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Existence and Genericity Problems for Dynamical Systems with Nonzero Lyapunov Exponents

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Abstract—This is a survey-type article whose goal is to review some recent results on existence of hyperbolic dynamical systems with discrete time on compact smooth manifolds and on coexistence of hyperbolic and non-hyperbolic behavior. It also discusses two approaches to the study of genericity of systems with nonzero Lyapunov exponents.

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1. INTRODUCTION

Deterministic chaos — a controversial term coined by Yorke — means the appearance of complicated "chaotic" motions in purely deterministic dynamical systems. It recognizes one of the greatest discoveries in the recent theory of smooth dynamical systems. It is now well-understood that chaotic behavior is caused by the instability of trajectories that forces them to separate. If the phase space of the system is compact, the trajectories mix together because there is not enough room to get them separated. This is one of the main reasons why systems with unstable trajectories on compact phase spaces exhibit chaotic behavior.

Intuitively, instability means that the behavior of trajectories that start in a small neighborhood of a given one resembles that of the trajectories in a small neighborhood of a hyperbolic fixed point. This can be described by saying that the tangent space along the orbit $f^n(x)$ admits an invariant splitting

$$T_{f^n(x)}M = E^s(f^n(x)) \oplus E^u(f^n(x))$$

with contraction along the *stable subspace* E^s and expansion along the *unstable subspace* E^u. This is the case of *complete* hyperbolicity.

A more general version of *partial* hyperbolicity requires that there exists an invariant splitting of the tangent space along the orbit

$$T_{f^n(x)}M = E^s(f^n(x)) \oplus E^c(f^n(x)) \oplus E^u(f^n(x))$$

with contraction along E^s, expansion along E^u and contraction and/or expansion with smaller rates along the *central subspace* E^c (this allows the case when there is no contraction and/or expansion along E^c, e.g., $df|E^c(x)$ is an isometry for every $x \in M$).

Further, one should distinguish *uniform* and *nonuniform* hyperbolicity. In the former case the asymptotic rates of contraction and expansion along invariant subspaces are uniformly bounded in x on an invariant compact subset in the phase space (in particular on the whole phase space). In the latter case the set of hyperbolic trajectories has full measure with respect to some f-invariant measure ν and the asymptotic rates of contraction and expansion along invariant subspaces depend on x. Thus nonuniformly hyperbolicity is a property of the system as well as of an invariant measure.

In studying chaotic systems one should separate the following two cases:

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1. Conservative systems: a diffeomorphism f of a compact smooth Riemannian manifold M preserving volume or a smooth measure ν (i.e., a measure, which is equivalent to volume);

2. Dissipative systems: a diffeomorphism f of a compact smooth Riemannian manifold M possessing a *trapping region*, i.e., an open set U such that $\overline{f(U)} \subset U$; the set $\Lambda = \bigcap_{n \geqslant 0} f^n(U)$ is a said to be a *(strange) attractor* for f.

In this paper we will only deal with the case of conservative systems with discrete time although many of the results presented here can be extended to systems with continuous time as well as to some dissipative systems.

In what follows we shall formally introduce all the known forms of hyperbolicity (complete and partial as well as uniform and nonuniform), describe some of the ergodic properties of hyperbolic dynamical systems and present some recent results on existence of hyperbolic systems on compact spaces. For nonuniformly completely hyperbolic systems we will discuss a very interesting phenomenon of coexistence in somewhat strong sense of hyperbolic and non-hyperbolic behavior.

We stress that nonuniform hyperbolicity can be expressed in terms of the Lyapunov exponent of the system — an important characteristic of the system first introduced by Lyapunov. It can easily be computed while studying the system numerically. Thus nonuniform hyperbolicity provides a rigorous mathematical basis for the study of chaotic motions in deterministic systems. In fact, measuring Lyapunov exponents is one of the central issues in studying chaotic dynamical systems — they are intrinsic observables that allow one to quantify a number of different physical properties such as sensitivity to initial conditions, local entropy production and the dimension of the attractor (see [1]).

