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GEODESIC FLOWS ON CLOSED RIEMANNIAN MANIFOLDS WITHOUT FOCAL POINTS

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Abstract. In this paper it is proved that a geodesic flow on a two-dimensional compact manifold of genus greater than 1 with Riemannian metric without focal points is isomorphic with a Bernoulli flow. This result generalizes to the multidimensional case. The proof is based on establishing some metric properties of flows with nonzero Ljapunov exponents (the K-property, etc.), and also the construction of horospheres and leaves on a very wide class of Riemannian manifolds, together with a study of some of their geometric properties.

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Introduction

Studies connected with geodesic flows on manifolds have not only a long history (which is expounded in detail in [1], §3), but have served as the impetus for the creation of a general theory of U-systems* (cf. [1], [2] and [7]). In the case when the manifold is compact and has negative curvature, the geodesic flow is a U-flow,* so it is topologically transitive, has the K-property, etc. (cf. [1]). We shall be interested in properties of geodesic flows on manifolds without focal points. Topological properties of such flows were studied by Eberlein [14]. In particular, he proved that a geodesic flow on a compact manifold satisfying the axiom of uniform visibility (cf. §5) and without conjugate points is topologically transitive. The first result establishing some metric properties of geodesic flows on two-dimensional manifolds without focal points was obtained by Kramli [5]. In [11] we generalized his theorem, proving that geodesic flows on compact surfaces without focal points of genus greater than one are ergodic. In the present paper we shall extend the study of metric properties of geodesic flows, relying, as in [11], on results of two kinds.

First is the description of metric properties of dynamical systems (i.e. diffeomorphisms and flows) preserving a measure equivalent with the smooth Lebesgue measure and having nonzero Ljapunov exponents on a set of positive measure (cf. [9]–[12]). However for flows in [11] only ergodicity on a set of positive measure was proved. In part I of the

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**Editor's note.* In English one speaks of Anosov systems and flows.

present paper we shall prove under some additional hypotheses the K-property and the Bernoulli property on a set of positive measure. Our result is analogous to the theorem on the alternative for U-flows (cf. [1], Theorem 14).

Second is the construction and description of the properties of horospheres. It is well known (cf. [2]) how to construct horospheres in Lobačevskiĭ space. With the help of the invariant contracting and expanding foliations for geodesic flows one can define horospheres on any compact manifold of variable negative curvature. Eberlein (cf. [16] and [22]), considering level lines of a Busemann function, constructed horospheres on manifolds of nonpositive curvature; he also studied some of their geometric properties (cf. also [21]). In part II of the present paper we shall construct horospheres on manifolds without conjugate points, satisfying a certain condition which we call the axiom of asymptoticity (cf. §5). This condition will be satisfied, for example, if the manifold has no focal points or satisfies the axiom of uniform visibility. We shall also study a series of geometric properties of horospheres (cf. §7).

We note that in [11] results related to the construction of horospheres were only formulated. In this paper they are proved in detail.

On the basis of the results obtained, in part III we shall continue the study of metric properties of geodesic flows begun in [11]. We shall show that geodesic flows on compact surfaces without focal points of genus greater than 1 are isomorphic with Bernoulli flows.

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PART I. METRIC PROPERTIES OF FLOWS WITH NONZERO LJAPUNOV EXPONENTS

§1. Preliminary information and results

1.1. In the present paper, a flow f^t on a smooth Riemannian manifold M is considered; f^t is defined by a vector field X and preserves a finite measure ν, equivalent to the measure induced by some Riemannian metric. The smoothness of M and the Riemannian metric can be assumed to be of class C^∞ without loss of generality; the scalar product and norm in the tangent space T_xM are denoted by $\langle\ ,\ \rangle_x$ and $\|\cdot\|_x$ (sometimes the index x will be omitted). On the tangent bundle TM there is defined a measurable function

$$\chi^+(x, v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|df^t v\|, \quad v \in T_xM,$$

called the Ljapunov characteristic exponent (cf. [3], [8] and [10]) (the number $\chi^+(x, v)$ is called *the characteristic exponent of the vector v at the point x*).

Our basic assumption is that the set $\Lambda = \{x \in M\colon\ \chi^+(x, v) \neq 0$ for any $v \in T_xM$, $v \neq \alpha X$, $\alpha \in \mathbf{R}\}$, which is measurable and invariant with respect to f^t, has positive measure. In the present section some metric properties of the flow on the set Λ are described.

1.2. PROPOSITION 1.1 (cf. [10], §1, and [11], §9). *There exist a measurable set $\widetilde{\Lambda} \subset \Lambda$, $\nu(\widetilde{\Lambda}) = \nu(\Lambda)$, measurable functions $\lambda(x)$, $C(x, \epsilon)$ and $K(x, \epsilon)$, $x \in \widetilde{\Lambda}$, $\epsilon > 0$, and a measurable family of subspaces E_{1x}, $E_{2x} \subset T_x M$ such that for any $t \in \mathbf{R}$*

1. $C(f^t(x),\varepsilon)\leqslant C(x,\varepsilon)e^{\varepsilon|t|},\quad K(f^t(x),\varepsilon)\geqslant K(x,\varepsilon)e^{-\varepsilon t}$;
2. $0<\lambda(x)<1,\ \lambda(f^t(x))=\lambda(x)$;
3. $T_xM=E_{1x}\oplus\{\alpha X(x)\}\oplus E_{2x},\quad \alpha\in\mathbf{R}$,

$$df^tE_{ix}=E_{if^t(x)},\quad i=1,2;$$

4. $\chi^+(x,v)<0$ *for each* $v\in E_{1x}$,
   $\chi^+(x,v)>0$ *for any* $v\in E_{2x}$;
5. *for any* $t>0$

$$\begin{aligned}&\|df^tv\|\leqslant C(x,\varepsilon)(\lambda(x))^te^{\varepsilon t}\|v\|,\\ &\|df^{-t}v\|\geqslant C^{-1}(x,\varepsilon)(\lambda(x))^{-t}e^{-\varepsilon t}\|v\|\end{aligned}\qquad (v\in E_{1x}),$$

analogous inequalities are valid for $v\in E_{2x}$;

6. *the angle* $\gamma(x)$ *between the subspaces* E_{1x} *and* E_{2x} *admits the estimate* $\gamma(x)\geqslant K(x,\epsilon)$.

1.3. We set for any integer $s>1$

$$\widetilde{\Lambda}_s=\{x\in\widetilde{\Lambda}:\lambda(x)\leqslant 1-\frac{1}{s},\ \text{where } s \text{ is the smallest number satisfying this inequality}\}.$$

It is obvious that $\widetilde{\Lambda}_s$ is a measurable set which is invariant with respect to f^t, while $\bigcup_{s>1}\widetilde{\Lambda}_s=\widetilde{\Lambda}$, and if $s_1\neq s_2$, then $\widetilde{\Lambda}_{s_1}\cap\widetilde{\Lambda}_{s_2}=\varnothing$. For $x\in\widetilde{\Lambda}_s$, we set $\epsilon(x)=\epsilon_s=(1/100)\ln(1+2/s)$. The function $\epsilon(x)$ is measurable and invariant with respect to f^t. For $l\geqslant 1$ we set

$$\widetilde{\Lambda}_s^l=\{x\in\widetilde{\Lambda}_s: C(x,\varepsilon(x))\leqslant l,\ K^{-1}(x,\varepsilon(x))\leqslant l\}.$$

The set $\widetilde{\Lambda}_s^l$ is measurable, while $\bigcup_{l\geqslant 1}\widetilde{\Lambda}^l_s=\widetilde{\Lambda}_s$ and $\widetilde{\Lambda}^l_s\subset\widetilde{\Lambda}_s^{l+1}$. One can show (cf. [11], Proposition 4.6) that the subspaces E_{1x} and E_{2x} depend continuously on the point x in the set $\widetilde{\Lambda}_s$. We also set

$$\widetilde{\Lambda}_{k,s}(\widetilde{\Lambda}^l_{k,s})=\{x\in\widetilde{\Lambda}_s(\widetilde{\Lambda}^l_s):\dim E_{1x}=k\}.$$

PROPOSITION 1.2. (cf. [10], Theorem 1.3.1). 1. *For any* $x\in\overline{\widetilde{\Lambda}}{}^l_{k,s}$ *(the bar denotes closure) there exist subspaces* E_{ix}, $i=1,2$, *satisfying Proposition* 1.1, *while* $\lambda(x)=1-1/s$, $C(x,\epsilon(x))=l$ *and* $K(x,\epsilon(x))=l^{-l}$.

2. *The subspaces* E_{ix}, $i=1,2$, *depend continuously on the point* x *in the set* $\overline{\widetilde{\Lambda}}{}^l_{k,s}$.

1.4. In this subsection local stable and unstable manifolds are defined for flows. Let us assume that $f^t\in C^r$, $r\geqslant 2$. We set $\kappa(x)=(1-1/s)e^{5\epsilon_s}$ for $x\in\widetilde{\Lambda}_s$. If $\delta(x)$ is a measurable function on the set $\widetilde{\Lambda}_s$, we write

$$\begin{aligned}&B^i(\delta(x))=\{u\in E_{ix}:\|u\|_x<\delta(x)\},\quad i=1,2,\\ &B^{i+2}(\delta(x))=\{u\in E_{ix}\oplus\{\alpha X(x)\}:\|u\|_x<\delta(x),\ \alpha\in\mathbf{R}\},\\ &U(x,\delta(x))=\exp_xB^1(\delta(x))\times B^2(\delta(x)).\end{aligned}$$

Also let $\rho(x,y)$ denote the distance in M induced by the Riemannian metric.

PROPOSITION 1.3. (cf. [10], §2.2). *There exist measurable functions $\delta'(x)$, $\delta(x)$ and $A(x)$, $x \in \widetilde{\Lambda}_s$, and a family of maps $\varphi(x)$: $B^1(\delta(x)) \longrightarrow B^4(\delta(x))$ of class C^{r-1}, depending measurably on $x \in \widetilde{\Lambda}_s$, satisfying the following conditions:*

1. *The set $V^-(x) = \{\exp_x(u, \varphi(x)u)\colon\ u \in B^1(\delta(x))\}$ is a submanifold of M of class C^{r-1}.*
2. $x \in V^-(x)$, $T_x V^-(x) = E_{1x}$.
3. *For any $y \in V^-(x)$ and $t > 0$ we have $f^t(y) \in U(f^t(x), \delta'(f^t(x)))$ and*

$$\rho(f^t(x), f^t(y)) \leqslant A(x)\ (\varkappa(x))^t \rho(x, y). \tag{1.1}$$

4. *For any $t > 0$*

$$\delta'(f^t(x)) > \delta'(x)\, e^{-5\varepsilon(x)t}, \quad \delta_s'^l = \inf_{x \in \widetilde{\Lambda}_s^l} \delta'(x) > 0,$$
$$\delta(f^t(x)) \geqslant \delta(x)\, e^{-10\varepsilon(x)t}, \quad \delta_s^l = \inf_{x \in \widetilde{\Lambda}_s^l} \delta(x) > 0,$$
$$A(f^t(x)) \leqslant A(x)\, e^{5\varepsilon(x)t}, \quad A_s^l = \sup_{x \in \widetilde{\Lambda}_s^l} A(x) < \infty.$$

5. *$f^t(V^-(x)) \cap U(f^t(x), \delta(f^t(x))) \subset V^-(f^t(x))$ for any $t \in \mathbf{R}$.*

The submanifold $V^-(x)$ is called the *local stable manifold* passing through the point x. Analogously one defines the *local unstable manifold*

$$V^+(x) = \{\exp_x(u, \psi(x)\,u) : u \in B^2(\delta(x))\},$$

where $\psi(x)$: $B^2(\delta(x)) \longrightarrow B^3(\delta(x))$ is a map of class C^{r-1}. The manifold $V^+(x)$ has properties 2–5 of Proposition 1.2 (with t changed to $-t$).

PROPOSITION 1.4 (cf. [10], Theorem 2.3.1 and Remark 2.3.1).

1. *If $x, y \in \widetilde{\Lambda}_s$, $y \in U(x, \tfrac14\delta(x))$ and $y \notin V^-(x)$, then*

$$V^-(x) \cap V^-(y) \cap U\left(y, \frac{1}{4}\delta(y)\right) = \varnothing.$$

2. *If $x \in \widetilde{\Lambda}^l_s$, $x_i \in \widetilde{\Lambda}^l_s$, $i = 1, 2, \dots$, and $x_i \longrightarrow x$, then $V^-(x_i) \cap U(x, q) \longrightarrow V^-(x) \cap U(x, q)$ in the C^1-topology, where $0 < q < \delta^l_s$.*
3. *For any $x, y \in \widetilde{\Lambda}^l_{k,s}$ and $y \in U(x, \tfrac12\delta^l_s)$,*

$$V^-(y) \cap U(x, \delta_s^l) \supset \left\{\exp_x(u, \psi_y(u)) : u \in B^1\left(\frac{1}{2}\delta_s^l\right)\right\},$$

where ψ_y: $B^1(\tfrac12\delta^l_s) \longrightarrow B^4(\delta^l_s)$ is a map of class C^{r-1}, while

$$\max_{y \in \widetilde{\Lambda}_s^l \cap U\left(x, \frac{1}{2}\delta_s^l\right)}\ \max_{u \in B^1\left(\frac{1}{2}\delta_s^l\right)} [\|\psi_y(u)\| + \|d\psi_y(u)\|] \leqslant 1.$$

The analogous assertion is valid for the manifold $V^+(y)$.

Let $\widetilde{\delta}(x)$ be a positive measurable function on the set $\widetilde{\Lambda}$. For $x \in \widetilde{\Lambda}$ and $\tau > 0$ we set

$$V^{-0}(x) = \bigcup_{|t|<\tau} V^-(f^t(x)),$$
$$\widetilde{V}^{-0}(x) = \bigcup_{y \in V^-(x) \cap U(x, \widetilde{\delta}(x))} \left(\bigcup_{|t|<\tau} f^t(y)\right).$$

Analogously one defines submanifolds $V^{+0}(x)$ and $\widetilde{V}^{+0}(x)$.

PROPOSITION 1.5 (cf. [11], Theorem 9.1). *There exists a measurable function* $\widetilde{\delta}(x) = \delta_T(x), 0 < \widetilde{\delta}(x) \leqslant \delta(x), x \in \widetilde{\Lambda}$, *satisfying* (1.2), *such that* $\widetilde{V}^{-0}(x) \subset V^{-0}(x)$ *and* $\widetilde{V}^{+0}(x) \subset V^{+0}(x)$.

1.5. Let W^1 and W^2 be two smooth submanifolds, transverse to the local stable manifolds passing through points $y \in \widetilde{\Lambda}^l_{k,s} \cap U(x, \frac{1}{4}\delta^l_s)$. There exist sets $\widetilde{W}^1 \subset W^1$ and $\widetilde{W}^2 \subset W^2$ for which the succession map p: $\widetilde{W}^1 \longrightarrow \widetilde{W}^2$ is defined. Namely, if $y \in W^1 \cap V^-(w)$ and $w \in \widetilde{\Lambda}^l_s \cap U(x, \frac{1}{4}\delta^l_s)$, then $p(y) = W^2 \cap V^-(w)$.

PROPOSITION 1.6 (cf. [10], Theorem 3.2.1). *There exist constants* q^l_s *and* J^l_s *satisfying the following conditions*:

1. *Any succession map constructed as indicated above is absolutely continuous in the neighborhood* $U(x, q^l_s)$.
2. *The jacobian* $J(p)(y)$ (y *is a point of density of the set* $\widetilde{W}^1 \cap \Lambda^l_{k,s}$) *satisfies*

$$|J(p)(y) - 1| \leqslant J^l_s \max_{y \in U\left(x, \frac{1}{8}\delta^l_s\right)} d(E_{2x}, E_{2y}).$$

Similarly one can construct a succession map using local unstable manifolds and prove the assertion analogous to Proposition 1.6.

In what follows, one denotes by ν^-_x, ν^+_x, ν^{-0}_x and ν^{+0}_x measures on the manifolds $V^-(x)$, $V^+(x)$, $V^{-0}(x)$ and $V^{+0}(x)$ respectively, induced by the Riemannian metric.

1.6. *Ergodicity.*

THEOREM 1.1 (cf. [11], Theorem 7.2). *There exist measurable sets* Λ_i, $i = 0, 1, 2, \ldots,$ *such that*

1. $\Lambda_i \cap \Lambda_j = \varnothing$, $i \neq j$, $\bigcup_i \Lambda_i = \Lambda$, $\Lambda_i \subset \Lambda_{k,s}$ *for some* k *and* s;
2. $\nu(\Lambda_0) = 0$, $\nu(\Lambda_i) > 0$ *for* $i > 0$;
3. $f^t(\Lambda_i) = \Lambda_i$ *for any* $t \in \mathbf{R}$;
4. *the flow* $f^t|\Lambda_i$ *is ergodic for* $i > 0$.

1.7. We shall assume that the reader is familiar with the basic concepts of general measure theory and ergodic theory, and also with the concepts connected with measurable partitions, entropy, the K-property, and the Bernoulli property ([13], [19]).

