}_\tau$. Since it preserves the y-coordinate using (B.2), we can write that

$$\widetilde{h}_\tau(x, y, z) = (r \cos \sigma, y, r \sin \sigma),$$

where $\sigma = \sigma(\tau, r, \theta, y) = \theta + \tau\psi(y)\rho(r)$. Therefore, the differential

$$D\widetilde{h}_\tau \colon E^u_F(w) \oplus E^c_F(w) \to E^u_F(g_{0\tau}(w)) \oplus E^c_F(g_{0\tau}(w))$$

can be written in the matrix form

$$D\widetilde{h}_\tau(w) = \begin{pmatrix} A(\tau,w) & B(\tau,w) \\ C(\tau,w) & D(\tau,w) \end{pmatrix} = \begin{pmatrix} r_x\cos\sigma - r\sigma_x\sin\sigma & r_y\cos\sigma - r\sigma_y\sin\sigma \\ r_x\sin\sigma + r\sigma_x\cos\sigma & r_y\sin\sigma + r\sigma_y\cos\sigma \end{pmatrix},$$

where

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad r_z = \frac{\partial r}{\partial z} = \frac{y}{r} = \sin\theta,$$

$$\sigma_x = \frac{\partial\sigma}{\partial x} = \frac{-z}{r^2} + \frac{z}{r}\tau\widetilde{\rho}_r(y,r) = \frac{\sin\theta}{r} + \tau\widetilde{\rho}_r(y,r)\cos\theta,$$

$$\sigma_z = \frac{\partial\sigma}{\partial z} = \frac{x}{r^2} + \frac{x}{r}\tau\widetilde{\rho}_r(y,r) = \frac{\cos\theta}{r} + \tau\widetilde{\rho}_r(y,r)\sin\theta,$$

and $\widetilde{\rho}(y,r) = \psi(y)\rho(r)$. It is easy to check that

$$\begin{aligned} A = A(\tau,w) &= 1 - \tau r\widetilde{\rho}_r\sin\theta\cos\theta - \frac{\tau^2\widetilde{\rho}^2}{2} - \tau^2 r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta + O(\tau^3), \\ B = B(\tau,w) &= -\tau\widetilde{\rho} - \tau r\widetilde{\rho}_r\sin^2\theta - \tau^2 r\widetilde{\rho}\widetilde{\rho}_r\sin\theta\cos\theta + O(\tau^3), \\ C = C(\tau,w) &= \tau\widetilde{\rho} + \tau r\widetilde{\rho}_r\cos^2\theta - \tau^2 r\widetilde{\rho}\widetilde{\rho}_r\sin\theta\cos\theta + O(\tau^3), \\ D = D(\tau,w) &= 1 + \tau r\widetilde{\rho}_r\sin\theta\cos\theta - \frac{\tau^2\widetilde{\rho}^2}{2} - \tau^2 r\widetilde{\rho}\widetilde{\rho}_r\sin^2\theta + O(\tau^3). \end{aligned} \tag{B.8}$$

By Lemma B.7 below, we have

$$L_\tau = \log\eta - \int_M \log(D(\tau,w) - \eta B(\tau,w)\alpha_\tau(g_{0\tau}(w)))dw.$$

By Lemma B.8, we have

$$\left.\frac{dL_\tau}{d\tau}\right|_{\tau=0} = 0, \qquad \left.\frac{d^2L_\tau}{d\tau^2}\right|_{\tau=0} < 0.$$

So we can choose τ so small that $L_\tau \neq \log\eta$. □

LEMMA B.7.

$$L_\tau = \log\eta - \int_M \log(D(\tau,w) - \eta B(\tau,w)\alpha_\tau(g_{0\tau}(w)))dw.$$

PROOF OF LEMMA B.7. Since $g_{0\tau} = h_0\circ F\circ\widetilde{h}_\tau = F\circ\widetilde{h}_\tau$, we have that

$$D_\tau(w) = Dg_{0\tau}(w)|E^u_{0\tau}(w)\oplus E^c_{0\tau}(w) = \begin{pmatrix} \eta A(\tau,w) & \eta B(\tau,w) \\ C(\tau,w) & D(\tau,w) \end{pmatrix}.$$

By (B.6),

$$D_\tau(w)\begin{pmatrix} 1 \\ \alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \eta A(\tau,w) + \eta B(\tau,w)\,\alpha_\tau(w) \\ C(\tau,w) + D(\tau,w)\,\alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \kappa_\tau(w) \\ \kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w)) \end{pmatrix}. \tag{B.9}$$