One of the main goals of this paper is to outline two approaches to the study of genericity of hyperbolic dynamical systems. The first one is to find out whether these systems form a "large" set in the space of all C^r, $r \geqslant 1$, diffeomorphisms preserving a given smooth measure. The other one is to answer the question whether a given one-parameter family of dynamical systems has a large set of parameters corresponding to hyperbolic systems.

Uniform (complete or partial) hyperbolicity survives under small perturbation of the system and therefore genericity of uniformly hyperbolic systems effectively amounts to their presence on a given compact space. This may not be the case: many compact manifolds do not carry uniformly hyperbolic systems. On the other hand, the presence of dynamical systems with nonzero Lyapunov exponents imposes no restrictions on the topology of the phase space and in fact, they can be constructed on any compact manifold (of dimension $\geqslant 2$). This makes the genericity problem for nonuniformly hyperbolic systems especially interesting.

2. UNIFORM HYPERBOLICITY

Instability of trajectories can be described by various forms of the *hyperbolicity conditions*. We begin with the strongest version known as uniform hyperbolicity.

2.1. Uniform Complete Hyperbolicity

See [2, 3] and also [4]. A diffeomorphism f of a compact Riemannian manifold M is said to be *uniformly completely hyperbolic* or to be *Anosov* if for each $x \in M$ there are a decomposition of the tangent bundle $TM = E^s \oplus E^u$ into two continuous df-invariant subbundles and constants $C > 0$, $0 < \lambda < 1 < \mu$ such that for $x \in M$ and $n \geqslant 0$,

1. $\|df^n v\| \leqslant C\lambda^n \|v\|$ for any $v \in E^s(x)$;

2. $\|df^{-n} v\| \leqslant C\mu^{-n} \|v\|$ for any $v \in E^u(x)$.[1)]

[1)] The requirement that the subbundles E^s and E^u are continuous can be dropped as continuity actually follows from uniform estimates (1) and (2).

The subspaces $E^s(x)$ and $E^u(x)$ are called *stable* and *unstable subspaces* at x respectively. They form two continuous subbundles (distributions) of the tangent bundle (one can show that indeed, they are Hölder continuous). Each of these subbundles is integrable to a continuous foliation of M. Thus we obtain two transversal *stable* W^s and *unstable* W^u foliations of M. By the classical Hadamard–Perron theorem, the leaves of these foliations can be characterized as follows

$$W^s(x) = \{y \in M : d(f^n(y), f^n(x)) \to 0,\ n \to \infty\},$$
$$W^u(x) = \{y \in M : d(f^n(y), f^n(x)) \to 0,\ n \to -\infty\}$$

(the convergence in fact, is exponential). Let us stress that the unstable leaf $W^u(x)$ at the point x is defined as being the stable leaf at x for the inverse map f^{-1}. [2)] However, if the two points $y, z \in W^u(x)$ are moved forward by the dynamics, the distance between them (measured in the intrinsic metric in $W^u(x)$) increases exponentially to infinity.

The stable foliation satisfies the crucial *absolute continuity property*, i.e., given a set of positive volume, its intersection with almost every leaf of the foliation has positive leaf-volume (and the unstable foliation satisfies a similar property). [3)]

2.2. Uniform Partial Hyperbolicity

See [5, 6] and also [7]. One can obtain a weaker version of uniform hyperbolicity by relaxing the requirement that stable and unstable subbundles generate the whole tangent bundle. A diffeomorphism f of a compact Riemannian manifold M is said to be *uniformly partially hyperbolic* if for each $x \in M$ there are a decomposition of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into three continuous df-invariant subbundles and numbers $C > 0$ and $0 < \lambda < \lambda' \leqslant 1 \leqslant \mu' < \mu$ such that for every $x \in M$ and $n \geqslant 0$,

1. $\|df^n v\| \leqslant C\lambda^n \|v\|$ for $v \in E^s(x)$;
2. $C^{-1}(\lambda')^n \|v\| \leqslant \|df^n v\| \leqslant C(\mu')^n \|v\|$ for $v \in E^c(x)$;
3. $\|df^{-n} v\| \leqslant C\mu^{-n} \|v\|$ for $v \in E^u(x)$.