In this section, we shall dwell only on the concept of metric transitivity (cf. [1], §5). It will be assumed that the measure ν is normalized. Let ξ be a measurable partition of the space (M, ν). The partition ξ is called *metrically transitive* if there exists no measurable ξ-set of intermediate measure (i.e. a ξ-set A for which $0 < \nu(A) < 1$).

1.8. Now we introduce the concept of a parallelepiped at a point $w \in \widetilde{\Lambda}^l_{k,s}$, which is used in the following constructions, and we shall study some of its properties.

A measurable set Π is called a δ-*parallelepiped at the point* $w \in \widetilde{\Lambda}^l_{k,s}$ if it satisfies the following conditions:

1. $w \in \Pi \subset \tilde{\tilde{\Lambda}}_{k,s}^{l_1} \cap B(w, \delta)$ $(l_1 = \psi(l, k, s)$ [cf. [11], Theorem 7.4(1)].
2. $\tilde{V}^{-0}(y) \cap V^+(z) \in \Pi$ for any $y, z \in \Pi$.

We shall say that the measurable set $A \subset M$ intersects the parallelepiped Π leafwise if for any $w \in A \cap \Pi$ we have $V^+(w) \cap \Pi \subset A \cap \Pi$.

The following assertions are proved just as in [11] (cf. Lemmas 8.1 and 8.4).

PROPOSITION 1.7. *For any number δ, $0 < \delta < \delta_s^l/8$, and any $w \in \tilde{\tilde{\Lambda}}_s^l$ one can find a number $r > 0$, independent of w, and a δ-parallelepiped Π at w, such that $\tilde{\tilde{\Lambda}}_{k,s}^l \cap B(w, r) \subset \Pi$.*

PROPOSITION 1.8. *Let Π be a δ-parallelepiped, α a finite partition of M with piecewise-smooth boundaries, and $\beta > 0$. There exists an $N_1 > 0$ such that for any $N' \geqslant N \geqslant N_1$ and β-almost any element $A \in \bigvee_N^{N'} f^t\alpha|\tilde{\Lambda}_{k,s}$ one can find a set $E \subset A$, intersecting Π leafwise, for which $\nu(E)(\nu(A))^{-1} \geqslant 1 - \beta$.*

1.9. "*Measurable foliations*". We fix $k > 0$ and $s > 1$, and for $x \in \Lambda_{k,s}$ we set

$$W^-(x) = \bigcup_{-\infty<t<\infty} f^{-t}(V^-(f^t(x))), \quad W^+(x) = \bigcup_{-\infty<t<\infty} f^{-t}(V^+(f^t(x))). \tag{1.4}$$

The following assertions are proved in [12].

THEOREM 1.2 (cf. [12], Theorem 3). *For any $x, y \in \tilde{\Lambda}_{k,s}$ and $t \in \mathbf{R}$ one has the following assertions*:

1. $W^-(x) \cap W^-(y) = \varnothing$, *if* $y \notin W^-(x)$.
2. $W^-(x) = W^-(y)$, *if* $y \in W^-(x)$.

3. *$W^-(x)$ is an immersed k-dimensional submanifold in M of class C^{r-1} without boundary.*

4. $f^t(W^-(x)) = W^-(f^t(x))$.

5. *If $y \in W^-(x)$, then $\rho_{f^t(W^-(x))}(f^t(x), f^t(y)) \longrightarrow 0$ as $t \longrightarrow \infty$ (here $\rho_{f^t(W^-(x))}$ is the distance in the submanifold $f^t(W^-(x))$ induced by the Riemannian metric).*

6. *Assertions 1–5 remain valid if in them one replaces $W^-(x)$ by $W^+(x)$ (and in 5 lets t tend to $-\infty$).*

Let x be a density point of the set $\tilde{\Lambda}_{k,s}^l$, and $A \subset \tilde{\Lambda}_{k,s}^l \cap U(x, \delta_s^l/8)$ a measurable set of positive measure. For $y \in A$ we denote by $n_i(y)$, $i = 1, 2, \ldots$, the sequence of moments at which the semitrajectory $\{f^n(y)\}$, $n \geqslant 0$, lands in the set A.

THEOREM 1.3 (cf. [12], Theorem 4). *For almost any $y \in A$*

$$W^-(y) = \bigcup_{i=1}^{\infty} f^{-n_i(y)}(V^-(f^{n_i(y)}(y))).$$

The sets $\Lambda_i \cap W_i(x)$, $x \in \Lambda_i$ (cf. Theorem 1.1) form a partition of Λ_i, which we denote by ζ_i.

(1) The function ψ is constructed in [11] for the case of a diffeomorphism. The proof for the case of a flow differs only in that the role of $V^-(x)$ is played by $\tilde{V}^{-0}(x)$.

THEOREM 1.4 (cf. [12], Theorem 1). *There exists a partition* η *of the set* Λ_i *which has the following properties*:

1. *For almost any* $x \in \Lambda_i$ *the set* $C_\eta(x)$ *is* mod 0 *an open subset of* $W^-(x)$.
2. $f^1\eta \geqslant \eta$.
3. $\bigvee_{k=0}^{\infty} f^k\eta = \varepsilon$.
4. $\bigwedge_{k=-\infty}^{0} f^k\eta = \nu(\xi_i)$.

1.10. In conclusion we give some notation which is used constantly in this paper.

H is the universal Riemannian covering manifold of M ($\dim M = m$).

SM (SH) is the $(2m-1)$-dimensional manifold in TM (TH) consisting of line elements, i.e. pairs (x, v), $x \in M(H)$, $v \in T_x M$ ($T_x H$), where $\|v\| = 1$ (sometimes the line element will be denoted simply by v).

$\gamma(t)$, $\gamma_v(t)$ and $\gamma_{xy}(t)$ are respectively a geodesic in M or H (parametrized by arc length), the geodesic defined by the line element v (i.e. $\gamma_v(0) = v$), and a geodesic joining x and y (in this case we assume that $x = \gamma_{xy}(0)$ and $y = \gamma_{xy}(t)$, where $t > 0$).

$S^{m-1}(x, t)$ is the $(m-1)$-dimensional sphere in H with center at the point x and radius t, which is a submanifold of H of class C^{r-1} (r is the smoothness class of the Riemannian metric).

Graph(φ) is the graph of the map φ.

The phrase "Proposition 4.4(1)" means assertion 1 of Proposition 4.4.

§2. The K-property

THEOREM 2.1. *Assume that the flow* $f^t|\Lambda_i$ *has continuous spectrum* (*cf.* [13], §2). *Then it is a K-flow.*

PROOF. We shall show that the partition ξ_i is metrically transitive. For the case of a U-system this assertion was proved by Anosov (cf. [1], Theorem 13); we shall adhere to basically the same scheme of argument. One can assume that $\Lambda_i \subset \Lambda_{k,s}$. We set $\Lambda^l = \overline{\Lambda}^l_{k,s} \cap \Lambda_i$.

Let us assume that there exists a measurable ξ_i-set A of intermediate measure.

LEMMA 2.1. *There exists a Borel* ξ_i*-set* B *of intermediate measure.*

PROOF. For some $l > 0$ we have $\nu(A \cap \Lambda^l) > 0$. By virtue of Proposition 1.6 there exists a point $x \in A \cap \Lambda^l$ such that the set $C = V^+(x) \cap A \cap \Lambda^l$ is measurable and $\nu_x^+(C) > 0$. There exists a closed set $D \subset C$ such that $\nu(D) > 0$. According to Proposition 1.6 the set $B = \bigcup_{y \in D} W^-(y)$ has positive measure. We denote by Q the set of rational numbers. It is easy to see that in (1.4) the union can be taken over $t \in Q$. We have

$$B = \bigcup_{y \in D} \bigcup_{t \in Q} f^{-t}(V^-(f^t(y))) = \bigcup_{t \in Q} f^{-t}\Big(\bigcup_{y \in f^t(D)} V^-(y)\Big).$$

Since the set $f^t(D)$ is closed, by virtue of Proposition 1.4(2) the set $\bigcup_{y \in f^t(D)} V^-(y)$ is Borel, and consequently B is Borel. The lemma is proved.

The following assertion is proved in the same way as Lemma 21.3 in [1].

LEMMA 2.2. *There exists a* $\tau > 0$ *such that the set* $C = \bigcup_{|t|\leq\tau} f^t(B)$ *has intermediate measure.*

We proceed to the proof of the theorem. We choose numbers ϵ and l such that

$$0<\varepsilon<\frac{1}{4}\nu(C), \tag{2.1}$$

$$\nu(\Lambda_i\setminus\Lambda^l)\leq\varepsilon\nu(\Lambda_i). \tag{2.2}$$

By virtue of Proposition 1.7 (cf. also [11], Lemma 8.3) there exists a covering of the set Λ^l by parallelepipeds $\Pi_1, \ldots, \Pi_m$ with centers at the points $x_1, \ldots, x_m$, whose diameters are less than $\frac{1}{2}\tau$. (The number τ is defined in Lemma 2.2. One can show that each Π_j is contained mod 0 in the set Λ_i; cf. [11], Theorem 7.1.) From the continuity of the spectrum of the flow f^t on the set Λ_i follows (cf. Lemma 21.1 in [1]) the existence of a number $\Delta > 0$ and a sequence of numbers $t_n \longrightarrow \infty$ such that for any $j = 1, \ldots, m$

$$\nu(f^{-t_n}(B)\cap\Pi_j)\geq\Delta. \tag{2.3}$$

We denote by $\widetilde{\nu}_j$ the measure in the quotient space Π_j/ξ_j^{-0}, where ξ_j^{-0} is the partition of Π_j by the submanifolds $V^{-0}(x)$ (the measure $\widetilde{\nu}_j$ is absolutely continuous with respect to the measures ν_x^+, where $x \in \Pi_j$). From (2.2), the definition of the set C (cf. Lemma 2.2) and Proposition 1.6 follows the existence of a $\delta > 0$ such that

$$\widetilde{\nu}_j(\{y\in\Lambda^l : V^{-0}(y)\subset f^{-t_n}(C)\cap\Pi_j\})>\delta. \tag{2.4}$$

We choose any number γ such that $0 < \gamma < \frac{1}{4}\min\{\delta, \epsilon\}$. There exists a measurable finite partition $\alpha = \{A_1, \ldots, A_l\}$ of the manifold M, each element of which has piecewise smooth boundary, and an α-set D such that

$$\nu(C\,\Delta D)\leq\gamma. \tag{2.5}$$

We consider the parallelepiped Π_1. From Proposition 1.8 for $\beta \leq \frac{1}{2}\min_{1\leq k\leq l} A_k$ follows the existence of a $T > 0$ such that for any $t \geq T$ one can find a set F_t satisfying the condition

$$\nu(F_t)\leq\gamma \tag{2.6}$$

and such that the set $P_t = f^{-t}(D)\setminus F_t$ intersects Π_1 leafwise. It follows from (2.4)–(2.6) that for any $t_n \geq T$ one can find a $y_n \in \Pi_1$ such that the set $Q_n = \widetilde{V}^{-0}(y_n) \cap P_{t_n} \cap \Pi_1$ is measurable and

$$\nu_{y_n}^{-0}(Q_n)>\left(1-\gamma-\frac{\gamma}{\delta}J_s^l\right)\nu_{y_n}^{-0}(V^{-0}(y_n)\cap\Pi_1). \tag{2.7}$$

From what was said above it follows that

$$(\bigcup_{z\in Q_n} V^+(z))\cap \Pi_1 \subset P_{t_n}\cap \Pi_1 \subset f^{-t_n}(D)\cap \Pi_1.$$

Hence from (2.5), (2.7) and Proposition 1.6(2) it follows that for $t_n \geqslant T$

$$\nu(f^{-t_n}(C)\cap \Pi_1)\cdot(\nu(\Pi_1))^{-1} \geqslant 1-\left(2\gamma+\frac{\gamma}{\delta}J_s^l\right)J_s^l.$$

One argues analogously for the parallelepipeds $\Pi_2, \ldots, \Pi_m$. After this, taking account of (2.2) we get that for sufficiently large t_n

$$\nu(f^{-t_n}(C)\cap \Lambda_i)\cdot(\nu(\Lambda_i))^{-1} \geqslant 1-m\gamma(2+J_s^l\delta^{-1})J_s^l-\varepsilon.$$

Since the number γ can be chosen arbitrarily small (independently of m and δ), this inequality contradicts (2.1). Thus, we have proved that $\nu(\xi_i)=\nu$. Now Theorem 2.1 follows from Theorem 1.4, the definition of a K-system and a result of D. Rudolph [20].

§3. The Bernoulli property

THEOREM 3.1. *Assume that the flow $F^t = f^t|\Lambda_i$ has continuous spectrum. Then it is isomorphic with a Bernoulli flow.*

The method of proof of this theorem goes back to Ornstein [19] and is a simple modification of the construction used in [11] to prove the analogous assertion in the case of a diffeomorphism. We shall only indicate those changes which must be made in the proof presented there (cf. [11], Theorem 8.1). Our construction is based on the concept of δ-parallelepiped, introduced in §1.8. We fix $\epsilon > 0$, and let $l > 0$ be such that

$$\nu(\Lambda_l\cap \tilde{\Lambda}_{k,s}^l)\cdot(\nu(\Lambda_l))^{-1} \geqslant 1-\varepsilon.$$

LEMMA 3.1. (cf. [11], Lemma 8.5). *For any $\delta > 0$ there exists a δ_1, $0 < \delta_1 \leqslant \delta$, such that for any δ_1-parallelepiped Π and any set $E \subset \Pi$, $\nu(E) > 0$, intersecting Π leafwise, one can find a bijective map θ: $E \xrightarrow{\text{onto}} \Pi$ (with respect to the normalized measures on E and Π), satisfying the following conditions:*

1) *The jacobian $J(\theta)(y)$ of the map θ at the point $y \in E$ satisfies $|J(\theta)(y) - 1| \leqslant \delta$.*

2) $\rho(F^t(y), F^t(\theta(y))) \leqslant \delta,\ t > 0,\ y\in E.$

PROOF. Let Π be a δ-parallelepiped at the point w, and let $w_1 \in \Pi$. We consider the succession map p_{w,w_1} of a measurable subset of $\tilde{V}^{-0}(w)$ onto a measurable subset of $\tilde{V}^{-0}(w_1)$, defined by means of the local stable manifolds $V^+(y)$, $y \in \Pi$ (cf. §1.5). If δ_1 is sufficiently small, then on the basis of (1.3) the jacobian $J(p_{w,w_1})$ satisfies

$$|J(p_{w,w_1})-1| \leqslant \frac{1}{3}\delta. \tag{3.1}$$

Lowering the number δ_1, one can assume by virtue of (1.1) that for any $y_1, y_2 \in V^-(w)$ and $t > 0$

$$\rho(F^t(y_1), F^t(y_2)) \leqslant \delta. \tag{3.2}$$

Hence (3.2) holds for any $y_1, y_2 \in \tilde{V}^{-0}(w)$ and $t > 0$. Let E be a measurable set of positive measure which intersects Π leafwise, and let

$$\theta_0 : E \cap \widetilde{V}^{-0}(w) \xrightarrow{\text{onto}} \Pi \cap \widetilde{V}^{-0}(w)$$

be any bijective measure-preserving map (keeping in mind the properly normalized measure ν_w^{-0}). If $y \in E$, then

$$z = V^+(y) \cap \widetilde{V}^{-0}(w) \in \Pi \cap \widetilde{V}^{-0}(w).$$

Moreover, $z \in E$, so that $z \in E \subset \widetilde{V}^{-0}(w)$. Hence the formula

$$\theta(y) = V^+(\theta_0(z)) \cap \widetilde{V}^{-0}(y) = p_{w,y} \circ \theta_0 \circ p_{wy}^{-1}(y)$$

properly defines the map θ. The first assertion follows from (3.1), and the second from (3.2) and the condition $\theta(y) \in \widetilde{V}^{-0}(y)$ for $y \in E$. The lemma is proved.

The proof proceeds further just as in [11] (cf. the proof of Theorem 8.1).

PART II. THE CONSTRUCTION OF HOROSPHERES AND LEAVES FOR GEODESIC FLOWS

§4. Preliminary information. The structure of the variational equation for geodesic flows

In this section we give a brief survey of the concepts and results relating to Riemannian manifolds without conjugate points and without focal points, and to geodesic flows on these manifolds. Here it will be assumed that the reader is acquainted with the elements of Riemannian geometry; in particular, we shall not dwell on such concepts as Riemannian connection, vector field along a curve, parallel translation, geodesic, etc., which can be found, for example, in the book [4]. The circle of questions we are interested in is connected with the limit solutions of the Jacobi equation constructed by Eberlein which play an important role in the study of variational equations for the geodesic flow.