Since $\widetilde{h}_\tau$ is volume preserving, $AD - BC = 1$ and therefore,

$$A + B\alpha = \frac{1}{D} + \frac{B}{D}(C + D\alpha).$$

Comparing the components in (B.9), we obtain

$$\begin{aligned}\kappa_\tau(w) =& \eta(A(\tau, w) + B(\tau, w)\, \alpha_\tau(w))\\ =& \eta\left(\frac{1}{D(\tau, w)} + \frac{B(\tau, w)}{D(\tau, w)}(C(\tau, w) + D(\tau, w)\, \alpha_\tau(w))\right)\\ =& \eta\left(\frac{1}{D(\tau, w)} + \frac{B(\tau, w)}{D(\tau, w)}(\kappa_\tau(w)\, \alpha_\tau(g_{0\tau}(w)))\right).\end{aligned}$$

Solving for $\kappa_\tau(w)$, we get

$$\kappa_\tau(w) = \frac{\eta}{D(\tau, w) - \eta B(\tau, w)\, \alpha_\tau(g_{0\tau}(w))}.$$

The desired result follows from (B.7). □

Lemma B.8.

$$\left.\frac{dL_\tau}{d\tau}\right|_{\tau=0} = 0, \qquad \left.\frac{d^2 L_\tau}{d\tau^2}\right|_{\tau=0} < 0. \tag{B.10}$$

Proof of Lemma B.8. In order to simplify notations we set $D'_\tau = \frac{\partial D}{\partial \tau}$, $B'_\tau = \frac{\partial B}{\partial \tau}$, $C'_\tau = \frac{\partial C}{\partial \tau}$, $D''_{\tau\tau} = \frac{\partial^2 D}{\partial \tau^2}$, and $B''_{\tau\tau} = \frac{\partial^2 B}{\partial \tau^2}$. Since the function $\alpha_\tau(w)$ is differentiable over τ (see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) by Lemma B.7, we find

$$\frac{dL_\tau}{d\tau} = -\int_M \frac{D'_\tau - \eta B'_\tau \alpha(g_{0\tau}(w)) - \eta B \frac{\partial \alpha_\tau(w)}{\partial \tau}(g_{0\tau}(w))}{D(\tau, w) - \eta B(\tau, w)\alpha_\tau(w)(g_{0\tau}(w))}\, dw$$

and therefore,

$$\begin{aligned}\frac{d^2 L_\tau}{d\tau^2} =& \int_M \left(\frac{D'_\tau - \eta B'_\tau \alpha(g_{0\tau}(w)) - \eta B(\tau, w)\frac{\partial \alpha_\tau(w)}{\partial \tau}(g_{0\tau}(w))}{D(\tau, w) - \eta B(\tau, w)\alpha_s(g_{0\tau}(w))}\right)^2 dw\\ &- \int_M \frac{E(\tau, w)}{D(\tau, w) - \eta B(\tau, w)\, \alpha_\tau(g_{0\tau}(w))} dw,\end{aligned}$$

where

$$\begin{aligned}E(\tau, w) =& D''_{\tau\tau} - \eta B''_{\tau\tau}\alpha(g_{0\tau}(w))\\ &- \eta B(\tau, w)\frac{\partial^2 \alpha_\tau(w)}{\partial \tau^2}(g_{0\tau}(w)) - 2\eta B'_\tau \frac{\partial \alpha_\tau(w)}{\partial \tau}(g_{0\tau}(w)).\end{aligned}$$

Note that for all $w \notin \Omega_2$,

$$A(\tau, w) = D(\tau, w) = 1, \quad C(\tau, w) = B(\tau, w) = 0$$

and for all $w \in M$,

$$A(0, w) = D(0, w) = 1, \quad C(0, w) = B(0, w) = 0, \quad \alpha_0(w) = 0.$$

It follows that

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0} = \int_{\Omega_2} D'_\tau \, dw, \tag{B.11}$$

and also that

$$\frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} = \int_{\Omega_2} \left[(D'_\tau)^2 - D''_{\tau\tau} + 2\eta B'_\tau \frac{\partial \alpha_\tau(w)}{\partial \tau}(g_{0\tau}(w)) \right]_{\tau=0} dw. \tag{B.12}$$

By (B.8), we obtain that

$$D'_\tau(0,w) = r\widetilde{\rho}_r(r) \sin\theta \cos\theta$$

and hence,

$$\int_{\Omega_2} D'_\tau dw = 0.$$

Therefore, (B.11) implies the equality in (B.10).