The subspaces $E^s(x)$, $E^c(x)$ and $E^u(x)$ are called *stable, central* and *unstable subspaces* at x respectively. They form three continuous subbundles (distributions) of the tangent bundle (one can show that indeed, they are Hölder continuous). The subbundles $E^s(x)$ and $E^u(x)$ are integrable to continuous transversal foliations, W^s and W^u of M called *stable* and *unstable* foliations. The central distribution may not be integrable. The stable and unstable foliations satisfy the absolute continuity property.

3. UNIFORM HYPERBOLICITY: EXAMPLES, EXISTENCE, ERGODIC PROPERTIES AND GENERICITY

3.1. Anosov Maps

A simple example of an Anosov map is a linear hyperbolic automorphism A of an n-dimensional torus T^n, i.e., a linear map of $\mathbb{R}^n$, given by a matrix A with integer entries and such that $\det A = 1$ and the eigenvalues λ_i satisfy $|\lambda_i| \neq 1$.

The existence of an Anosov diffeomorphism on a compact smooth manifold imposes strong requirements on the topology of the manifold: there are two continuous nonsingular foliations, the action on the fundamental group is hyperbolic, etc. In particular, there is no Anosov diffeomorphism on the 2-sphere. Anosov diffeomorphisms are known to exist on tori (see the above example) and factors of compact nilpotent Lie groups (see [7]). It is conjectured that there is no Anosov diffeomorphism on any other manifold.

Anosov diffeomorphisms of class C^1 form an open set in the space $\mathrm{Diff}^1(M)$ (or $\mathrm{Diff}^1(M, \nu)$ where ν is an invariant smooth measure on M). In particular, there is an open set of Anosov C^1 maps on any torus.

Anosov diffeomorphisms are “chaotic”. More precisely, the following statement holds.

[2)] Similarly, the unstable subspace $E^u(x)$ at x is defined as being the stable subspace at x for the inverse map f^{-1}.

[3)] Indeed, the stable foliation satisfies a stronger version of the absolute continuity property: the conditional measure generated by volume on almost every leaf of the foliation is equivalent to the leaf-volume with bounded and strictly positive density.

Theorem 1 (see [4]). *An Anosov diffeomorphism f of a compact smooth connected Riemannian manifold, preserving a smooth measure ν, is ergodic. If in addition, f is topologically mixing (i.e., for any nonempty open sets U and V there exists $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for any $n \geqslant N$) then f is a Bernoulli automorphism.*

Chaotic behavior associated with Anosov maps is robust. More precisely, an Anosov diffeomorphism f of a compact smooth connected Riemannian manifold, preserving a smooth measure ν, is *stably ergodic*, i.e., any sufficiently small perturbation of f, preserving ν, is ergodic.

3.2. Partially Hyperbolic Maps

A simple example of a partially hyperbolic map is the direct product of the identity map (of some manifold) and a linear hyperbolic automorphism of a torus.

The presence of a partially hyperbolic diffeomorphism on a compact smooth manifold imposes some requirements on the topology of the manifold.

Theorem 2 (see [8]). *A compact 3-dimensional manifold, whose fundamental group is finite, does not carry a partially hyperbolic diffeomorphism whose central distribution is integrable; in particular, there is no partially hyperbolic diffeomorphisms on the 3-sphere.*

Partially hyperbolic diffeomorphisms of class C^1 form an open set in the space $\mathrm{Diff}^1(M)$ (or $\mathrm{Diff}^1(M,\nu)$ where ν is an invariant smooth measure).

Partially hyperbolic maps may not be ergodic as the above example shows. To guarantee ergodicity we impose some restrictions on the system.

We say that two points $x, y \in M$ are *accessible* if there exists a collection of points $z_1, \dots, z_n$ such that $x = z_1$, $y = z_n$ and $z_k \in W^s(z_{k-1})$ or $z_k \in W^u(z_{k-1})$ for $k = 2, \dots, n$. Accessibility is a transitive relation. We say that f has the *accessibility property* if there is only one accessibility class, i.e., any two points are accessible.

We say that f has the *essential accessibility property* if any accessibility class has either measure one or zero (with respect to an invariant smooth measure on M).

We say that f is *center bunched* if $\lambda < \mu'\mu^{-1}$ or if $\mu' > \lambda'\lambda^{-1}$.