4.1. *Jacobi fields.* Everywhere in this chapter M denotes a complete smooth Riemannian m-dimensional manifold, equipped with a Riemannian metric of class C^3. A *Jacobi field* is a vector field along a geodesic γ, satisfying the Jacobi equation

$$Y'' + R_{XY}X = 0, \tag{4.1}$$

where the primes denote covariant differentiation along γ (another notation is $\partial/\partial t$), R is the curvature tensor and $X = \dot\gamma(t)$ is the unit tangent vector field along γ. We denote by $J(\gamma)$ the $2n$-dimensional space of Jacobi fields along γ. Let $\{l_i(t)\}$, $i = 1, \ldots, m$, be the system of vector fields along γ obtained by parallel transport of an orthonormal system at the point $\gamma(0)$, where $l_m(t) = \dot\gamma(t)$. Then (4.1) can be written in matrix form:

$$\frac{d^2}{dt^2} Y(t) + R(t)Y(t) = 0, \tag{4.2}$$

where $Y(t)$ is an m-dimensional vector and $R(t) = (R_{ij}(t))$ is the matrix

$$R_{ij}(t) = \langle R_{l_m(t) l_i(t)} l_n(t), l_j(t)\rangle, \qquad i, j = 1, \ldots, m.$$

Let π: $TM \longrightarrow M$ be the natural projection and K: $T(TM) \longrightarrow TM$ be the map of the Riemannian connection. It is known (cf. [17], §1) that for any $v \in TM$

$$T_vTM = \operatorname{Ker} d\pi \oplus \operatorname{Ker} K, \quad \dim \operatorname{Ker} d\pi = \dim \operatorname{Ker} K.$$

We introduce a scalar product in the space T_vTM by setting

$$\langle \xi, \eta \rangle_v = \langle d\pi\xi, d\pi\eta \rangle_{\pi(v)} + \langle K\xi, K\eta \rangle_{\pi(v)}.$$

In this metric the subspaces Ker $d\pi$ and Ker K are orthogonal.

Let $v \in TM$ and $\xi \in T_v TM$, and let γ_v be the geodesic with initial vector v. We define a Jacobi field Y_ξ by the initial conditions $Y_\xi(0) = d\pi\xi$ and $Y'_\xi(0) = K\xi$.

The map $\xi \longrightarrow Y_\xi$ is a linear isomorphism of T_vTM onto $J(\gamma_v)$ (cf. [17], Proposition 1.7).

Let $Z(s), -\epsilon \leqslant s \leqslant \epsilon$, be a curve in TM with $Z(0) = \dot\gamma(0)$. We consider a variation $r(t, s)$ of the geodesic γ of the form

$$r(t, s) = \exp(tZ(s)), \quad t \geqslant 0, \quad -\varepsilon \leqslant s \leqslant \varepsilon. \tag{4.3}$$

The variation $r(t, s)$ is called *geodesic* if the curve $\alpha(s) = \pi(Z(s))$ is a geodesic and the vectors $Z(s)$ and $\dot\alpha(s)$ are orthogonal.

PROPOSITION 4.1 (cf. [6], Lemma 14.3). *The vector field*

$$Y(t) = \frac{\partial}{\partial s} r(t, s)\big|_{s=0} \tag{4.4}$$

along the geodesic γ is a Jacobi field along γ.

4.2. *Conjugate and focal points.* The points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are called *conjugate* if there exists a Jacobi field $Y \not\equiv 0$ along γ such that $Y(t_1) = Y(t_2) = 0$. The points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are called *focal* if there exists a Jacobi field Y along γ such that $Y(t_1) = 0$, $Y'(t_1) \neq 0$ and $(d/dt)(\|Y(t)\|^2)|_{t=t_2} = 0$.

PROPOSITION 4.2 (cf. [4]). 1. *The points $\gamma(t_1)$, $t_1 > 0$, and $\gamma(0)$ are conjugate if and only if there exists a variation $r(t, s)$ of the form* (4.3), *where $r(0, s) = \gamma(0)$, $-\epsilon \leqslant s \leqslant \epsilon$, for which the point $\gamma(t_1)$ is the limit of points of intersection of the geodesic $\gamma(t)$ and $r(t, s)$ (s fixed) as $s \longrightarrow 0$.*

2. *The points $\gamma(t_1)$, $t_1 > 0$, and $\gamma(0)$ are focal if and only if there exists a geodesic variation $r(t, s)$ of the form* (4.3) *for which $\gamma(t_1)$ is the limit of points of intersection of $\gamma(t)$ and $r(t, s)$ (s fixed) as $s \to 0$.*

PROPOSITION 4.3 (cf. [1], §22). *If on the geodesic γ no two points are conjugate, then for any $t_1, t_2 \in \mathbf{R}$, $v_1 \in T_{\gamma(t_1)}M$ and $v_2 \in T_{\gamma(t_2)}M$ there exists a unique Jacobi field $Y(t)$ along γ such that $v_1 = Y(t_1)$ and $v_2 = Y(t_2)$.*

One says that a Riemannian manifold has no conjugate (focal) points if on each geodesic no two points are conjugate (focal).

It is easy to see that if a Riemannian manifold has no focal points, then it also has no conjugate points. One can also prove that if a manifold has nonpositive curvature, then it has no focal points (cf. [17], §1).

We shall also mention some results describing geometric properties of manifolds without conjugate points and without focal points.

PROPOSITION 4.4. *Assume that the manifold M has no conjugate points.*

1. *The universal Riemannian covering H of M is diffeomorphic with* $\mathbf{R}^m$ by means of *the map* $\exp_x$, $x \in M$ (*cf.* [4], §7).

2. *Geodesics in H realize the distance between any two of their points. Any two geodesics in H intersect in no more than one point* (*cf.* [4], §7.2).

3. *If the curvature of M at any point and in any two-dimensional direction is greater than or equal to* $-a^2$, *then any two intersecting geodesics* $\gamma_1(t)$ *and* $\gamma_2(t)$ *in H diverge, i.e.* $\rho(\gamma_1(t), \gamma_2(t)) \longrightarrow \infty$ *as* $t \longrightarrow \infty$ (*cf.* [14], *p.* 168).

4. *Let* $S^{m-1}(x, t)$ *be an* $(m-1)$*-dimensional sphere in H. Then for any* $y \in S^{m-1}(x, t)$ *the geodesic* $\gamma_{xy}(s)$ *is orthogonal to* $S^{m-1}(x, t)$ (*cf.* [6], *Lemma* 10.5).

PROPOSITION 4.5 (cf. [18]). *If the manifold M has no focal points, then the sphere* $S^{m-1}(x, t)$ *is a (strictly) convex set.*

4.3. *Limit solutions.* Starting with this subsection we shall assume that the manifold M has no conjugate points. We consider the matrix Jacobi equation corresponding to (4.2) ($D(t)$ is an $m \times m$ matrix):

$$\frac{d^2}{dt^2} D(t) + R(t) D(t) = 0. \tag{4.5}$$

PROPOSITION 4.6 (cf. [17], §2). *Let* $D_s(t)$, $s > 0$, *be the solution of* (4.5) *with boundary conditions* $D_s(0) = I$ *and* $D_s(s) = 0$. *Then there exists a solution* $D^-(t)$ *of this equation satisfying the conditions*

$$D^-(0) = I, \quad D^-(t) = \lim_{s\to\infty} D_s(t),$$
$$\frac{d}{dt} D^-(t)\big|_{t=0} = \lim_{s\to\infty} \frac{d}{dt} D_s(t)\big|_{t=0}, \quad \det(D^-(t)) \neq 0$$

for any $t \in \mathbf{R}$.

The solution $D^-(t)$ is called the *negative limit solution* of equation (4.5). Analogously one constructs the *positive limit solution* $D^+(t)$ of (4.5).

For any $v \in SM$, if $V(v)$ is the vector field defined by the geodesic flow in SM (see §4.5, below), we set

$$X^-(v) = \{\xi \in T_vSM : \langle \xi, V(v)\rangle = 0,\ Y_\xi(t) = D^-(t)\, d\pi\xi\},$$
$$X^+(v) = \{\xi \in T_vSM : \langle \xi, V(v)\rangle = 0,\ Y_\xi(t) = D^+(t)\, d\pi\xi\}.$$

The subspaces $X^-(v)$ and $X^+(v)$ are called respectively the *stable* and *unstable subspaces* of $T_v SM$.

PROPOSITION 4.7 (cf. [17], Propositions 2.4 and 2.11). 1. *For any* $v \in SM$, $X^-(v)$ *and* $X^+(v)$ *are vector subspaces of* $T_v SM$ *of dimension* $m - 1$.

2. $d\pi X^-(v) = d\pi X^+(v) = \{w \in T_{\pi(v)}M$: *w is orthogonal to* $v\}$.

3. *Let* τ: $SM \longrightarrow SM$ *be the involution* $\tau(v) = -v$. *Then* $X^+(-v) = d\tau X^-(v)$ *and* $X^-(-v) = d\tau X^+(v)$.

4. *If the curvature of M at any point and in any two-dimensional direction is greater than or equal to* $-a^2$, $a > 0$, *then for any* $\xi \in X^-(v)$ (*or* $\xi \in X^+(v)$)

$$\|K\xi\| \leqslant a\|d\pi\xi\|. \quad (4.6)$$

PROPOSITION 4.8 (cf. [17], §3). *Assume that the Riemannian manifold has no focal points. Then for any Jacobi field* Y_ξ, $\xi \in X^-(v)$ *(respectively* $\xi \in X^+(v)$*), the function* $\|Y_\xi(t)\|$ *is nonincreasing (respectively, nondecreasing).*

PROPOSITION 4.9 (cf. [17], Proposition 2.7). *Let the curvature of the manifold M at any point and in any two-dimensional direction be greater than or equal to* $-a^2$, $a > 0$, *and let* $Y(t)$ *be a perpendicular Jacobi field along the geodesic* $\gamma(t)$, *i.e.* $\langle Y(t), \dot\gamma(t)\rangle = 0$, *while* $Y(0) = 0$. *Then for any* $t > 0$

$$\|Y'(t)\| \leqslant a \coth(at)\|Y(t)\|. \quad (4.7)$$

4.4. *Isometries.* The constructions described in the preceding subsections can be carried over to the universal Riemannian covering H of the manifold M. In particular, for each $v \in SH$ one can construct stable and unstable subspaces $X^-(v)$, $X^+(v) \subset T_v SH$. The fundamental group $\pi_1(M)$ (we shall not indicate the base point explicitly) of M acts by isometries on the covering H.

PROPOSITION 4.10 (cf. [17]). *For any* $v \in SH$ *and* $\varphi \in \pi_1(M)$

$$d(d\varphi)X^-(v) = X^-(d\varphi v), \quad d(d\varphi)X^+(v) = X^+(d\varphi v).$$

4.5. *Geodesic flows and Jacobi fields.* The geodesic flow f^t is the flow in the manifold SM given by the formula (cf. [1], §22) $f^t(v) = \dot\gamma_v(t)$ for $v \in SM$. The flow f^t can be carried over to a flow (also denoted by f^t) on SH. We denote by V the vector field on SM (respectively SH) defined by f^t.

PROPOSITION 4.11 (cf. [17], Propositions 1.7, 2.4 and 2.12). *Let* $v \in SM$ *and* $\xi \in T_v SM$ *(respectively* $v \in SM$ *and* $\xi \in T_v SH$*). Then the following assertions are true*:

1. $Y_\xi(t) = d\pi \circ df^t\xi$, $\quad Y'_\xi(t) = K \circ df^t\xi$.
2. $\|df^t\xi\|^2 = \|Y_\xi(t)\|^2 + \|Y'_\xi(t)\|^2$.
3. *If* $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$, *then* $Y_\xi(t) \neq 0$.
4. $df^t X^-(v) = X^-(f^t(v))$, $\quad df^t X^+(v) = X^+(f^t(v))$.
5. $\xi \in X^-(v)$ *(respectively* $\xi \in X^+(v)$*) if and only if* $\langle \xi, V(v)\rangle = 0$ *and* $\|d\pi \circ df^t\xi\| \leqslant$ const *for* $t > 0$ *(respectively for* $t < 0$*).*

§5. Axiom of asymptoticity

5.1. *Formulation of the axiom of asymptoticity.* Starting with this section it is assumed that the manifold M has no conjugate points and the curvature of M at any point and in any two-dimensional direction is greater than or equal to $-a^2$. We also stress that geodesics are parametrized by arc-length. Geodesics γ_1 and γ_2 on the universal Riemannian covering H are called *asymptotic for* $t > 0$ if one can find a constant $C > 0$ such that $\rho(\gamma_1(t), \gamma_2(t)) \leqslant C$ for any $t > 0$. Analogously one can define *asymptotic geodesics for*

$t < 0$ (geodesics which are asymptotic for $t > 0$ are simply said to be *asymptotic*). The relation of asymptoticity for $t > 0$ (respectively for $t < 0$) is an equivalence relation. A class of equivalent elements is called an *infinitely distant point*, and the set of equivalence classes is called the *absolute* and is denoted by $H(\infty)$. The class of geodesics asymptotic to $\gamma(t)$ for $t > 0$ (for $t < 0$) is denoted by $\gamma(+\infty)$ (respectively $\gamma(-\infty)$).

The action of the fundamental group $\pi_1(M)$ on H can be extended to $H(\infty)$. Namely, if $\varphi \in \pi_1(M)$ and $p = \gamma_v(+\infty) \in H(\infty)$, then $\varphi(p) = \gamma_{d\varphi v}(+\infty) = (\varphi\gamma_v)(+\infty)$.

We choose an arbitrary point $x \in H$, a vector $v \in SH$, a sequence of line elements $v_n \longrightarrow v$, a sequence of points $x_n \longrightarrow x$ and a sequence of numbers $t_n \longrightarrow +\infty$. We denote by γ_n a geodesic joining the points x_n and $\gamma_{v_n}(t_n)$. It is easy to see that the sequence of vectors $\dot{\gamma}_n(0)$ is compact, so that the sequence of geodesics has a limit geodesic.

DEFINITION 5.1. The manifold M satisfies the *axiom of asymptoticity* if for any choice of x_n, $x \in H$, v_n, $v \in SH$, $x_n \longrightarrow x$, $v_n \longrightarrow v$ and $t_n \longrightarrow \infty$ any limit geodesic of the above-constructed sequence of geodesics γ_n is asymptotic to the geodesic γ.

It follows from Proposition 4.4(3) that the sequence of geodesics γ_n has a unique limit geodesic, i.e., it converges.

PROPOSITION 5.1. *If the manifold M satisfies the axiom of asymptoticity, then for any geodesic γ and any point $x \in H$ one can find a unique geodesic γ' passing through x and asymptotic with γ.*

PROOF. We choose an arbitrary sequence of numbers $t_n \longrightarrow +\infty$ and consider the sequence of geodesics $\gamma_n = \gamma_{x\gamma(t_n)}(t)$. It is easy to see that the sequence of geodesics γ_n has a limit geodesic, which by the axiom of asymptoticity is asymptotic to γ. The uniqueness of the asymptotic geodesic passing through a given point follows from Proposition 4.4(3).

5.2. THEOREM 5.1. *If the manifold M has no focal points, then it satisfies the axiom of asymptoticity.*

PROOF. We choose arbitrary points $x, y \in H$ and a geodesic γ passing through x, where $y \notin \gamma$. We fix $t_0 > 0$ and consider the family of spheres

$$\{S^{m-1}(\gamma(t_0), s) : 0 \leqslant s \leqslant \rho(y, \gamma(t_0)) = t_1\}.$$

In what follows the parameter on the geodesic $\gamma(t)$ will be reckoned from the point $w = \gamma(t_0 - t_1)$ in the direction of the point $z = \gamma(t_0)$. We consider the geodesic segment $\delta_{t_1}(u)$, $0 \leqslant u \leqslant a$, on the sphere $S^{m-1}(z, t_1)$, $\delta_{t_1}(a) = y$, $\delta_{t_1}(0) = w$, and on equipping it with unit vectors orthogonal to $S^{m-1}(z, t_1)$ and directed to the center, we get a smooth curve $Z(u, t_1)$ in SH. We consider the variation $r(u, s) = \exp(sZ(u, t_1))$, $0 \leqslant s \leqslant t_1$. The points of intersection of the geodesics $\gamma^u(s) = r(u, s)$ with $S^{m-1}(z, s)$ (s fixed) form a smooth curve $\delta_s(u)$, $\delta_s(0) = \gamma(s)$, $\delta_s(a) = \gamma_{yz}(s)$. We denote by $l(s)$ the length of the curve $\delta_s(u)$.

LEMMA 5.1. *For any $t_0 > 0$*

$$l(s) \leqslant l(0), \quad 0 \leqslant s \leqslant t_1. \tag{5.1}$$

PROOF. Let $Y_u(s)$ be the Jacobi field along the geodesic $\gamma^u(s)$ generated by the variation $r(u, s)$ (cf. (4.4)), $0 \leqslant u \leqslant a$. It is obvious that $Y_u(t_1) = 0$ and $\langle Y_u(s), \dot{\gamma}^u(s)\rangle = 0$. Since M has no focal points, by virtue of Proposition 11.6

$$\|Y_u(s_1)\| \geqslant \|Y_u(s_2)\|, \tag{5.2}$$

if $s_1 \leqslant s_2$. We have

$$l(s) = \int_0^a \|Y_u(s)\| du. \tag{5.3}$$

It follows from (5.2) and (5.3) that $l(s_1) \geqslant l(s_2)$ if $0 \leqslant s_1 \leqslant s_2 \leqslant t$, whence (5.1) follows. The lemma is proved.