We now proceed with the inequality in (B.10). Applying Lemma B.9 below we obtain that

$$\frac{\partial \alpha}{\partial \tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta} + \sum_{n=1}^{\infty} \frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^{n+1}}.$$

It follows that

$$\begin{aligned} 2\eta B'_\tau(0,w)\frac{\partial \alpha}{\partial \tau}(g_{0\tau}(w))\Big|_{\tau=0} =& 2B'_\tau(0,w)C'_\tau(0,w) \\ &+ 2B'_\tau(0,w)\sum_{n=1}^{\infty} \frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^n}. \end{aligned}$$

First, we evaluate the term

$$\mathcal{F}(w) = D'_\tau(0,w)^2 - D''_{\tau\tau}(0,w) + 2B'_\tau(0,w)C'_\tau(0,w).$$

Using (B.8), we find that

$$\begin{aligned} \mathcal{F}(w) =& (r\widetilde{\rho}_r \sin\theta\cos\theta)^2 + (\widetilde{\rho}^2 + 2r\widetilde{\rho}\widetilde{\rho}_r \sin^2\theta) \\ &- 2(\widetilde{\rho} + r\widetilde{\rho}_r \sin^2\theta)(\widetilde{\rho} + r\widetilde{\rho}_r\cos^2\theta) \\ =& -\widetilde{\rho}^2 - (r\widetilde{\rho}_r \sin\theta\cos\theta)^2 - 2r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta. \end{aligned} \tag{B.13}$$

Recall that $\Omega_2 = B^{uc}(\bar{q}, \varepsilon_0) \times B^s(\bar{q}, \varepsilon_0)$ and $\widetilde{\rho}(r) = 0$ if $r \geq \varepsilon$. We have

$$\int_{\Omega_2} 2r\widetilde{\rho}\widetilde{\rho}_r \cos^2\theta \, dw = \int_{-\varepsilon_0}^{\varepsilon_0} dy \int_0^{2\pi} 2\cos^2\theta \, d\theta \int_0^{\varepsilon} r^2\widetilde{\rho}\widetilde{\rho}_r \, dr. \tag{B.14}$$

Since $0 = \widetilde{\rho}(0) = \widetilde{\rho}(\varepsilon)$ (by the definition of the function ρ), we find that

$$\int_0^\varepsilon r^2\widetilde{\rho}\widetilde{\rho}_r \, dr = \frac{1}{2}r^2\widetilde{\rho}^2\Big|_0^\varepsilon - \int_0^\varepsilon r\widetilde{\rho}^2 \, dr = -\int_0^\varepsilon r\widetilde{\rho}^2 \, dr.$$

We also have that

$$\int_0^{2\pi} 2\cos^2\theta \, d\theta = \int_0^{2\pi} d\theta. \tag{B.15}$$

It follows from (B.14)–(B.15) that

$$-\int_{\Omega_2} 2r\widetilde{\rho}\widetilde{\rho}_r \cos^2\theta\, dw = \int_{\Omega_2} r\widetilde{\rho}^2\, dw \le \varepsilon \int_{\Omega_2} \widetilde{\rho}^2\, dw. \tag{B.16}$$

Arguing similarly one can show that

$$-\int_{\Omega_2} (r\widetilde{\rho}_r \sin\theta\cos\theta)^2\, dw = -\frac{1}{8}\int_{\Omega_2} (r\widetilde{\rho}_r)^2\, dw \tag{B.17}$$

Thus we conclude using (B.13), (B.16), and (B.17) that

$$\int_{\Omega_2} \mathcal{F}(w)\, dw \le -(1-\varepsilon)\int_{\Omega_2} \widetilde{\rho}_r^2\, dw - \frac{1}{8}\int_{\Omega_2} (r\widetilde{\rho}_r)^2\, dw < 0. \tag{B.18}$$

We now evaluate the remaining term

$$\mathcal{G}(w) = \sum_{n=1}^{\infty} \frac{1}{\eta^i} \int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0, g_{00}^{-n}(w))\, dw.$$