Theorem 3 (see [9]). *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure. Assume that f is (essentially) accessible and center bunched. Then f is ergodic and in fact, has the K-property.*

Let f be a partially hyperbolic diffeomorphism, which preserves a smooth measure ν and has the accessibility property. Any sufficiently small perturbation g of f is partially hyperbolic but it is not known whether g has the accessibility property. If this is the case then by Theorem 3, g is ergodic (provided g preserves ν), since center-bunching is an open property. We obtain the following result establishing robustness of chaotic behavior associated with partially hyperbolic systems. We say that a partially hyperbolic C^2 diffeomorphism f is *stably (essentially) accessible* if so is any sufficiently small (in the C^1 topology) C^2 perturbation g of f (assuming g preserves ν).

Theorem 4. *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure ν. Assume that f is stably (essentially) accessible and center bunched. Then f is stably ergodic (indeed, stably a K-diffeomorphism).*

A different approach to study ergodicity of partially hyperbolic maps is suggested in [10] and is based on examining Lyapunov exponents in the central direction (see the definition of the Lyapunov exponent below). This is the case of *mixed hyperbolicity* — a combination of uniform partial hyperbolicity and nonzero Lyapunov exponents in the central subspace.

Theorem 5 (see [10]). *Let f be a C^2 partially hyperbolic diffeomorphism a of compact smooth Riemannian manifold M preserving a smooth measure ν. Assume that f satisfies the following properies:*

1. *it is (essentially) accessible;*

2. *there exists an invariant set A of positive ν-measure such that the Lyapunov exponents $\chi(x, v)$ is negative for all $x \in A$ and $v \in E^c(x)$ (or $\chi(x, v)$ is positive for all $x \in A$ and $v \in E^c(x)$).*

Then f is ergodic and in fact, is Bernoulli. Furthermore, the map f is stably ergodic (*indeed, stably Bernoulli*).

A similar result has been recently obtained in [11] in the case when $\dim E^c = 2$ and the Lyapunov exponents in the central subspace are nonzero (and of different signs).

4. NONUNIFORM HYPERBOLICITY

We describe a weaker version of hyperbolicity. One of its main features is that hyperbolic trajectories form a subset of full measure (with respect to an f-invariant smooth measure) in the manifold but not every trajectory is hyperbolic.

4.1. Nonuniform Complete Hyperbolicity

See [12, 13] and also [14, 15]. Let f be a C^1 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure ν. We say that f is *nonuniformly completely hyperbolic* if for ν-almost every $x \in M$ one can find a decomposition of the tangent space at x

$$T_x M = E^s(x) \oplus E^u(x)$$

and if there are positive Borel functions $\lambda(x)$ and $\mu(x)$ and for every $\varepsilon_0 > 0$ positive Borel functions $\varepsilon(x)$, $C(x)$ and $K(x)$ on M such that for ν-almost every $x \in M$,

1. the subspaces $E^s(x)$ and $E^u(x)$ depend measurably on x and are invariant under the differential df, i.e., $df E^s(x) = E^s(f(x))$ and $df E^u(x) = E^u(f(x))$;
2. $0 < \lambda(x)e^{\varepsilon(x)} < 1 < \mu(x)e^{-\varepsilon(x)}$ and the functions λ, μ and ε are invariant under f, i.e.,
$$\lambda(f(x)) = \lambda(x), \quad \mu(f(x)) = \mu(x), \quad \varepsilon(f(x)) = \varepsilon(x);$$
3. $\|df^n v\| \leqslant C(x)\lambda(x)^n \|v\|$ for $v \in E^s(x)$ and $n \geqslant 0$;
4. $\|df^{-n} v\| \leqslant C(x)\mu(x)^{-n} \|v\|$ for $v \in E^u(x)$ and $n \geqslant 0$;
5. the angle $\angle(E^s(x), E^u(x)) \geqslant K(x)$;
6. for every $m \in \mathbb{Z}$,
$$C(f^m(x)) \leqslant e^{\varepsilon(x)|m|} C(x), \quad K(f^m(x)) \geqslant e^{-\varepsilon(x)|m|} K(x).$$

The last property means that the estimates in (3)–(5) can deteriorate but with subexponential rate.