LEMMA 5.2. *For any* $t_0 > 0$ *we have* $\rho(x, \gamma(t_1)) \leqslant \rho(x, y)$.

PROOF. Depending on the disposition of the points x, z and $w = \gamma(0)$, two cases are possible.

1. The point w lies between x and z. It follows from the triangle inequality that $\rho(x, w) + \rho(w, z) \leqslant \rho(x, y) + \rho(y, z)$. Since $\rho(w, z) = \rho(y, z) = t_1$, it follows that $\rho(x, w) \leqslant \rho(z, y)$.

2. The point x lies between w and z. It follows from the triangle inequality that $\rho(z, x) = \rho(z, y) - \rho(x, y)$. Since $\rho(z, x) = t_1 - \rho(x, w)$ and $\rho(z, y) = t_1$, it follows that $\rho(x, w) \leqslant \rho(x, y)$. The lemma is proved.

LEMMA 5.3. *There exists an* $\epsilon > 0$ *such that for* $t_0 \geqslant 1$ *it follows from the condition* $\rho(x, y) \leqslant \epsilon$ *that* $l(t_1) \leqslant 4\rho(x, y)$.

PROOF. We join the points $w = \gamma(0)$ and y by a geodesic segment $\bar{\gamma}(\tau)$ ($\bar{\gamma}(0) = w$) and through the points $\bar{\gamma}(\tau)$ and z we draw a geodesic which intersects $S^{m-1}(z, t_1)$ at a point $\Delta(\tau)$. We denote the variation obtained in this way by $\bar{r}(\tau, s)$ ($\bar{r}(\tau, 0) = \Delta(\tau)$; $\bar{r}(\tau, t_1) = z$). Let $\bar{Y}_\tau(s)$ be the Jacobi field corresponding to $\bar{r}(\tau, s)$. We have $\bar{Y}_\tau(t_1) = 0$ and

$$\langle \bar{Y}_\tau(s), \frac{\partial}{\partial s}\bar{r}(\tau, s)\rangle = 0.$$

According to Proposition 4.9 there exists a $C > 0$ such that $\|\bar{Y}'_\tau(s)\| \leqslant C\|\bar{Y}_\tau(s)\|$ for any $t_0 \geqslant 1$ and $0 \leqslant s \leqslant t_1$. We also have that $\gamma(\tau) = r(\tau, s(\tau))$, where $s(\tau) \geqslant 0$ for $0 \leqslant \tau \leqslant \rho(y, w)$. Moreover, for any $\alpha > 0$ one can find an $\epsilon > 0$ (independent of t_0) such that if $\rho(x, y) \leqslant \epsilon$, then $s(\tau) \leqslant \alpha$ for any $0 \leqslant \tau \leqslant \rho(y, w)$ (because by virtue of Lemma 5.2, $\rho(y, w) \leqslant \rho(x, y) + \rho(x, w) \leqslant 2\rho(x, y)$). Using the inequality

$$|\|\bar{Y}_\tau(s)\|'| \leqslant \|\bar{Y}'_\tau(s)\|,$$

we get

$$\|\bar{Y}_\tau(s(\tau))\| = \|\bar{Y}_\tau(0)\| + \int_0^{s(\tau)} \|\bar{Y}_\tau(s)\|' ds \geqslant \|\bar{Y}_\tau(0)\|$$

$$- \int_0^{s(\tau)} \|\bar{Y}'_\tau(s)\| ds \geqslant \|\bar{Y}_\tau(0)\| - Cs(\tau)\|\bar{Y}_\tau(0)\| \geqslant \frac{1}{2}\|\bar{Y}_\tau(0)\|,$$

if the number α is chosen small enough that $1 - C\alpha \geqslant ½$. It follows that $\|\dot{\Delta}(\tau)\| = \|\overline{Y}_\tau(0)\| \leqslant 2\|\overline{Y}_\tau(s(\tau))\| \leqslant 2\|\dot{\gamma}(\tau)\| = 2$. Hence $l(t_1) \leqslant$ the length of $\Delta(\tau) \leqslant 2\rho(w, y) \leqslant 4\rho(x, y)$. The lemma is proved.

Let $v_n \longrightarrow v$, $x_n \longrightarrow x$ and $t_n \longrightarrow \infty$. If the points x and $\pi(v)$ are sufficiently close, then from Lemmas 5.1–5.3 follows the existence of $\tau_n \longrightarrow 0$ such that the function $\rho(\gamma_{v_n}(t + \tau_n), \gamma_n(t))$ is nonincreasing for $t \in [0, t_n]$ and is bounded above:

$$4\rho(x_n, \pi(v_n)) \leqslant 5\rho(x, \pi(v))$$

(if n is sufficiently large). Hence follows the validity of the axiom of asymptoticity for sufficiently close points x and $\pi(v)$. Since the relation of asymptoticity is an equivalence relation, from this follows the validity of the axiom of asymptoticity for any $x \in H$ and $v \in SH$. The theorem is proved.

5.3. *The axiom of visibility.* We shall say that a Riemannian manifold satisfies the *axiom of uniform visibility* (cf. [14]–[16] and [22]) if for any $\epsilon > 0$ there exists an $R = R(\epsilon)$ such that from each point $x \in H$, any geodesic segment γ for which $\rho(x, \gamma) \geqslant R$ is visible at an angle less than ϵ.

PROPOSITION 5.2 (cf. [14], Theorems 4.2 and 5.1). *A compact two-dimensional manifold M of genus greater than* 1 *satisfies the axiom of uniform visibility.*

PROPOSITION 5.3 (cf. [14], Propositions 1.13 and 1.7). *If the compact manifold M satisfies the axiom of uniform visibility, then the following assertions hold*:

1. *One can introduce a topology on the absolute $H(\infty)$ and construct a homeomorphic map of the closed unit ball in $\mathbf{R}^m$ onto the set $H \cup H(\infty)$, which associates the interior of the ball with the set H and the sphere S^{m-1} with the absolute $H(\infty)$.*

2. *For any two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ there exists a geodesic $\gamma(t)$ such that $\gamma(+\infty) = \gamma_1(+\infty)$ and $\gamma(-\infty) = \gamma_2(-\infty)$.*

3. *If $p, q \in H(\infty)$ and U and V are open neighborhoods of p and q respectively* (*cf. assertion* 1), *then there exists an isometry φ of the space H such that $\varphi(U) \subset V$.*

From the results of [14] (cf. Lemma 1.6) there also follows

PROPOSITION 5.4. *If the manifold M satisfies the axiom of uniform visibility, then it also satisfies the axiom of asymptoticity.*

THEOREM 5.2 (cf. [14], Theorem 3.7). *If the compact manifold M satisfies the axiom of uniform visibility, then the geodesic flow in SM is topologically transitive.*

§6. Invariant foliations for a geodesic flow

6.1. THEOREM 6.1. *If the smooth m-dimensional compact manifold M with Riemannian metric of class C^3 has no conjugate points and satisfies the axiom of asymptoticity, then the distributions X^- and X^+ are integrable, and their maximal integral submanifolds form continuous C^1-foliations W^- and W^+ respectively of the manifold SH* (*for the definition of continuous foliation cf.* [1], §4).

PROOF. We shall only prove the integrability of the distribution $X^-(v)$, since for

$X^+(v)$ the argument is analogous. We consider the sphere $S^{m-1}(x, t)$, $x \in H$, $t > 0$. Let $y \in S^{m-1}(x, t)$, $w \in T_y S^{m-1}(x, t)$, $\|w\| = 1$, and let $\delta_t(s)$, $-\epsilon \leqslant s \leqslant \epsilon$, be a geodesic segment on the sphere such that $\delta_t(0) = y$ and $\dot{\delta}_t(0) = w$. We consider the variation

$$r_t(u, s) = \gamma_{\delta_t(s)x}(u), \quad \gamma_{\delta_t(s)x}(0) = \delta_t(s).$$

By virtue of Proposition 4.1(1) the vector field

$$Y_t(u) = \frac{\partial}{\partial s} r_t(u, s)\big|_{s=0} \tag{6.1}$$

is a Jacobi field along the geodesic $\gamma_{yx}(u)$ satisfying

$$Y_t(0) = w, \quad Y_t(t) = 0. \tag{6.2}$$

By virtue of Proposition 4.4(4)

$$\langle Y_t(u), \dot{\gamma}_{yx}(u)\rangle = 0, \quad 0 \leqslant u \leqslant t. \tag{6.3}$$

Let $n_t(s)$ be a vector at the point $\delta_t(s)$, normal to $S^{m-1}(x, t)$. Since $\langle\dot{\delta}_t(s), n_t(s)\rangle = 0$ for any $s \in [-\epsilon, \epsilon]$, one has

$$\langle \dot{\delta}_t'(s), n_t(s)\rangle = -\langle \dot{\delta}_t(s), n_t'(s)\rangle, \quad -\varepsilon \leqslant s \leqslant \varepsilon. \tag{6.4}$$

Since the parameter u is the length of the geodesic segment, one has

$$\frac{\partial}{\partial u} r_t(u, s)\big|_{u=0} = \dot{\gamma}_{\delta_t(s)x}(0) = n_t(s).$$

From what was said above and Lemma 8.7 of [6] it follows that

$$Y'(0) = \frac{\partial}{\partial u}\frac{\partial}{\partial s} r_t(u, s)\Big|_{\substack{s=0,\\ u=0}} = \frac{\partial}{\partial s}\frac{\partial}{\partial u} r_t(u, s)\Big|_{\substack{u=0,\\ s=0}} = \frac{\partial}{\partial s} n_t(s)\big|_{s=0}.$$

Hence the curvature $K(t, w)$ of the curve $\delta_t(s)$ for $s = 0$ admits the estimate

$$|K(t, w)| = |\langle \dot{\delta}_t'(s), n_t(s)\rangle|_{s=0}| \leqslant \|w\| \|Y_t'(0)\| = \|Y_t'(0)\|.$$

Since by virtue of (6.2) and (6.3) the field $Y_t(u)$ satisfies the hypotheses of Proposition 4.9, from this and (4.7) follows

$$|K(t, w)| \leqslant a \coth(at). \tag{6.5}$$

Let $v \in SH$, $\pi(v) = x$. We fix $\epsilon > 0$ and we consider the geodesic area element orthogonal to the vector v (i.e. the set of geodesic segments of length 2ϵ whose initial vector is orthogonal to v), which we shall denote by $\Pi(v, \epsilon)$ (this is a submanifold of H of class C^2). In a neighborhood of x (for sufficiently small ϵ) the set $\exp_x^{-1}(S^{m-1}(\gamma_v(t), t))$ can be represented as the graph of a function $\varphi_t(u)$, $u \in \exp_x^{-1}(\Pi(v, \epsilon))$, of class C^2, where $\varphi_t(0) = 0$ and $d\varphi_t(0)/du = 0$ (of course $\varphi_t(u)$ depends on x and v, but we shall not indicate this dependence explicitly in the notation). We identify $T_x H$ with $\mathbf{R}^n$ and $\exp_x^{-1}(\Pi(v, \epsilon))$ with a neighborhood of zero in $\mathbf{R}^{n-1}$. We denote by $g_{ij}(y)$ the components of the Riemannian metric in $T_y M$, $y \in M$, and let $g_{ij}^*(y) = (\exp_x^{-1})^* g_{ij}(y)$. For sufficiently small ϵ (by the compactness of M, ϵ can be chosen independently of x) by virtue of the properties of the map $\exp_x$ (cf. [4], §2) we have for $y \in B(x, \epsilon)$, $y = (y_1, \ldots, y_m)$,

$$\frac{1}{2}|g_{ij}(y)| \leqslant |\overset{*}{g}_{ij}(y)| \leqslant 2|g_{ij}(y)|,$$
$$\frac{1}{2}\left|\frac{\partial}{\partial y_k} g_{ij}(y)\right| \leqslant \left|\frac{\partial}{\partial y_k} \overset{*}{g}_{ij}(y)\right| \leqslant 2\left|\frac{\partial}{\partial y_k} g_{ij}(y)\right|.$$

Hence and from (6.5) follows the boundedness of the curvature of the surface which is the graph of the function $\varphi_t(u)$, $u \in \exp_x^{-1}(\Pi(v, \epsilon))$ (uniformly for $t \geqslant 1$ and $v \in SH$). Hence there exists a constant $C > 0$, independent of x and v, such that for $t \geqslant 1$ and $u \in \exp_x^{-1}(\Pi(v, \epsilon))$ we have

$$\left\| \frac{d^2}{du^2} \varphi_t(u) \right\| \leqslant Ca. \tag{6.6}$$

Hence the family of functions $\{\varphi_t(u)\}$ is relatively compact in the C^1-topology. Let φ be the limit function of this family.

LEMMA 6.1. 1. $\varphi \in C^1$.

2. *If $y \in$ Graph(φ), then a geodesic passing through the point $\exp_x y$, orthogonal to the submanifold $\widetilde{W}(x) = \exp_x$ Graph(φ), is asymptotic to the geodesic $\gamma_v(t)$.*

3. *If $\delta_t(s)$, $-\epsilon \leqslant s \leqslant \epsilon$, is a geodesic segment on the submanifold $\widetilde{W}(x)$, where $\delta_t(0) = x$, and $Z(s)$ is an assignment along this segment of vectors normal to $\widetilde{W}(x)$, then a variation of the form (4.3) defines by (4.4) a Jacobi field $Y(t) = Y_\xi(t)$, where $\xi \in X^-(v)$ and $d\pi\xi = \delta_t(0)$.*

PROOF. Assertion 1 follows from the construction of the function φ.

2. The function φ is the limit in the C^1-topology of the functions φ_{t_n}, where t_n is some sequence of numbers, $t_n \longrightarrow \infty$. If $y \in$ Graph(φ), then there exists a sequence of points $y_n \longrightarrow y$, where $y_n \in$ Graph(φ). We set $z_n = \exp_x y_n$ and $z = \exp_x y$. Since the geodesic $\gamma_n = \gamma_{z_n \gamma_v(t_n)}$ is orthogonal to the sphere $S^{m-1}(\gamma_v(t_n), t_n)$ at the point z_n, the sequence of geodesics γ_n converges to a geodesic orthogonal to the submanifold $\widetilde{W}(x)$ at z. Since M satisfies the axiom of asymptoticity, this geodesic is asymptotic to $\gamma_v(t)$.

3. We choose an arbitrary vector $w \in T_x M$, orthogonal to v, and let $\delta_t(s)$, $-\epsilon \leqslant s \leqslant \epsilon$, be a geodesic segment on the sphere $S^{m-1}(\gamma_v(t), t)$ such that $\delta_t(0) = x$ and $\dot{\delta}_t(0) = w$. Taking an orthonormal system of parallel vector fields along the geodesic $\gamma_v(t)$, we write the Jacobi field (6.1) in the form $Y_t(u) = D_t(u)w$, where $D_t(u)$ is a solution of the matrix equation (4.5) with boundary conditions $D_t(0) = I$ and $D_t(t) = 0$. On the basis of Proposition 4.6 the limit

$$\lim_{t_n \to \infty} Y_{t_n}(u) = \lim_{t_n \to \infty} D_{t_n}(u)\, w = D_t(u)\, w = Y_\xi(u)$$

exists. The lemma is proved.

We consider a point $y \in \widetilde{W}(x)$ and denote by $\gamma(y, t)$ a geodesic passing through y and orthogonal to $\widetilde{W}(z)$, where we assume that $\gamma(y, 0) = y$. By virtue of Proposition 5.1 and Lemma 6.1(2) the geodesics $\gamma(y_1, t)$ and $\gamma(y_2, t)$ do not intersect if $y_1 \neq y_2$. Hence the set

$$U(x) = \bigcup_{y \in \widetilde{W}(x)} \bigcup_{-\varepsilon < t < \varepsilon} \gamma(y, t)$$

is an open neighborhood of x. To each point $z \in U(x)$ we assign the vector $v(z)$ which is the unit tangent vector to the geodesic $\gamma(y, t)$, passing through z. Let $\hat{\varphi}(u)$ be any other limit function of the family $\{\varphi_t(u)\}$, and let $\hat{W}(x) = \exp_x \operatorname{Graph}(\hat{\varphi})$. According to Lemma 6.1, $x \in \hat{W}(x)$, and if $z \in \hat{W}(x)$, then a geodesic γ passing through z and orthogonal to the submanifold $\hat{W}(x)$ is asymptotic to $\gamma_v(t)$. Hence $\gamma = \gamma(y, \cdot)$, where y is the point of intersection of γ and $\widetilde{W}(x)$. Hence $\hat{W}(x)$ is orthogonal to the field $v(z)$. Introducing any local coordinates in a neighborhood of x, considering the intersections of $\hat{W}(x)$ and $\widetilde{W}(x)$ with all possible planes passing through v, and using the classical uniqueness theorem in the theory of differential equations, we conclude that $\varphi(u) = \hat{\varphi}(u)$, $u \in \exp_x^{-1}(\Pi(v, \epsilon))$. In particular, $\varphi_t \longrightarrow \varphi$ as $t \longrightarrow \infty$.