Since the map $g_{00} = F$ preserves the Riemannian volume we obtain that

$$\begin{aligned}\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0, g_{00}^{-n}(w))\, dw &\le \int_{\Omega_2} B'_\tau(0,w)^2\, dw + \int_{\Omega_2} C'_\tau(0, g_{00}^{-n}(w))^2\, dw \\ &= \int_{\Omega_2} B'_\tau(0,w)^2\, dw + \int_{\Omega_2} C'_\tau(0,w)^2\, dw\end{aligned}$$

Applying (B.8), we find that

$$\begin{aligned}&\int_{\Omega_2} B'_\tau(0,w)^2\, dw + \int_{\Omega_2} C'_\tau(0,w)^2\, dw \\ &= \int_{\Omega_2} (\widetilde{\rho} + r\widetilde{\rho}_r \sin^2\theta)^2\, dw + \int_{\Omega_2} (\widetilde{\rho} + r\widetilde{\rho}_r \cos^2\theta)^2\, dw \\ &\le 4\left(\int_{\Omega_2} \widetilde{\rho}^2\, dw + \int_{\Omega_2} r^2\widetilde{\rho}_r^2\, dw\right).\end{aligned}$$

It follows that for sufficiently large $N > 0$ (which does not depend on ε)

$$\sum_{i=N}^{\infty} \frac{1}{\eta^i} \int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0, g_{00}^{-i}(w))\, dw \le \frac{1}{10}\left(\int_{\Omega_2} \widetilde{\rho}^2\, dw + \int_{\Omega_2} r^2\widetilde{\rho}_r^2\, dw\right). \tag{B.19}$$

Note that if $g_{00}^{-n}\Omega_2 \cap \Omega_2 = \varnothing$, then $B'_\tau(0,w)C'_\tau(0, g_{00}^{-n}(w)) = 0$ for all w. Hence,

$$\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0, g_{00}^{-n}(w))\, dw = 0.$$

We may choose the point q and a small ε such that $g_{00}^{-n}\Omega_2 \cap \Omega_2 = F^{-n}\Omega_2 \cap \Omega_2 = \varnothing$ for all $n = 1, 2, \ldots, N$. It follows from (B.12), (B.18), and (B.19) that

$$\begin{aligned}\frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} &= \int_{\Omega_2}\mathcal{F}(w)\,dw + \int_{\Omega_2}\mathcal{G}(w)\,dw \\ &\le -\left(\frac{9}{10}-\varepsilon\right)\int_{\Omega_2}\tilde{\rho}^2\,dw - \frac{1}{40}\int_{\Omega_2} r^2\tilde{\rho}_r^2\,dw < 0.\end{aligned}$$

The desired result follows. □

LEMMA B.9.

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \sum_{n=0}^{\infty}\frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^{n+1}}.$$

PROOF OF LEMMA B.9. Define

$$R(\tau, w, \alpha) = \frac{C(\tau,w)+D(\tau,w)\alpha}{\eta(A(\tau,w)+B(\tau,w)\alpha)}.$$

It follows from (B.6) that

$$\alpha_\tau(g_{0\tau}(w)) = R(\tau, w, \alpha_\tau(w)). \tag{B.20}$$

By (B.6) and (B.8), we have

$$\frac{\partial R}{\partial\tau}\Big|_{\tau=0} = \frac{(C'_\tau + D'_\tau\alpha)(A+B\alpha)+(C+D\alpha)(A'_\tau+B'_\tau\alpha)}{\eta(A+B\alpha)^2}\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta}.$$

Since $AD - BC = 1$,

$$\frac{\partial R}{\partial\alpha}\Big|_{\tau=0} = \frac{AD-BC}{\eta(A+B\alpha)^2}\Big|_{\tau=0} = \frac{1}{\eta}.$$

It follows from (B.20) that

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta} + \frac{1}{\eta}\cdot\frac{\partial\alpha}{\partial\tau}(w)\Big|_{\tau=0}.$$

Since this inequality holds for any w, replacing w with $g_{0\tau}^{-1}(w)$ we obtain

$$\frac{\partial\alpha}{\partial\tau}(w)\Big|_{\tau=0} = \frac{C'_\tau(0, g_{0\tau}^{-1}(w))}{\eta} + \frac{1}{\eta}\cdot\frac{\partial\alpha}{\partial\tau}(g_{0\tau}^{-1}(w))\Big|_{\tau=0}.$$

The result follows by induction. □

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