The subspaces $E^s(x)$ and $E^u(x)$ are called *stable* and *unstable subspaces* at x respectively. Unlike the case of uniform hyperbolicity they are only defined on a set of full measure and depend measurably on x. As in the case of uniform hyperbolicity the unstable subspace $E^u(x)$ at a point x is defined as being the stable subspace at x for the inverse map f^{-1}.

One can construct, for almost every $x \in M$, the *global stable* and *unstable manifolds* at x, $W^s(x)$ and $W^u(x)$. They form "measurable" foliations of M, which are transversal to each other. The unstable leaf $W^u(x)$ at a point x is defined as being the stable leaf at x for the inverse map f^{-1}. However, unlike the global unstable leaves for uniformly hyperbolic systems, there is no reason to assume that the size of the unstable leaf actually increases when the leaf is moved forward by the dynamics — it is an open problem whether there is a nonuniformly hyperbolic system which possesses an unstable leaf of "bounded size".

4.2. Nonuniform Partial Hyperbolicity

See [12] and also [15]. Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure ν. We say that f is is *nonuniformly partially hyperbolic* if for ν-almost every $x \in M$ one can find a decomposition of the tangent space at x

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

and if there are positive Borel functions $\lambda(x)$, $\lambda'(x)$, $\mu(x)$ and $\mu'(x)$ and for every $\varepsilon_0 > 0$ positive Borel functions $\varepsilon(x)$, $C(x)$ and $K(x)$ on M such that for ν-almost every $x \in M$,

1. the subspaces $E^s(x)$, $E^c(x)$ and $E^u(x)$ depend measurably on x and are invariant under the differential df, i.e., $df E^s(x) = E^s(f(x))$, $df E^c(x) = E^c(f(x))$ and $df E^u(x) = E^u(f(x))$;

2. the functions λ, λ', μ', μ and ε are invariant under f, i.e.,
$$\lambda(f(x)) = \lambda(x), \quad \lambda'(f(x)) = \lambda'(x), \quad \mu'(f(x)) = \mu'(x),$$
$$\mu(f(x)) = \mu(x), \quad \varepsilon(f(x)) = \varepsilon(x)$$
and they satisfy
$$0 < \lambda(x)e^{\varepsilon(x)} < \lambda'(x) \leqslant 1 \leqslant \mu'(x) < \mu(x)e^{-\varepsilon(x)};$$

3. $\|df^n v\| \leqslant C(x)\lambda(x)^n\|v\|$ for $v \in E^s(x)$;

4. $C(x)^{-1}(\lambda'(x))^n\|v\| \leqslant \|df^n v\| \leqslant C(x)(\mu'(x))^n\|v\|$ for $v \in E^c(x)$;

5. $\|df^{-n} v\| \leqslant C(x)\mu(x)^{-n}\|v\|$ for $v \in E^u(x)$;

6. the angles satisfy
$$\angle(E^c(x), E^s(x)) \geqslant K(x), \ \angle(E^c(x), E^u(x)) \geqslant K(x),$$
$$\angle(E^s(x), E^u(x)) \geqslant K(x);$$

7. for every $m \in \mathbb{Z}$,
$$C(f^m(x)) \leqslant e^{\varepsilon(x)|m|}C(x), \quad K(f^m(x)) \geqslant e^{-\varepsilon(x)|m|}K(x).$$

The last property means that the estimates in (3)–(6) can deteriorate but with subexponential rate.

The subspaces $E^s(x)$, $E^c(x)$ and $E^u(x)$ are called *stable, central* and *unstable subspaces* at x respectively. They are defined on a set of full measure and depend Borel measurably on x.

4.3. Nonuniform Hyperbolicity and Lyapunov Exponents

Nonuniform hyperbolicity can be characterized in more "practical" terms using the Lyapunov exponent of the system. The latter was introduced by Lyapunov in his seminal work on stability of solutions of ordinary differential equations (see [16]; some crucial results in this direction were also obtained by Perron, see [17, 18]; a good account of their work in this direction can be found in [19]). An adaptation of the notion of the Lyapunov exponent to dynamical systems was made by Oseledets [20], by Pesin [12, 21] and by Ruelle [22, 23] (see also books [14, 15] for a contemporary account of this work).