We denote by $\widetilde{\mathfrak{S}}^-(v)$ an assignment to the submanifold $\widetilde{W}(x)$ of orthonormal vectors directed to the same side as the vector v. It follows from Lemma 6.1 that $\widetilde{\mathfrak{S}}^-(v)$ is an $(m-1)$-dimensional submanifold of SH of class C^1, which has the following characteristic property:

there exists a neighborhood $U(v)$ such that if $w \in \widetilde{\mathfrak{S}}^-(v) \cap U(v)$, then the geodesic $\gamma_w(t)$ is orthogonal to the submanifold $\pi(\widetilde{\mathfrak{S}}^-(v))$ and is asymptotic to the geodesic $\gamma_v(t)$. (6.7)

It also follows from Lemma 6.1(3) that

$$v \in \widetilde{\mathfrak{S}}^-(v), \quad T_v\widetilde{\mathfrak{S}}^-(v) = X^-(v). \tag{6.8}$$

LEMMA 6.2. *The submanifold $\widetilde{\mathfrak{S}}^-(v)$ depends continuously in the C^1-topology on $v \in SH$.*

PROOF. Let $v_n, v \in SH$, $v_n \longrightarrow v$ as $n \longrightarrow \infty$. We set $\gamma_n(t) = \gamma_{v_n}(t)$, $x_n = \pi(v_n)$, $x = \pi(v)$ and $\gamma(t) = \gamma_v(t)$, and we consider the sphere $S^{m-1}(x_n, t)$. By virtue of (6.6), for sufficiently small $\epsilon > 0$ and $t \geqslant 1$ the surface $W_{n,t} = \exp_{x_n}^{-1}(S^{m-1}(x_n, t))$ can be represented as the graph of some function of class C^2, denoted by $\psi(n, t, u)$, $u \in \exp_{x_n}^{-1}(\Pi(v_n, \epsilon))$, and the surface $W_n = \exp_{x_n}^{-1}(\pi(\widetilde{\mathfrak{S}}^-(v_n)))$ as the graph of some function of class C^1, denoted by $\psi(n, u)$, $u \in \exp_{x_n}^{-1}(\Pi(v_n, \epsilon))$, where $\psi(n, t) \longrightarrow \psi(n)$ as $t \longrightarrow +\infty$ in the C^1-topology for any $n > 0$. It follows that for sufficiently small $\epsilon > 0$ the surfaces $W_{n,t}$ and W_n can be represented as the graphs of functions denoted by $\psi(n, t, u)$ and respectively $\overline{\psi}(n, u)$, where $u \in \exp_x^{-1}(\Pi(v, \epsilon))$. Moreover, the family of functions $\{\overline{\psi}(n, t), \overline{\psi}(n)\}$ is compact in the C^1-topology. Let $\overline{\psi}$ be the limit function of the family of functions $\{\overline{\psi}(n)\}$. We choose sequences of numbers $n_k \longrightarrow \infty$ and $t_k \longrightarrow \infty$ such that $\overline{\psi}(n_k, t_k) \longrightarrow \overline{\psi}$. We set

$$W = \{\exp_x(u, \overline{\psi}(u)) : u \in \exp_x^{-1}(\Pi(v, \varepsilon))\}.$$

Let $y \in W$. We choose a sequence of points $y_k \in W_{n_k, t_k}$ such that $y_k \longrightarrow y$ as $k \longrightarrow \infty$. By virtue of the axiom of asymptoticity the geodesics $\overline{\gamma}_k$, joining the points y_k and $\gamma_{n_k}(t_k)$, converge to some geodesic $\bar{\gamma}$ asymptotic to γ. Moreover, $\bar{\gamma}$ is orthogonal to the submanifold W (because the geodesics $\tilde{\gamma}_k$ are orthogonal to W_{n_k,t_k}). Hence and from property (6.7) follows the required assertion. The lemma is proved.

An immediate consequence of property (6.7) is

LEMMA 6.3. *For any $v \in SH$ and $w \in \tilde{\mathfrak{S}}^-(v)$*

$$\tilde{\mathfrak{S}}^-(v) \cap U(v) \subset \tilde{\mathfrak{S}}^-(v).$$

This lemma allows one to "paste together" the submanifolds $\tilde{\mathfrak{S}}^-(v)$ passing through different line elements $v \in SH$. Namely, we shall call line elements v and w *equivalent* (cf. [2], §2), if one can find line elements $v_1 = v, v_2, \ldots, v_p = w$ such that $v_i \in \tilde{\mathfrak{S}}^-(v_{i-1})$, $i = 2, \ldots, p$. From Lemmas 6.2, 6.3 and condition (6.8) it follows that the set of equivalence classes forms the required continuous C^1-foliation. Theorem 6.1 is proved.

Let $v \in SH$. We denote by $\mathfrak{S}^-(v)$ ($\mathfrak{S}^+(v)$) the leaf of the foliation $\mathfrak{S}^-$ ($\mathfrak{S}^+$) containing the line element v.

DEFINITION 6.1. The leaf $\mathfrak{S}^-(v)$ (respectively $\mathfrak{S}^+(v)$) is called *the stable* (respectively *unstable*) *horosphere passing through the line element v.*

Some properties of horospheres are established in the following assertions.

PROPOSITION 6.1. 1. $\mathfrak{S}^-(-v) = \mathfrak{S}^+(v)$, *and* $\mathfrak{S}^+(-v) = \mathfrak{S}^-(v)$.

2. *If $\varphi \in \pi_1(M)$, then $d\varphi\, \mathfrak{S}^-(v) = \mathfrak{S}^-(d\varphi v)$ and $d\varphi\, \mathfrak{S}^+(v) = \mathfrak{S}^+(d\varphi v)$.*

3. *There exists a $\delta > 0$ such that, for any $v \in SH$,*

a) *if $w \in \mathfrak{S}^-(v) \cap B(v, \delta)$, then the geodesic $\gamma_w(t)$ is orthogonal to the submanifold $\pi(\mathfrak{S}^-(v) \cap B(v, \delta))$ and is asymptotic to the geodesic $\gamma_v(t)$;*

b) *if $w \in \mathfrak{S}^+(v) \cap B(v, \delta)$, then $\gamma_w(t)$ is orthogonal to $\pi(\mathfrak{S}^+(v) \cap B(v, \delta))$ and is asymptotic to $\gamma_{-v}(t)$.*

PROOF. Assertion 1 follows from Proposition 4.7; assertion 2 from Proposition 4.10 and the construction of the leaf $\mathfrak{S}^-(v)$; assertion 3 from assertion 1, property (6.7) and the continuity of the foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$.

PROPOSITION 6.2. 1. $\mathfrak{S}^-(v)$ *is a connected $(m-1)$-dimensional closed submanifold of SH.*

2. *For any $t \in \mathbf{R}$, $t \neq 0$, we have $f^t(\mathfrak{S}^-(v)) \cap \mathfrak{S}^-(v) = \varnothing$.*

PROOF. It is obvious that the set $\mathfrak{S}^-(v)$ is connected.

LEMMA 6.4. *For any $y \in \pi(\mathfrak{S}^-(v))$ there exist sequences of numbers $t_n \longrightarrow +\infty$ and points y_n such that $y_n \longrightarrow y$ and $y_n \in S^{m-1}(\gamma_v(t_n), t_n)$.*

PROOF. The set of points satisfying the assertion of the lemma is nonempty (it contains the point $\pi(v)$) and closed in $\pi(\mathfrak{S}^-(v))$ (this is easily proved by a diagonal process). We shall show that this set is open. In fact, arguing just as in the proof of Lemma 6.2, we get that if $y_n \longrightarrow y$, then the family of submanifolds

$$\{S^{m-1}(\gamma_n(t_n), t_n) \cap B(\pi(v), \varepsilon)\}$$

is compact in the C^1-topology for some $\epsilon > 0$. If W is the limit submanifold then by virtue of the axiom of asymptoticity any geodesic orthogonal to W is asymptotic to $\gamma_v(t)$. Hence it follows from (6.7) that $W \subset \pi(\mathfrak{S}^-(v))$. Now the lemma follows from the connectedness of $\pi(\mathfrak{S}^-(v))$.

Let $y, z \in \pi(\mathfrak{S}^-(v))$ and $t_n \longrightarrow \infty$. On the basis of Lemma 6.4 there exists a sequence $z_n \longrightarrow z$ such that $z_n \in S^{m-1}(\gamma_v(t_n), t_n)$. We consider a vector w_n orthogonal to $S^{m-1}(\gamma_v(t_n), t_n)$ at the point $z_n = \pi(w_n)$ and directed to the center, and a vector $w \in \mathfrak{S}^-(v)$, $\pi(w) = z$. We have $w_n \longrightarrow w$. Hence $p_n = \pi(f^t(w_n)) \longrightarrow \pi(f^t(w)) = p$ and $p_n \in S^{m-1}(\gamma_v(t_n + t), t_n)$. We consider the sphere $S^{m-1}(\gamma_v(t_n + t), t_n + t)$. Each point of such a sphere is distant by t from the sphere $S^{m-1}(\gamma_v(t_n + t), t_n)$. By virtue of Lemma 6.4 there exist $y_n \in S^{m-1}(\gamma_v(t_n + t), t_n + t)$ such that $y_n \longrightarrow y$. Hence from what was said above it follows that $\rho(y, p) \geqslant t$. Since the points y and z were chosen arbitrarily, assertion 2 follows from this. We shall prove assertion 1. If $w \in \mathfrak{S}^-(v) \cap U(v)$ and $w \in \tilde{\mathfrak{S}}(v)$ (cf. (6.7)), then $\gamma_w(t)$ intersects $\pi(\mathfrak{S}^-(v))$ at some point $\gamma_w(t_0)$, where $f^{t_0}(w) \in \mathfrak{S}^-(v)$, which contradicts assertion 2. The proposition is proved.

THEOREM 6.2. *If the compact manifold M with Riemannian metric of class C^3 has no conjugate points and satisfies the axiom of asymptoticity, then the distributions X^- and X^+ are integrable and their integral submanifolds form continuous C^1-foliations (denoted as before by $\mathfrak{S}^-$ and $\mathfrak{S}^+$) of the manifold SM.*

THEOREM 6.3. *The foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ are invariant with respect to the geodesic flow f^t (considered in SM or SH).*

PROOF. Let $v \in SM$ (or $v \in SH$) and $w \in \tilde{\mathfrak{S}} = f^t(\mathfrak{S}^-(v)) \cap B(f^t(v), \delta)$ (cf. Proposition 6.1(3)). It is obvious that the geodesics $\gamma_v(s)$ and $\gamma_w(s)$ are asymptotic. From Propositions 4.11(4) and 4.7(2) it follows that $\gamma_w(s)$ is orthogonal to $\pi(\tilde{\mathfrak{S}})$. Hence, by virtue of Proposition 6.1(3), $\tilde{\mathfrak{S}} = \mathfrak{S}^-(f^t(v)) \cap B(f^t(v), \delta)$. One argues analogously for the foliation $\mathfrak{S}^+$. The theorem is proved.

For $v \in SM$ (or $v \in SH$) we denote by $Z(v)$ the one-dimensional subspace of $T_v SM$ (respectively $T_v SH$) generated by the vector $V(v)$. Also let $\mathfrak{S}^0$ denote the smooth foliation of the manifold SM or SH formed by the trajectories of the geodesic flow.

THEOREM 6.4. *If the manifold M has no conjugate points and satisfies the axiom of asymptoticity, then the pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^0$ is integrable in the sense of* [1] *(cf.* §4), *and leaves of the corresponding foliation (denoted by $\mathfrak{S}^{-0}$) are integral submanifolds of the distribution $X^- \oplus Z$. The foliation $\mathfrak{S}^{-0}$ is invariant with respect to the flow f^t, and its leaves have the following properties*:

1. *$w \in \mathfrak{S}^{-0}(v)$ if and only if the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ are asymptotic for $t > 0$.*
2. *$d\varphi(\mathfrak{S}^{-0}(v)) = \mathfrak{S}^{-0}(d\varphi v)$ for any $v \in SH$ and $\varphi \in \pi_1(M)$.*

The pair of foliations $\mathfrak{S}^+$ and $\mathfrak{S}^0$ has analogous properties (the corresponding foliation is denoted by $\mathfrak{S}^{+0}$).

PROOF. For $v \in SM$ we set

$$\mathfrak{S}^{-0}(v) = \bigcup_{w \in \mathfrak{S}^-(v)} \bigcup_{-\infty < t < \infty} f^t(w).$$

From Theorem 6.3 it follows that $\mathfrak{S}^{-0}(v) = \bigcup_{-\infty<t<\infty} \mathfrak{S}^{-}(f^t(v))$. Hence for sufficiently small $\delta > 0$ the set $\mathfrak{S}^{-0}(v) \cap B(v, \delta)$ is a submanifold of SH of class C^1, and $T_v \mathfrak{S}^{-0}(v) = X^-(v) \oplus Z(v)$. From Proposition 6.1(3) it follows that for any $w \in \mathfrak{S}^{-0}(v)$ the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ are asymptotic for $t > 0$. Moreover, there exists a $\delta > 0$ such that, for any $v \in SH$ and $w \in B(v, \delta)$ such that $\gamma_v(+\infty) = \gamma_w(+\infty)$, we have $w \in \mathfrak{S}^{-0}(v)$. If $v, w \in SH$ and $\gamma_v(+\infty) = \gamma_w(+\infty)$, then we choose $w_0 = v, w_1, \dots, w_p = w$ such that $w_i \in B(w_{i-1}, \delta)$ and $\gamma_{w_i}(+\infty) = \gamma_{w_{i-1}}(+\infty)$, $i = 1, \dots, p$. It follows from what was said above that $w \in \mathfrak{S}^{-0}(v)$. Assertion 1 is proved. Assertion 2 follows from Proposition 6.1(2). The theorem is proved.

DEFINITION 6.2. The leaf $\mathfrak{S}^{-0}(v)$ (respectively $\mathfrak{S}^{+0}(v)$) is called *the stable (unstable) leaf passing through the line element v.*

§7. Some consequences

7.1. THEOREM 7.1. *If the compact manifold M has no conjugate points and satisfies the axiom of asymptoticity, then the distributions X^- and X^+ (considered in SM or SH) are continuous.*

The proof follows from Theorems 6.1 and 6.2.

It follows from the results of the preceding section and also Theorem 5.1 and Propositions 5.2 and 5.4 that *horospheres and leaves can be constructed on any compact Riemannian manifold satisfying one of the following conditions*:

1. *The manifold has no focal points.*(2)
2. *The manifold has no conjugate points and satisfies the axiom of visibility.*
3. *The manifold has no conjugate points and* $\dim M = 2$ *(actually, if the genus of M is 2 or greater, then M satisfies the axiom of visibility; in the case of the torus any metric without conjugate points, as is known* [23], *coincides with the standard metric).*

7.2. *Limit spheres.* Let $v \in SH$, $x = \pi(v)$ *and* $p = \gamma_v(+\infty)$.

DEFINITION 7.1. The set $L(x, p) = \pi(\mathfrak{S}^-(v))$ is called *the limit sphere with center at the point p, passing through the point x* (we note that $p \notin L(x, p)$).

Immediately from the definition of limit sphere and Propositions 5.1, 6.1 and 6.2 we get

THEOREM 7.2. *Assume that the compact Riemannian manifold M has no conjugate points and satisfies the axiom of asymptoticity.*

1. *The limit sphere $L(x, p)$ is an $(m-1)$-dimensional submanifold of H of class C^1* ($m = \dim M$). *The set $L(x, p)$ is closed in H.*

2. *For any $x \in H$ and $p \in H(\infty)$ there exists a unique limit sphere with center at p, passing through x.*

3. *The leaf $\mathfrak{S}^-(v)$ is the limit sphere $L(x, p)$ ($x = \pi(v)$, $p = \gamma_v(+\infty)$) equipped with orthogonal unit vectors directed to the same side as the vector v. The leaf $\mathfrak{S}^+(v)$ is the limit sphere $L(x, q)$ ($q = \gamma_v(-\infty) = \gamma_{-v}(+\infty)$) fitted with orthogonal unit vectors directed to the same side as $-v$.*

(2) *Added in proof.* The existence and some properties (cf. our Proposition 7.3 and Theorem 7.5(c)) of limit spheres on manifolds without focal points are obtained in [24] from consideration of the Busemann function.

4. *For any* $v, w \in SH$ *such that* $\gamma_v(+\infty) = \gamma_v(-\infty)$, *the geodesic* $\gamma_w(t)$ *intersects the limit sphere* $L(\pi(v), \gamma_v(+\infty))$ *in a unique point.*

5. *If* $\varphi \in \pi_1(M)$, *then* $\varphi(L(x, p)) = L(\varphi(x), \varphi(p))$.