Recall that given a diffeomorphism f of a smooth Riemannian manifold M, the Lyapunov exponent of f at a point $x \in M$ of a vector $v \in T_xM$ is defined by the formula

$$\chi(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df_x^n v\|.$$

This means that for every sufficiently small $\varepsilon > 0$ and any sufficiently large n,

$$\|df_x^n v\| \sim \exp(\chi(x,v) \pm \varepsilon)n.$$

Therefore if $\chi(x,v) > 0$ the differential asymptotically expands v with an exponential rate and if $\chi(x,v) < 0$ the differential asymptotically contracts v with an exponential rate. In other words the subspaces

$$E^s(x) = \{v \in T_xM : \chi(x,v) < 0\}, \quad E^u(x) = \{v \in T_xM : \chi(x,v) > 0\}$$

can be viewed as **candidates** for stable and unstable subspaces at x. Whether they are **actually** stable and unstable subspaces at x depends on a subtle condition, called *Lyapunov–Perron regularity*, that the differential df must satisfy along the trajectory of x (see [12] and also [14, 15]). Regularity is a crucial notion in the stability theory originated in works of Lyapunov [16] and Perron [17, 18]. The celebrated Multiplicative ergodic theorem of Oseledets [20] claims that almost every point with respect to an invariant Borel measure is Lyapunov–Perron regular.

It follows that nonuniform complete hyperbolicity is equivalent to the fact that the Lyapunov exponent is not equal to zero for **every** vector v and almost every point x with respect to an invariant smooth measure ν (see [12, 20] and also [15, 24, 25]); in this case ν is called a *hyperbolic measure*. Similarly, nonuniform partial hyperbolicity is equivalent to the fact that the Lyapunov exponent $\chi(x,v)$ is not equal to zero for **some** vector(s) v and almost every point x with respect to an invariant smooth measure ν (see [12] and also [15]).

4.4. Relations Between Nonuniform and Uniform Versions of Hyperbolicity

There are three principle relations between nonuniform and uniform versions of hyperbolicity. For the sake of discussion we consider the case of a $C^{1+\alpha}$ diffeomorphism f of a compact manifold M preserving a smooth measure ν and we only discuss the complete version of hyperbolicity. The first of these relations provides an increasing sequence of compact (not invariant) subsets that are uniformly hyperbolic for f and exhaust M almost surely.. The second one gives a sequence of invariant measures supported on compact invariant uniformly hyperbolic sets that exhaust the metric entropy $h_\nu(f)$. Finally, the last relation describes "Anosov rigidity": if f is nonuniformly hyperbolic on a compact invariant subset of M, then this subset is a uniformly hyperbolic set for f. More precise description of these three relations is as follows.

1. See [12] (and also [14, 15]). The sets

$$\tilde{R}_n := \left\{x \in M : \lambda(x) \leqslant 1 - \frac{1}{n} < 1 + \frac{1}{n} \leqslant \mu(x),\, C(x) \leqslant n,\, K(x) \geqslant \frac{1}{n}\right\},\ n \geqslant 1$$

are nested and exhaust the whole M up to a set of zero measure, i.e., $\tilde{R}_n \subset \tilde{R}_{n+1}$ and $\bigcup_{n\geqslant 1} \tilde{R}_n = M \pmod 0$. These sets are uniformly hyperbolic for f, i.e. the estimates (3)–(5) in the definition of nonuniform hyperbolicity are uniform on $\tilde{R}_n$. The invariant decomposition of the tangent space into stable and unstable subspaces can be extended from $\tilde{R}_n$ to its closure $R_n = \overline{\tilde{R}_n}$. The sets R_n are nested and exhaust the whole M up to a set of zero measure. They are uniformly hyperbolic sets for f, compact but not f-invariant.

2. See [26] (and also [15]). Assume that the measure ν is ergodic and that $h_\nu(f) > 0$. Then there is a sequence of f-invariant measures ν_n supported on uniformly hyperbolic sets Λ_n (horseshoes) such that $\nu_n \to \nu$ as $n \to \infty$ in the weak* topology and $h_{\nu_n}(f) \to h_\nu(f)$.

3. See [27]. We present two results. In what follows $K \subset M$ is a compact f-invariant subset for a C^1 diffeomorphism of a compact manifold M.

1. Assume that the Lyapunov exponent $\chi(x, v)$ is nonzero for *every* $x \in K$ and $v \in T_xM$. Then there exists $\varepsilon > 0$ such that $|\chi(x, v)| > \varepsilon$ for $x \in K$ and $v \in T_xM$. Moreover, the nonwandering set $NW(K)$ of K is uniformly hyperbolic.

2. Let $K \subset M$ be a compact f-invariant set that admits two transverse f-invariant Baire cone families C_x and D_x on $TM|K$ such that for *every* $x \in K$ we have $\chi(x, v) > 0$ for $v \in C_x$ and $\chi(x, v) < 0$ for $v \in D_x$.[4)] Then K is a uniformly hyperbolic set for f. In particular, if $K = M$, then f is an Anosov diffeomorphism.

5. NONUNIFORM HYPERBOLICITY: EXISTENCE, ERGODIC PROPERTIES, EXAMPLES

The first example of a dynamical systems with continuous time was constructed in [28]. We shall describe an example due to Katok [29] (see also [14, 15]) of a diffeomorphism of the 2-torus with nonzero Lyapunov exponents, which is not an Anosov map. Starting with the hyperbolic toral automorphism A given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

consider the disk D_r centered at zero of radius r. Let (s_1, s_2) be the coordinates in D_r obtained from the eigendirections of A. The map A is the time-1 map of the local flow in D_r generated by the system of ordinary differential equations:

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda.$$

We obtain the desired map by slowing down A near the origin.

Fix small $r_1 < r_0$ and consider the time-1 map g generated by the system of ordinary differential equations in D_{r_1}:

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$
$$\dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$

where ψ is a real-valued function on $[0, 1]$ satisfying:

1. ψ is a C^∞ function except for the origin;

2. $\psi(0) = 0$ and $\psi(u) = 1$ for $u \geqslant r_0$ where $0 < r_0 < 1$;

3. $\psi'(u) > 0$ for every $0 < u < r_0$;

4. $\int_0^1 \frac{du}{\psi(u)} < \infty$.

We have that $g(D_{r_2}) \subset D_{r_1}$ for some $r_2 < r_1$ and that g is of class C^∞ in $D_{r_1} \setminus \{0\}$ and coincides with A in some neighborhood of the boundary ∂D_{r_1}.

The map G, given as $G(x) = g(x)$ if $x \in D_{r_1}$ and $G(x) = A(x)$ otherwise, defines a homeomorphism of the torus, which is a C^∞ diffeomorphism everywhere except for the origin. The map G has the following properties:

(G1) G has nonzero Lyapunov exponents almost everywhere;

(G2) G preserves a probability measure $d\tilde{\nu} = \kappa_0^{-1} \kappa \, d\nu$ where ν is area, the density κ is a positive C^∞ function and it is infinite at 0 and $\kappa_0 > 0$ is the normalizing factor.

[4)] A cone $C_x \subset T_xM$ of angle $\alpha(x) \geqslant 0$ around a subspace $E(x)$ is the set of vectors $v \in T_xM$ such that $\angle(v, E(x)) \leqslant \theta(x)$. Two cones C_x and D_x are transverse if there are subspaces $L_1 \subset C_x$ and $L_2 \subset D_x$ such that L_1 and L_2 are transverse. A family of cones C_x is continuous if the defining subspaces $E(x)$ and angles $\alpha(x)$ depend continuously on x and it is f-invariant if $df(C_x) \subseteq C_{f(x)}$. A function on a separable metric space is a Baire function of class 1 or 0 if it is the pointwise limit of continuous functions.

We change the coordinate system in the torus by a map ϕ such that the map $f = \phi \circ G \circ \phi^{-1}$ preserves area. Set

$$\phi(s_1, s_2) = \frac{1}{\sqrt{\kappa_0 \tau}} \left(\int_0^\tau \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2)$$

($\tau = {s_1}^2 + {s_2}^2$) in D_{r_1} and ϕ is identity otherwise. One can show that f is an area-preserving C^∞ diffeomorphism with nonzero Lyapunov exponents almost everywhere. It is called the *Katok map*.