We consider the limit sphere $L(x, p)$, $x \in H$, $p \in H(\infty)$. For $y \in L(x, p)$ we denote by $\gamma(y, t)$ the geodesic passing through y orthogonal to $L(x, p)$ and parametrized so that $\gamma(y, 0) = 0$ and $\gamma(y, +\infty) = p$.

PROPOSITION 7.1. *Let* $x, y \in H$ *and* $p \in H(\infty)$, *and let the number* t_0 *be such that* $\gamma(x, t_0) \in L(y, p)$. *Then*

$$\rho(L(x, p), L(y, p)) = \rho(x, \gamma(x, t_0)) = |t_0|.$$

The proof follows directly from Definition 7.1 and Theorem 6.4.

For $x \in H$ and $p \in H(\infty)$ we set

$$B^-(x, p) = \bigcup_{y \in L(x,p)} \bigcup_{t>0} \gamma(y, t), \quad B^+(x, p) = \bigcup_{y \in L(x,p)} \bigcup_{t<0} \gamma(y, t).$$

The set $B^-(x, p)$ is called the *interior of the limit sphere* or the *(open) limit ball with center at p, passing through x.* The set $B^+(x, p)$ is called the *exterior of the limit sphere.* From Theorem 7.2 follows

PROPOSITION 7.2. 1. $H = B^-(x, p) \cup B^+(x, p) \cup L(x, p)$.

2. *The sets* $B^-(x, p)$ *and* $B^+(x, p)$ *are open and simply-connected.*

7.3. *Elements of uniqueness.* On manifolds without focal points limit spheres have certain additional properties which we shall study here.

PROPOSITION 7.3. *If the manifold M has no focal points, then a limit sphere is a convex set.*

PROOF. Since a limit sphere is locally a limit of spheres (cf. Lemma 6.1), the assertion follows from Proposition 4.5.

We shall call a line element $v \in SH$ an *element of nonuniqueness* if there exists a vector $w \in SH$ such that

$$\gamma_v(+\infty) = \gamma_w(+\infty), \quad \gamma_v(-\infty) = \gamma_w(-\infty). \tag{7.1}$$

The other $v \in SH$ are called *elements of uniqueness.*

We denote by $\rho_{\mathfrak{S}^-(v)}$ the distance in the submanifold $\mathfrak{S}^-(v)$ induced by the Riemannian metric.

THEOREM 7.3. *Assume that the manifold M has no focal points.*

1. *If* v *is an element of nonuniqueness, then there exists a vector* $w \in SH$, *orthogonal to* v, *and a geodesic segment* $\delta(s)$, $0 \leqslant s \leqslant a$, *in SH such that* $\delta(0) = \pi(v)$, $\dot{\delta}(0) = w$, $\delta(s) \in L(\pi(v), \gamma_v(+\infty))$ *for any* $s \in [0, a]$, *and the union of the geodesics passing through points of* $\delta(s)$ *orthogonal to* $L(\pi(v), \gamma_v(+\infty))$ *is the image under a global geodesic isometric imbedding of the strip* $\{(s, t) \in \mathbf{R}^2\colon\ 0 \leqslant s \leqslant a, -\infty < t < \infty\}$.

2. *If v is an element of uniqueness, then for any $w \in W^-(v)$ the function*

$$\psi(t) = \rho_{\mathfrak{S}^-(f^t(v))}(f^t(v), f^t(w))$$

is monotone increasing, and tends to $+\infty$ as $t \longrightarrow -\infty$.

PROOF. 1. We consider the set

$$\Pi(v) = \bigcup_{-\infty<t<\infty} \bigcup_{z\in SH,\ \langle z,v\rangle=0} \gamma_z(t),$$

which is an $(m-1)$-dimensional submanifold of H of class C^2, dividing H into two parts (actually, $\exp_{\pi(v)}^{-1}\Pi(v) = \mathbf{R}^{m-1}$). Since a limit sphere is a convex set (cf. Proposition 7.3), it is situated on one side of the submanifold $\Pi(v)$, namely, the one to which the vector v is directed.

LEMMA 7.1. *If v is an element of nonuniqueness, then there exists a point $y \in L(\pi(v), \gamma_v(+\infty))$, $y \neq \pi(v) = x$, such that the geodesic segment δ_{xy} in H joining x and y lies on $L(\pi(v), \gamma_v(+\infty))$.*

PROOF. We choose a vector $w \in SH$ such that (7.1) holds. Let y be the point of intersection of $\gamma_w(t)$ and $L(\pi(v), \gamma_v(+\infty))$. Without loss of generality one can assume that $y = \pi(w)$. It suffices to show that $B^-(x, \gamma_v(-\infty)) = B^-(y, \gamma_w(-\infty))$. Let, for example (cf. Proposition 7.2), $B^-(x, \gamma_v(-\infty)) \subset B^-(y, \gamma_w(-\infty))$ (strict inclusion) and $y \notin B^-(x, \gamma_v(-\infty))$. Then $B^-(y, \gamma_v(-\infty)) \cap B^+(x, \gamma_w(+\infty)) \neq \varnothing$. But this is impossible, because by virtue of the choice of y and (7.1), $B^-(y, \gamma_v(-\infty)) = B^-(x, \gamma_w(-\infty))$, and, by virtue of Theorem 7.2, $B^-(x, \gamma_w(-\infty)) \cap B^+(x, \gamma_w(+\infty)) = \varnothing$. Thus, $y \in \Pi(v)$. Hence it follows from Proposition 7.3 that $\delta_{xy} \in L(\pi(v), \gamma_v(+\infty))$. The lemma is proved.

We now draw through the points of the geodesic segment δ_{xy} geodesics orthogonal to $L(\pi(v), \gamma_v(+\infty))$. It follows from Definition 7.1 that these geodesics are asymptotic. We denote the two-dimensional manifold so obtained by P. Since the element $f^t(v)$ is an element of nonuniqueness, repeating the preceding argument we get that the curve $\pi(f^t(Z_{xy}))$ (Z_{xy} is the curve δ_{xy} fitted with unit vectors orthogonal to $L(\pi(v), \gamma_v(+\infty))$) is a geodesic in H and lies in P. Hence from the elementary geometry of geodesic quadrangles it follows that the curvature at any point $z \in P$ in the two-dimensional direction T_zP is equal to zero. Since M has no focal points, by virtue of Proposition 3.17 in [17] any Jacobi field generated by the above construction is parallel. It follows that the geodesics $\gamma(z_1, t)$ and $\gamma(z_2, t)$, where $z_1, z_2 \in \delta_{xy}$, are asymptotes, i.e. $\rho(\gamma(z_1, t), \gamma(z_2, \cdot)) = \rho(\gamma(z_1, \cdot), \gamma(z_2, t)) = \text{const.}$ for any $t \in \mathbf{R}$. Hence the imbedding x: $\mathbf{R} \times [0, a]$ (a is the length of δ_{xy}), $x(t, u) = \gamma(z, t)$, $z = \delta_{xy}(u)$, is the one desired (cf. [22], §5).

2. The proof of monotonicity of the function $\psi(t)$ is analogous to the proof of Theorem 5.1. We fix $t > 0$ and we consider the family of limit spheres

$$\{L(\gamma_v(s-t), \gamma_v(+\infty)), 0 \leqslant s \leqslant t\}.$$

We choose a geodesic segment $\delta_t(u) \in L(\gamma_v(-t), \gamma_v(+\infty))$, $0 \leqslant u \leqslant a$, $\delta_t(0) = \pi(f^{-t}(v))$, $\delta_t(a) = \pi(f^{-t}(w))$. Equipping it with unit vectors orthogonal to the limit sphere, we get a curve $Z_t(u)$ in SH. We consider the variation $r(u, s) = \exp(sZ_t(u))$, $0 \leqslant s \leqslant t$,

$0 \leqslant u \leqslant a$. The points of intersection of $\gamma^u(s) = r(u, s)$ with $L(\gamma_v(s - t), \gamma_v(+\infty))$ form a smooth curve $\delta_s(u)$, $\delta_s(0) = \gamma_v(s - t)$, $\delta_s(a) = \gamma_w(s - t)$. We denote by $l(s)$ the length of this curve. To prove the assertion it suffices to show that $\psi(s - t) \leqslant \psi(-t)$ for any s, $0 \leqslant s \leqslant t$. For this, we shall show that $l(s) \leqslant l(t)$ for any $s \in [0, t]$. We consider the Jacobi field $Y_u(s)$ along the geodesic $\gamma^u(s)$, generated by the variation $r(u, s)$ (cf. (4.4)). Since M has no focal points, it follows from the properties of limit spheres (cf. Theorem 6.1 and Lemma 6.1(3)) and Proposition 4.8 that $Y_u(s)$ satisfies (5.2). Hence, by virtue of (5.3), $l(s_1) \leqslant l(s_2)$ if $0 \leqslant s_1 \leqslant s_2 \leqslant t$. It follows that $\psi(s - t) \leqslant l(s) \leqslant l(t) \leqslant \psi(-t)$.

Now we shall show that $\psi(t) \longrightarrow \infty$ as $t \longrightarrow -\infty$. If the functions $\psi(t)$ were bounded as $t \longrightarrow -\infty$, then from the definition of limit sphere it would follow that $\rho(\gamma_v(-t), \gamma_w(-t)) \leqslant \psi(t) \leqslant$ const. for any $t > 0$, so that $\gamma_v(t)$ and $\gamma_w(t)$ would be asymptotic for $t < 0$. But this contradicts the fact that the line element v is an element of uniqueness. The theorem is proved.

7.4. *Strengthened axiom of uniform visibility.*

DEFINITION 7.2 (cf. [14]–[16] or [22], Axiom 2). A manifold M without conjugate points satisfies the *strengthened axiom of uniform visibility* if it satisfies the axiom of uniform visibility and for any two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ one can find a unique geodesic $\gamma(t)$ such that $\gamma(-\infty) = \gamma_1(-\infty)$ and $\gamma(+\infty) = \gamma_2(+\infty)$ (cf. Proposition 5.3(2)), or in other words, if any two points on the absolute can be joined by a unique geodesic; equivalent definition: *any element $v \in SH$ is an element of uniqueness.*

It follows from Theorem 7.3 that if the manifold M has no focal points and satisfies the strengthened axiom of uniform visibility, then the foliation $\mathfrak{S}^-$ *is expanding as $t \longrightarrow -\infty$* (i.e. for any $v \in SH$ and $w \in \mathfrak{S}^-(v)$ we have $\psi(t) \longrightarrow \infty$ as $t \longrightarrow -\infty$; cf. Theorem 7.3).

THEOREM 7.4. *If the compact two-dimensional manifold M has no focal points and satisfies the strengthened axiom of uniform visibility, then the foliation $\mathfrak{S}^-$ is contracting as $t \longrightarrow +\infty$.*

PROOF. We must show that for any $v \in SH$ and $w \in \mathfrak{S}^-(v)$ the function $\psi(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. From the hypotheses of the theorem, Proposition 7.3 and Theorem 7.3(2) it follows that the limit sphere $L(\pi(v), p)$, $p = \gamma_v(+\infty)$, is strictly convex, and the function $\psi(t)$ is monotone decreasing.

Let $t_n \longrightarrow +\infty$, $v_n = f^{t_n}(v)$, $w_n = f^{t_n}(w)$, $x_n = \pi(v_n)$ and $y_n = \pi(w_n)$. We choose a sequence of isometries φ_n such that the sequence of points $\{\varphi_n(x_n)\}$ lies in a compact domain in H. Since the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ are asymptotic, $\rho(x_n, y_n) \leqslant$ const. for any $n > 0$, so that the sequence of points $\{\varphi_n(y_n)\}$ is situated in a compact domain of H. Hence, without loss of generality, one can assume that $\varphi_n(x_n) \longrightarrow r$, $\varphi_n(y_n) \longrightarrow s$, $d\varphi_n v_n \longrightarrow \bar{v}$, $d\varphi_n w_n \longrightarrow \bar{w}$ and $\varphi_n(p) \longrightarrow q$. Since $\varphi_n(y_n) \in \varphi_n(L(x_n, p)) = L(\varphi_n(x_n), \varphi_n(p))$ (cf. Theorem 7.2(5)), one has $s \in L(r, q)$.

Let us assume that $\psi(t_n) \geqslant a > 0$. Then $\rho(x_n, y_n) \geqslant b > 0$, and hence

$$\rho(\varphi_n(x_n), \varphi_n(y_n)) \geqslant b.$$

Thus $\rho(r, s) \geqslant b$. On the other hand, it follows from the axiom of visibility that $\measuredangle_p(x_n, y_n) = 0$ (cf. [14], Lemma 1.6; $\measuredangle_a(b, c)$ denotes the angle under which the geodesic segment

bc is visible from the point a). Hence from the formula for the sum of the angles of a geodesic triangle with vertices at the points x_n, y_n and p (cf. [4], §3.6, formula (21)) it follows that $\measuredangle_{y_n}(x_n, p) + \measuredangle_{x_n}(y_n, p) \longrightarrow \pi$ as $n \longrightarrow \infty$. Thus, one can assume that

$$\measuredangle_{\varphi_n(y_n)}(\varphi_n(x_n), \varphi_n(p)) \to \alpha \geqslant \frac{\pi}{2}$$

which means $\measuredangle_s(r, q) \geqslant \pi/2$. But this contradicts the strict convexity of the limit sphere $L(s, q)$ and the condition $r \in L(s, q)$, $r \neq s$. The theorem is proved.

In conclusion, we mention two criteria for a manifold without focal points to satisfy the strengthened axiom of uniform visibility. For manifolds of nonpositive curvature these results were obtained in [22] (cf. §5).

THEOREM 7.5. *Assume that the compact Riemannian manifold M has no focal points. Then the following assertions are equivalent.*

1. *M does not satisfy the strengthened axiom of uniform visibility.*
2. *There exists a global geodesic isometric imbedding of the strip* $\{(x, y)\colon 0 \leqslant x \leqslant c, -\infty < y < \infty\}$ *in H for some $c > 0$ (cf. Proposition 5.2(1)).*
3. *There exists a global geodesic isometric imbedding of the rectangle* $\{(x, y)\colon 0 \leqslant x \leqslant c, 0 \leqslant y \leqslant T\}$ *for some $c > 0$ and any $T > 0$.*

PROOF. The equivalence of assertions 1 and 2 follows from assertion 1 of Theorem 7.3 and Definition 7.2. The proof of the equivalence of assertions 2 and 3 is a simple modification of the proof of Lemma 4.2 in [14].

7.5. *The topology of the absolute.* The results presented in this section are generalizations to the case of manifolds satisfying the axiom of asymptoticity of results obtained in [22] (cf. §2) for Hadamard manifolds and in [14] (cf. also Proposition 5.3(1)) for manifolds satisfying the axiom of visibility.

Let $x \in H$, $v \in SH$ and $\pi(v) = x$. We shall call the set $C(v, \epsilon) = \{p \in H \cup H(\infty)\colon \measuredangle_x(\gamma_v(+\infty), p) < \epsilon\}$ *the cone in $H \cup H(\infty)$ with vertex at x, axis v and angle ϵ*; and we shall call the set $T(v, \epsilon, r) = C(v, \epsilon) \setminus \{y \in H\colon \rho(x, y) \leqslant r\}$ *the truncated cone in $H \cup H(\infty)$ with vertex at x, axis v, angle ϵ and radius r.*

THEOREM 7.6. *Assume that the compact manifold M satisfies the axiom of asymptoticity. Then on $H \cup H(\infty)$ one can introduce a topology k satisfying the following conditions:*

1. *The restriction of k to H coincides with the topology in H induced by the Riemannian metric.*
2. *H is an open everywhere dense subset of $H \cup H(\infty)$; the sets $H(\infty)$ and $H \cup H(\infty)$ are compact.*
3. *For any $p \in H(\infty)$ the collection of cones containing p is a local basis for the topology k at p.*
4. *For any $p \in H(\infty)$ and $x \in H$ the collection of truncated cones containing p with vertex at x is a local basis for the topology k at p.*

PROOF. We fix a point $x \in H$ and we consider the map $\varphi_x\colon \bar{B} \longrightarrow H \cup H(\infty)$ (B is the open unit ball in T_xM) defined by

$$\varphi_x(y) = \begin{cases} \exp_x(1 - \|v_y\|)^{-1}, & y \in B, \\ \gamma_{v_y}(+\infty), & y \in \partial B, \end{cases}$$

where v_y is a vector in $T_x M$ with initial point at zero and end at y. By virtue of Proposition 4.4 the map $\varphi_x|B$ is a homeomorphism and $\varphi_x|\partial B$ is bijective. It follows from Proposition 5.1 that $\varphi_x|\partial B$ is a map onto the absolute $H(\infty)$. The topology of the closed ball $\overline{B}$ induces, with the help of the map φ_x, a topology on $H \cup H(\infty)$. We shall show that this topology is independent of the choice of x. Let $y \in H$. It suffices to show that for any closed set $A \subset H \cup H(\infty)$ the set $D = \varphi_y^{-1}(A)$ is also closed. This follows immediately from the following lemma.

LEMMA 7.2. *Let $x, y \in H$, $p, p_n \in H \cup H(\infty)$ and $\varphi_x^{-1}(p_n) \longrightarrow \varphi_x^{-1}(p)$. Then $\varphi_{y_n}^{-1}(p) \longrightarrow \varphi_y^{-1}(p)$.*

PROOF. If $p \in H$ (so that $p_n \in H$ for all sufficiently large n), then the assertion is obvious. Let $p \in H(\infty)$. We consider the following two cases:

1. $p_n \in H$. We consider the geodesics $\gamma_n = \gamma_{xp_n}$ and $\sigma_n = \sigma_{yp_n}$, and also geodesics γ and σ such that $\gamma(0) = x$, $\gamma(+\infty) = p$, $\sigma(0) = y$ and $\sigma(+\infty) = p$ (cf. Proposition 5.1). We set $v_n = \dot\gamma_n(0)$, $w_n = \dot\sigma_n(0)$, $v = \dot\gamma(0)$ and $w = \dot\sigma(0)$. Since $\varphi_x^{-1}(p_n) \longrightarrow \varphi_x^{-1}(p)$, one has $v_n \longrightarrow v$, and hence, by virtue of the axiom of asymptoticity, $w_n \longrightarrow w$.

2. $p_n \in H(\infty)$. By virtue of Proposition 5.1, there exist geodesics γ_n, σ_n, γ and σ such that $\gamma_n(0) = \gamma(0) = x$, $\sigma_n(0) = \sigma(0) = y$, $\gamma_n(+\infty) = \sigma_n(+\infty) = p_n$ and $\gamma(+\infty) = \sigma(+\infty) = p$. We set $v_n = \dot\gamma_n(0)$, $w_n = \dot\sigma_n(0)$, $v = \dot\gamma(0)$ and $w = \dot\sigma(0)$. We have $w_n \in \mathfrak{S}^{-0}(v_n)$. Since $v_n \longrightarrow v$, by virtue of the continuity of the foliations $\mathfrak{S}^{-0}$ (cf. Theorem 6.4) any limit point $\widetilde{w}$ of the sequence w_n lies in $\mathfrak{S}^{-0}(v)$. Hence $p = \gamma_v(+\infty) = \gamma_{\widetilde{w}}(+\infty)$, whence it follows that $\widetilde{w} = w$ (cf. Proposition 5.1) and $w_n \longrightarrow w$. The lemma is proved.

Assertions 1–4 follow immediately from our method of introducing the topology on the set $H \cup H(\infty)$. The theorem is proved.

PROPOSITION 7.4. *Let $v_n, v \in SH$, $v_n \longrightarrow v$ and $p = \gamma_v(+\infty)$. For any truncated cone $T(v, \epsilon, r)$ with vertex at the point $x = \pi(v)$ and axis v, there exist $N > 0$ and $T > 0$ such that for any $n \geqslant N$ and $t \geqslant T$ the point $\gamma_{v_n}(t)$ lies in $T(v, \epsilon, r)$.*

PROOF. Let us assume the contrary. Then for some $\epsilon > 0$ and $r > 0$ there exists a sequence of numbers $t_n \longrightarrow \infty$ such that $\gamma_{v_n}(t_n)$ does not lie in $T(v, \epsilon, r)$. We consider the sequence of geodesics σ_n joining the points x and $\gamma_{v_n}(t_n)$, and let σ be a limit geodesic (which exists, since the sequence of vectors $w_n = \dot\sigma_n(0)$ is compact). By virtue of the axiom of asymptoticity (applied in relation to x, $x_n = x$, $w_n = \dot\sigma_n(0)$ and $w = \dot\sigma(0)$) we get that the geodesics σ and γ are asymptotic, so that $w = v$ and $w_n \longrightarrow v$. Hence for sufficiently large n and t we get that $\varphi_x^{-1}(\sigma_n(t)) \subset \varphi_x^{-1}(T(v, \epsilon, r))$. But this is impossible, since $\gamma_{v_n}(t_n) \notin T(v, \epsilon, r)$. This contradiction proves the required assertion.

Directly from the results obtained we get

COROLLARY 7.1. *If $v_n, v \in SH$ and $v_n \longrightarrow v$, then $\gamma_{v_n}(+\infty) \longrightarrow \gamma_v(+\infty)$ as $n \longrightarrow \infty$.*

COROLLARY 7.2. *Let $x \in H$, $x_n, y_n, y \in H \cup H(\infty)$, $x_n \longrightarrow x$, $y_n \longrightarrow y$, $v_n = \dot\gamma_{x_n y_n}(0)$ and $v = \dot\gamma_{xy}(0)$. Then $v_n \longrightarrow v$ as $n \longrightarrow \infty$.*

COROLLARY 7.3. *The function* ψ_t: $SH \longrightarrow H \cup H(\infty)$ *defined by* $\psi_t(v) = \gamma_v(t)$ *is continuous for* $0 \leqslant t \leqslant \infty$.

COROLLARY 7.4. *The function* φ: $H \cup H(\infty) \longrightarrow SH$ *defined by* $\varphi(x, y) = \dot\gamma_{xy}(0)$ *is continuous.*

COROLLARY 7.5. *For any* $x \in H$ *and any geodesic* γ, $\lim_{t\to\infty}\varphi_x^{-1}(\gamma(t))$ *exists and is equal to* $\varphi_x^{-1}(\gamma(+\infty))$.

PART III. METRIC PROPERTIES OF GEODESIC FLOWS

§8. The condition of negativity of the characteristic exponents for a geodesic flow

It is well known (cf. [1] and [2]) that the geodesic flow f^t has a smooth measure, which we shall denote by μ. Let $v \in SM$. We consider some orthonormal system of parallel vector fields along $\gamma_v(t)$, and a vector w orthogonal to v. We set

$$w(t) = D^-(t)\, w\, (\| D^-(t)\, w\|)^{-1},$$

$$K_{v,w}(t) = \langle R_{\dot\gamma_v(t) w(t)} \dot\gamma_v(t),\ w(t)\rangle,$$

where $D^-(t)$ is the solution of (4.5) constructed in Proposition 4.6. We consider the set

$$\underset{\sim}{\Lambda} = \left\{ v \in SM:\ \text{for any } w \in SM, \text{ orthogonal to } v, \right.$$

$$\left. \overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t K_{v,w}(s)\, ds < 0 \right\}.$$

The set $\underset{\sim}{\Lambda}$ is measurable and invariant with respect to the flow f^t. Let χ^+ be the characteristic exponent of the dynamical system f^t (cf. §1.1).

THEOREM 8.1 (cf. [11], Theorem 10.5). *Assume that the Riemannian manifold M has no focal points. Then* $\chi^+(v, \xi) < 0$ ($\chi^+(v, \xi) > 0$) *if* $v \in \underset{\sim}{\Lambda}$ *and* $\xi \in X^-(v)$ ($\xi \in X^+(v)$).

In the two-dimensional case this theorem admits a converse.

THEOREM 8.2. *Assume that the Riemannian manifold M has no conjugate points and* $\dim M = 2$. *Let* $v \in SM$ *be such that* $\chi^+(v, \xi) < 0$ *for some (and consequently any)* $\xi \in X^-(v)$. *Then* $v \in \underset{\sim}{\Lambda}$.

PROOF. To prove the theorem we shall need

LEMMA 8.1. *Let* ψ: $\mathbf{R}^+ \longrightarrow \mathbf{R}$ *be a continuous function, where*

$$\sup_{t\geqslant 0} |\psi(t)| = a < \infty \quad \textit{and} \quad \overline{\lim} \frac{1}{t} \int_0^t \psi(s)\, ds = b < 0.$$

Then

$$\underline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \psi^2(s)\, ds = c > 0.$$

PROOF OF THE LEMMA. We choose a number $\epsilon \in (0, \min(a/2, -b/4))$ and consider the set $X = \{s \in [0, t]: |\psi(s)| \geqslant \epsilon\}$. There exists a $T > 0$ such that for any $t \geqslant T$

$$-\frac{b}{2} < -\frac{1}{t}\int_0^t \psi(s)\,ds = \frac{1}{t}\left[\int_X (-\psi(s))\,ds + \int_{[0,t]\setminus X} (-\psi(s))\,ds\right]$$

$$\leqslant \frac{1}{t} a \operatorname{mes}(X) + \frac{1}{t}\varepsilon\,(t - \operatorname{mes}(X)).$$

(Here mes denotes Lebesgue measure on the line.) Hence

$$\operatorname{mes}(X) \geqslant \left(-\frac{b}{2} - \varepsilon\right)(a - \varepsilon)^{-1} t \geqslant -\frac{bt}{2a}.$$

Thus

$$c \geqslant \varliminf_{t\to\infty} \frac{1}{t}\int_X \psi^2(s)\,ds \geqslant -\varepsilon^2 \frac{b}{2a}.$$

The lemma is proved.

We proceed to the proof of the theorem. We fix a vector w, $\|w\| = 1$, orthogonal to v, and we choose a vector $\xi \in X^-(v)$, $\|\xi\| = 1$, for which $d\pi\xi = w$. By means of the usual substitution the scalar Jacobi equation reduces to the scalar Ricatti equation (cf. [11], §10.5):

$$\dot z(t) + z^2(t) + K_{v,w}(t) = 0, \tag{8.1}$$

while according to Proposition 4.7(4)

$$\sup_{t\geqslant 0} |z(t)| \leqslant a < \infty,$$

and by the hypotheses of the theorem and (4.6)

$$\varlimsup_{t\to\infty} \frac{1}{t}\int_0^t z(s)\,ds = \varlimsup_{t\to\infty} \frac{1}{t}\ln\|Y_\xi(t)\| = \varlimsup_{t\to\infty}\frac{1}{t}\ln\|d\pi\circ df^t\xi\|$$

$$= \varlimsup_{t\to\infty}\frac{1}{t}\ln\|df^t\xi\| = \chi^+(v, \xi) < 0.$$

Hence, integrating (8.1) on the segment $[0, t]$ and using Lemma 8.1, we get that

$$\varlimsup_{t\to\infty}\frac{1}{t}\int_0^t K_{v,w}(s)\,ds = -\varliminf_{t\to\infty}\frac{1}{t}\int_0^t z^2(s)\,ds < 0.$$

The theorem is proved.

COROLLARY 8.1. *If the compact Riemannian manifold M has no focal points and* $\mu(\underset{\sim}{\Lambda}) > 0$, *then* $\mu(\Lambda) > 0$, *where* Λ *is the set defined in* §1.1.

§9. Ergodic properties of geodesic flows on manifolds without focal points

9.1. THEOREM 9.1. *Let M be a compact Riemannian manifold without focal points satisfying the axiom of uniform visibility. Then either* $\mu(\Lambda) = 0$, *or* $\mu(\Lambda) = 1$. *In the latter case the geodesic flow is isomorphic to a Bernoulli flow* (*the set* Λ *is defined in* §1.1).

PROOF. Let us assume that $\mu(\Lambda) > 0$. We consider the local stable and unstable manifolds $V^-(v)$ and $V^+(v)$, $v \in \widetilde{\Lambda}$. We denote by κ: $H \longrightarrow M$ the covering map. In what follows, a small circle over the notation for vectors, sets, etc. indicates that they are considered in the Riemannian universal covering H or in SH. Thus $\kappa(\mathring{v}) = v$, $\kappa(\mathring{V}^-(v)) = V^-(v)$, $\kappa(\mathring{\Lambda}) = \Lambda$.

LEMMA 9.1. *$\mathring{V}^-(\mathring{v}) \subset \mathring{\mathfrak{S}}^-(\mathring{v})$, $\mathring{V}^+(\mathring{v}) \subset \mathring{\mathfrak{S}}^+(\mathring{v})$ for any $\mathring{v} \in \mathring{\widetilde{\Lambda}}$.*

PROOF. Let $\mathring{w} \in \mathring{V}^-(\mathring{v})$. By virtue of (1.1)

$$\mathring{\rho}(\pi(f^t(\mathring{v})), \pi(f^t(\mathring{w}))) \to 0 \quad \text{as} \quad t \to \infty. \tag{9.1}$$

Hence, in particular, $\gamma_{\mathring{v}}(+\infty) = \gamma_{\mathring{w}}(+\infty)$. Let us assume that $\mathring{w} \notin \mathring{\mathfrak{S}}^-(\mathring{v})$. We consider the limit sphere $L(\pi(\mathring{v}), \gamma_{\mathring{v}}(+\infty))$. Let $\mathring{z}$ be the point of intersection of $\gamma_{\mathring{w}}(t)$ and $L(\pi(\mathring{v}), \gamma_{\mathring{v}}(+\infty))$. Then by virtue of Proposition 7.1

$$\mathring{\rho}(\pi(f^t(\mathring{v})), \pi(f^t(\mathring{w}))) + \mathring{\rho}(\mathring{z}, \pi(\mathring{w})) > 0,$$

which contradicts (9.1). The lemma is proved.

LEMMA 9.2. *For any $\delta > 0$ and $\alpha > 0$ one can find a number $T(\delta, \alpha)$ such that for any $\mathring{v} \in \mathring{\widetilde{\Lambda}}^l_{m-1,s}$ and $\mathring{w}_1, \mathring{w}_2 \in \mathring{V}^-(\mathring{v})$ such that $\rho_{\mathring{\mathfrak{S}}^-(\mathring{v})}(\mathring{w}_1, \mathring{w}_2) = \alpha$, and any $t \geqslant T(\delta, \alpha)$, one has the inequality*

$$\mathring{\rho}_{\mathring{\mathfrak{S}}^-(f^{-t}(\mathring{v}))}(f^{-t}(\mathring{w}_1), f^{-t}(\mathring{w}_2)) > \delta.$$

PROOF. If this is not so, then there exist $\delta > 0$, $\alpha > 0$ and sequences $\mathring{v}_n \in \mathring{\widetilde{\Lambda}}^l_{m-1,s}$, $\mathring{w}^n_1, \mathring{w}^n_2 \in \mathring{V}^-(\mathring{v}_n)$ and $k_n \longrightarrow \infty$ such that $\mathring{\rho}_{\mathring{\mathfrak{S}}^-(\mathring{v}_n)}(\mathring{w}^n_1, \mathring{w}^n_2) = \alpha$ and

$$\mathring{\rho}_{\mathring{\mathfrak{S}}^-(f^{-k_n}(\mathring{v}_n))}(f^{-k_n}(\mathring{w}^n_1), f^{-k_n}(\mathring{w}^n_2)) < \delta. \tag{9.2}$$

Since the set $\widetilde{\widetilde{\Lambda}}^l_{m-1,s}$ is compact (cf. Proposition 1.2), passing to a subsequence we get that $v_n \longrightarrow v \in \widetilde{\widetilde{\Lambda}}^l_{m-1,s}$. We choose a sequence of isometries $\varphi_n \in \pi_1(M)$ of the space H such that $d\varphi_n \mathring{v}_n \longrightarrow \mathring{v}$. Again passing to a subsequence, we can assume by virtue of Proposition 6.1(2) that $d\varphi_n \mathring{w}^n_1 \longrightarrow \mathring{w}_1$ and $d\varphi_n \mathring{w}^n_2 \longrightarrow \mathring{w}_2$, where $\mathring{w}_1, \mathring{w}_2 \in \mathring{V}^-(\mathring{v})$ and $\mathring{\rho}_{\mathring{\mathfrak{S}}^-(\mathring{v})}(\mathring{w}_1, \mathring{w}_2) = \alpha$. According to Theorem 7.3(1) the line element $\mathring{v}$ is an element of uniqueness. Moreover, by virtue of Proposition 2.3.1 of [10], $\chi^+(\mathring{w}, \mathring{\xi}) < 0$ for any $\mathring{w} \in \mathring{V}^-(\mathring{v})$ and $\mathring{\xi} \in T_{\mathring{w}}\mathring{V}^-(\mathring{v})$. Using Theorem 7.3(1) again, we conclude that any line element $\mathring{w} \in \mathring{V}^-(\mathring{v})$ is an element of uniqueness. Hence and from Theorem 7.3(2) follows the existence of a number $T(\delta, \alpha) > 0$ such that for any $t \geqslant T(\delta, \alpha)$

$$\mathring{\rho}_{\mathring{\mathfrak{S}}^-(f^{-t}(\mathring{v}))}(f^{-t}(\mathring{w}_1), f^{-t}(\mathring{w}_2)) > 2\delta.$$

Hence for sufficiently large n

$$a = \mathring{\rho}_{\mathring{\mathfrak{S}}^-(f^{-T(\delta,\alpha)}(\mathring{v}_n))}(f^{-T(\delta,\alpha)}(\mathring{w}^n_1), f^{-T(\delta,\alpha)}(\mathring{w}^n_2)) > \frac{3}{2}\delta.$$

Using the monotonicity of $\mathring{\rho}_{\mathring{\mathfrak{S}}^-(f^{-t}(\mathring{v}_n))}(f^{-t}(\mathring{w}^n_1), f^{-t}(\mathring{w}^n_2))$ on elements of uniqueness (cf. Theorem 7.3(2)), we get that for sufficiently large $k_n \geqslant T(\delta, \alpha)$

$$\mathring{\rho}_{\mathring{\mathfrak{S}}^{-}(f^{-k_n}(\mathring{v}_n))}(f^{-k_n}(\mathring{w}_1^n),\ f^{-k_n}(\mathring{w}_2^n)) \geqslant a > \frac{3}{2}\delta.$$

But this contradicts (9.2). The lemma is proved.

From Lemma 9.2 immediately follows

LEMMA 9.3. *For any $\delta > 0$ there exists a measurable function $T(v)$ such that for any $v \in \widetilde{\Lambda}$ and $t \geqslant T(v)$*

$$f^{-t}(V^{-}(v)) \supset B^{-}(f^{-t}(v), \delta).$$

Since for almost any $v \in \widetilde{\Lambda}^l_{m-1,s}$ the semitrajectory $\{f^t(v)\}$, $t \geqslant 0$, lands in the set $\widetilde{\Lambda}^l_{m-1,s}$ infinitely many times, from Lemma 9.3 and Theorem 1.3 we get

LEMMA 9.4. $\mathfrak{S}^{-}(v) = W^{-}(v)$ *for almost all* $v \in \widetilde{\Lambda}$.

We proceed to the proof of the theorem. Since the manifold M satisfies the axiom of uniform visibility, by virtue of Theorem 5.2 the flow f^t is topologically transitive. Thus from Lemma 9.4 it follows that f^t satisfies the hypotheses of Theorem 9.5 in [11]. From this theorem it follows that f^t is ergodic on Λ. We shall show that $\Lambda = SM$ (mod 0).

Let $\mathring{v} \in \mathring{\widetilde{\Lambda}}^l_{m-1,s}$. We consider a ball $\mathring{Q}$ on $L(\pi(\mathring{v}), \gamma_{\mathring{v}}(+\infty))$ with center at $\pi(\mathring{v})$. We identify $H \cup H(\infty)$ with the closed unit ball in $T_{\pi(\mathring{v})}H$. We consider the set $\mathring{A}_t(\mathring{v}) \subset H$, $t > 0$, bounded by $\mathring{Q}$, segments of the geodesics $\gamma_{\mathring{w}}(s)$, $\pi(\mathring{w}) \in \partial\mathring{Q}$, $-\mathring{w} \in \mathring{\mathfrak{S}}^{-}(\mathring{v})$, $0 \leqslant s \leqslant t$, and the domain $\mathring{Q}_t \subset L(\gamma_{\mathring{v}}(t), \gamma_{\mathring{v}}(+\infty))$, whose boundary is the set $\{\gamma_{\mathring{w}}(t), \pi(\mathring{w}) \in Q, -\mathring{w} \in \mathfrak{S}^{-}(\mathring{v})\}$. If $\mathring{Q}$ is chosen sufficiently small, so that $\mathring{Q} \subset \pi(\mathring{V}^{-}(\mathring{v}))$, then, as follows from Theorem 7.3(2), any line element $\mathring{w}$, $-\mathring{w} \in \mathring{\mathfrak{S}}^{-}(\mathring{v})$, for which $\pi(\mathring{w}) \in \mathring{Q}$, is an element of uniqueness. Hence any two geodesics $\gamma_{w_1}(t)$ and $\gamma_{w_2}(t)$ in H diverge if $\pi(\mathring{w}_i) \in \mathring{Q}$ and $-\mathring{w}_i \in \mathring{\mathfrak{S}}^{-}(\mathring{v})$, $i = 1, 2$.

We consider the domain $Q_\infty \subset H(\infty)$ whose boundary is the set $\{\gamma_{\mathring{w}}(+\infty), \pi(\mathring{w}) \in \partial\mathring{Q}, -\mathring{w} \in \mathring{\mathfrak{S}}^{-}(\mathring{v})\}$. We put

$$\mathring{B}_t(\mathring{v}) = \{\mathring{w} \in SH : \pi(\mathring{w}) \in \mathring{A}_t(\mathring{v}),\ \gamma_{\mathring{w}}(+\infty) \in Q_\infty\}.$$

We shall show that for almost all $\mathring{v} \in \mathring{\Lambda}$

$$\mathring{B}_\infty(\mathring{v}) \subset \mathring{\Lambda} \pmod 0. \tag{9.3}$$

We fix $t \in [0, \infty]$ and put

$$\mathring{C}_t(\mathring{v}) = \bigcup_{-t \leqslant s \leqslant t}\ \bigcup_{\mathring{w} \in \mathring{V}^{+}(-\mathring{v})} f^s(\mathring{w}), \quad \mathring{D}_t(\mathring{v}) = \bigcup_{\mathring{w} \in \mathring{C}_t(\mathring{v})} \mathring{\mathfrak{S}}^{-}(\mathring{w}).$$

It is easy to see that $\mathring{B}_\infty(\mathring{v}) \subset \mathring{D}_\infty(\mathring{v})$ for any $\mathring{v} \in \mathring{\widetilde{\Lambda}}$. But, by virtue of Lemma 9.4, $\mathring{D}_t(\mathring{v}) \subset \mathring{\widetilde{\Lambda}}$ (mod 0) for any $t > 0$, and hence $\mathring{D}_\infty(\mathring{v}) \subset \mathring{\widetilde{\Lambda}}$ (mod 0). Hence $\mathring{B}_\infty(\mathring{v}) \subset \mathring{\widetilde{\Lambda}}$ (mod 0).

Let $\mathring{U} \subset H$ and $V \subset H(\infty)$ be open balls. We consider the open set $\mathring{R} = \{\mathring{w} \in SH: \pi(\mathring{w}) \in \mathring{U}, \gamma_{\mathring{w}}(+\infty) \in V\}$. It follows from Proposition 5.3(3) that for any $\mathring{v} \in SH$ there exists an isometry φ of the space H such that $\mathring{\varphi}(R) \subset \mathring{B}_\infty(\mathring{v})$. Hence on the basis of Lemma 9.5 and the inclusion (9.3) we have $\mathring{R} \subset \mathring{\widetilde{\Lambda}}$ (mod 0). Thus we have proved that $\Lambda = SM$ (mod 0) and the geodesic flow in SM is ergodic. We shall show that it is isomorphic to a Bernoulli flow.

We note that from the condition $\mu(\Lambda) > 0$ and the Gauss-Bonnet formula (cf. [4], §7.8) it follows that the manifold M is not homeomorphic to a two-dimensional torus. Hence from a theorem of V. I. Arnol'd (cf. [1], §23) it follows that the flow f^t has no continuous eigenfunctions.

LEMMA 9.5 (cf. [1], Lemma 20.2). *If φ is a measurable eigenfunction of the flow f^t, then there exists a continuous eigenfunction ψ such that $\varphi = \psi$ (mod 0).*

PROOF. For some λ and almost all $v \in SM$ we have $\varphi(f^t(v)) = e^{2\pi i\lambda t}\varphi(v)$. Setting $t_0 = 1/\lambda$, we get that φ is invariant mod 0 with respect to the diffeomorphism f^{t_0}. Since $\Lambda = SM$ (mod 0), for almost all $v \in SM$ and $w \in \mathfrak{S}^-(v)$ we have

$$\rho_{\mathfrak{S}^-(f^{t_0 n}(v))}(f^{t_0 n}(v), f^{t_0 n}(w)) \to 0$$

as $n \longrightarrow \infty$. Now it is easy to show by the method of Hopf (cf. [2], Theorem 4.4, and [11], Theorem 6.1) that the function φ is constant mod 0 on stable horospheres; analogously one can show that φ is constant mod 0 on unstable horospheres. The rest of the proof of the lemma proceeds just as in the case of U-flows (cf. [1], §20, proof of Theorem 6).

Thus, we have proved that the flow f^t has no eigenfunctions in SM. Hence the assertion to be proved follows from Theorems 2.1 and 3.1.

In the two-dimensional case Theorem 9.1 can be strengthened.

THEOREM 9.2. *If the compact two-dimensional manifold M of genus greater than 1 has no conjugate points, then either $\mu(\Lambda) = 0$, or $\mu(\Lambda) = 1$. In the latter case the geodesic flow is isomorphic to a Bernoulli flow and $\Lambda = SM$ (mod 0) (the set Λ is defined in §1.1).*

PROOF. Since $\dim M = 2$, it follows from Theorems 8.1 and 8.2 that for $v \in \Lambda$ the conditions $\xi \in X^-(v)$ and $\chi^+(v, \xi) < 0$ are equivalent and $\underset{\sim}{\Lambda} \subset \Lambda$. By virtue of Theorem 1.1, the set Λ decomposes into a countable number of ergodic components of positive measure. Since $\dim M = 2$, Lemma 9.1 shows that the hypotheses of Theorem 9.5 of [11] are satisfied, so that each ergodic component is mod 0 an open set, and $\mathfrak{S}^-(v) = W^-(v)$ for almost all $v \in \Lambda$. Since the genus of M is at least 2, M satisfies the axiom of uniform visibility (cf. Proposition 5.2), and consequently the flow f^t is topologically transitive (cf. Theorem 5.2). Hence f^t is ergodic on Λ (cf. [11], §9).

We shall show that $\Lambda = SM$ (mod 0). From what was said above and the results of [11] (cf. §§7 and 9) it follows that, for almost all $\mathring{v} \in \mathring{\Lambda}$, almost all line elements $\mathring{w} \in \mathring{\mathfrak{S}}^-(\mathring{v})$ lie in the set $\mathring{\Lambda}$ (we are using the notation of the previous theorem). We fix a line element $\mathring{v} \in \mathring{\Lambda}$ with the indicated property. Let Q_∞ be a segment on the absolute $H(\infty)$ containing the point $\gamma_{\mathring{v}}(-\infty)$. We join the ends of this segment with the point $\gamma_{\mathring{v}}(+\infty)$ by geodesics (cf. Proposition 5.3(2)). Let $\mathring{x}$ and $\mathring{y}$ be the points of intersection of these geodesics and the limit sphere $L(\pi(\mathring{v}), \gamma_{\mathring{v}}(+\infty))$. From what was said above it follows that almost all line elements $\mathring{w}$ such that $-\mathring{w} \in \mathring{\mathfrak{S}}^-(\mathring{v})$ and $\pi(\mathring{w}) \in [\mathring{x}, \mathring{y}]$ lie in $\mathring{\Lambda}$. It remains to repeat the concluding part of the proof of Theorem 9.1. The theorem is proved.

In [11] (cf. Theorem 10.6) it is proved that if M is a compact two-dimensional manifold of genus greater than 1, then $\mu(\underset{\sim}{\Lambda}) > 0$. Hence from Theorem 9.2 and Corollary 8.1 follows

THEOREM 9.3. (cf. [11], Theorem 10.7). *If a compact two-dimensional manifold of genus greater than* 1 *has no focal points, then the geodesic flow is isomorphic to a Bernoulli flow.*

From the formula for the entropy of a flow (cf. [11], §5) and Theorem 9.2 follows

THEOREM 9.4. *If a geodesic flow on a compact two-dimensional manifold of genus greater than* 1 *without conjugate points has positive entropy, then it is isomorphic to a Bernoulli flow.*

We note that a geodesic flow on a two-dimensional torus with Riemannian metric without focal points has zero entropy (since this metric is equivalent to the standard one).

REMARK 9.1. The absence of focal points is used in the proof of Theorem 9.1 to assure that from the condition $\mathring{v} \in \mathring{\Lambda}$ one can derive with the help of Theorem 7.3(1) that $\mathring{v}$ is an element of uniqueness. However, if a priori it is known that any element $\mathring{v} \in SH$ is an element of uniqueness, the condition of absence of focal points can be discarded. Thus we arrive at the following theorem.

THEOREM 9.5. *If the manifold M is compact, has no conjugate points and satisfies the strengthened axiom of uniform visibility, than either* $\mu(\Lambda) = 0$, *or* $\mu(\Lambda) = 1$. *In the latter case the geodesic flow is isomorphic to a Bernoulli flow.*

REMARK 9.2. Modifying the proof of Lemma 9.5 and using Theorem 1.2(5), it is easy to prove the following assertion.

THEOREM 9.6. *Let f^t: $M \longrightarrow M$ be a flow of class C^2, preserving a measure ν equivalent to the Riemannian volume. Assume that* $\widetilde{\Lambda}_{k,s} = M$ (mod 0) *for some k and s (cf.* §1.3) *and there exist continuous foliations $\widetilde{W}^-$ and $\widetilde{W}^+$ of M such that* $\widetilde{W}^-|\Lambda_{k,s} = W^-|\Lambda_{k,s}$ *and* $\widetilde{W}^+|\Lambda_{k,s} = W^+|\Lambda_{k,s}$ *(cf.* §1.9). *Then any measurable eigenfunction of the flow f^t* mod 0 *is continuous.*

This theorem allows one to supplement Theorems 2.1 and 3.1 in the following way.

THEOREM 9.7. *Under the hypotheses of Theorem* 9.6 *either the flow f^t is isomorphic to a Bernoulli flow, or it can be represented as the suspension of a diffeomorphism of a compact manifold having almost everywhere nonzero Ljapunov characteristic exponents.*

PROOF. If the flow f^t has continuous spectrum, then it is isomorphic to a Bernoulli flow (cf. Theorem 3.1). If it has an eigenfunction, then it is mod 0 continuous (cf. Theorem 9.6). Considering its level lines and arguing just as in the case of U-flows (cf. [1], §20, the proof of Theorem 12), it is easy to show that these level lines are compact manifolds, and the succession diffeomorphism on each of these manifolds has almost everywhere nonzero Ljapunov exponents. The theorem is proved.

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BIBLIOGRAPHY

1. D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov 90 (1967); English transl., Proc. Steklov Inst. Math. **90** (1967).

2. D. V. Anosov and Ja. G. Sinaĭ, *Certain smooth ergodic systems*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 107–172; English transl. in Russian Math. Surveys **22** (1967).

3. B. F. Bylov et al., *Theory of Ljapunov exponents and its application to problems of stability*, "Nauka", Moscow, 1966. (Russian)

4. D. Gromoll, W. Klingenberg and W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math., Vol. 55, Springer-Verlag, Berlin and New York, 1968.

5. A. Kramli, *Geodesic flows on compact Riemannian surfaces without focal points*, Studia Sci. Math. Hungar. **8** (1973), 59–78. (Russian)

6. J. Milnor, *Morse theory*, Ann. of Math. Studies, no. 51, Princeton Univ. Press, Princeton, N.J., 1963.

7. Zbigniew Nitecki, *Differentiable dynamics*, M. I. T. Press, Cambridge, Mass., 1971.

8. V. I. Oseledec, *A multiplicative ergodic theorem*, Trudy Moskov. Mat. Obšč. **19** (1968), 179–210; English transl. in Trans. Moscow Math. Soc. **19** (1968).

9. Ja. B. Pesin, *Ljapunov characteristic exponents and ergodic properties of smooth dynamical systems with an invariant measure*, Dokl. Akad. Nauk SSSR **226** (1976), 774–777; English transl. in Soviet Math. Dokl. **17** (1976).

10. ———, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 1332–1379; English transl. in Math. USSR Izv. **10** (1976).

11. ———, *Characteristic Ljapunov exponents and smooth ergodic theory*, Uspehi Mat. Nauk **32** (1977), no. 4 (196), 55–112; English transl. in Russian Math. Surveys **32** (1977).

12. ———, *A description of the π-partition of a diffeomorphism with an invariant measure*, Mat. Zametki **22** (1977), 29–44; English transl. in Math. Notes **22** (1977).

13. V. A. Rohlin, *Lectures on the entropy theory of transformations with invariant measure*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 3–56; English transl. in Russian Math. Surveys **22** (1967).

14. Patrick Eberlein, *Geodesic flow in certain manifolds without conjugate points*, Trans. Amer. Math. Soc. **167** (1972), 151–170.

15. ———, *Geodesic flows on negatively curved manifolds*. I, Ann. of Math. (2) **95** (1972), 492–510.

16. ———, *Geodesic flows on negatively curved manifolds*. II, Trans. Amer. Math. Soc. **178** (1973), 57–82.

17. ———, *When is a geodesic flow of Anosov type*? I, J. Differential Geometry **8** (1973), 437–463.

18. ———, *When is a geodesic flow of Anosov type*? II, J. Differential Geometry **8** (1973), 565–577.

19. Donald S. Ornstein and Benjamin Weiss, *Geodesic flows are Bernoullian*, Israel J. Math. **14** (1973), 184–198.

20. Daniel Rudolph, *A two-valued step coding for ergodic flows*, Math. Z. **150** (1976), 201–220.

21. E. Heintze and H. C. Im Hof, *On the geometry of horospheres*, Preprint, Bonn, 1975; to appear in J. Differential Geometry.

22. P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109.

23. Eberhard Hopf, *Closed surfaces without conjugate points*, Proc. Nat. Acad. Sci. USA **34** (1948), 47–51.

24. Jost-Hinrich Eschenburg, *Horospheres and the stable part of the geodesic flow*, Math. Z. **153** (1977), 237–251.

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