This map is a basic element in constructing volume-preserving C^∞ diffeomorphisms with nonzero Lyapunov exponents on any manifold. More precisely, the following statement holds.

Theorem 6 (see [30]). *Any compact smooth Riemannian manifold M of dimension $\geqslant 2$ admits a C^∞ volume-preserving diffeomorphism f, which has nonzero Lyapunov exponents almost everywhere and is Bernoulli.*

A similar result for dynamical systems with continuous time was obtained in [31].

Nonuniformly completely hyperbolic dynamical systems are chaotic.

Theorem 7 (see [12] and also [14, 15]). *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and ν an f-invariant smooth hyperbolic measure on M. Then*

1. $M = \bigcup_{i \geqslant 0} \Lambda_i$, $\Lambda_i \cap \Lambda_j = \emptyset$;
2. $\nu(\Lambda_0) = 0$ *and* $\nu(\Lambda_i) > 0$ *for* $i > 0$;
3. $f|\Lambda_i$ *is ergodic for* $i > 0$;
4. *for each* $i > 0$ *there is* $n_i > 0$ *such that* $\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_{ij}$ *where* $\Lambda_{ij} \cap \Lambda_{ik} = \emptyset$ *if* $j \neq k$, $f(\Lambda_{ij}) = \Lambda_{i,j+1}$, *for* $1 \leqslant j < n_i$, $f^{n_i}(\Lambda_{i1}) = \Lambda_{i1}$ *and* $f^{n_i}|\Lambda_{i1}$ *is Bernoulli.*

There is an example of a nonuniformly completely hyperbolic C^∞ volume-preserving diffeomorphism of the 3-torus, which has countably (not finitely) many ergodic components (see [32]). This example shows that the above theorem cannot be improved.

Recall that for every $x \in M$, the Lyapunov exponent $\chi(x, \cdot)$ can take on finitely many values

$$\chi_1(x) \geqslant \cdots \geqslant \chi_p(x), \quad p = \dim M,$$

where the functions $\chi_i(x)$ are Borel measurable and invariant under the map f. If ν is a hyperbolic measure for f then $\chi_i(x) \neq 0$ for any $i = 1, \ldots, p$. The following result describes another important property of hyperbolic smooth measures.

Theorem 8 (see [12] and also [14, 15]). *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and ν an f-invariant smooth hyperbolic measure on M. Then the metric entropy of $h_\nu(f)$ can be computed by the following formula:*

$$h_\nu(f) = \int_M \sum_{i:\, \chi_i(x) > 0} \chi_i(x)\, d\nu.$$

6. NONUNIFORM HYPERBOLICITY: GENERICITY

It is one of the most challenging problems in the theory of smooth dynamical systems to establish genericity of dynamical systems with nonzero Lyapunov exponents. Theorem 6 can be viewed as an important first step in studying the genericity problem. In this regard we consider the following two conjectures.

Conjecture 1. *Let f be a $C^{1+\alpha}$ volume-preserving diffeomorphism of a compact smooth Riemannian manifold M (possibly with some zero Lyapunov exponents). Then arbitrarily close to f in Diff$^{1+\alpha}(M, m)$ (where m is volume) there is a diffeomorphism g, which has nonzero Lyapunov exponents on a set of positive volume.*

Conjecture 2. *Let f be a $C^{1+\alpha}$ volume-preserving diffeomorphism of a compact smooth Riemannian manifold M with nonzero Lyapunov exponents. Then there exists a neighborhood $U \subset \mathrm{Diff}^{1+\alpha}(M, m)$ of f and a residual set $B \subset U$ such that every $g \in B$ has nonzero Lyapunov exponents on a set of positive volume.*

The requirement that f is of class of smoothness $C^{1+\alpha}$ is crucial due to the following result.

Theorem 9 (see [33–36]). *Aside from Anosov diffeomorphisms a C^1 generic system on a compact surface has zero Lyapunov exponents.*

In order to make a progress in solving the above conjectures one can try first to analyze small perturbations of the Katok map of the 2-torus described above. Let us point out that any *gentle* perturbation of this map (i.e., a perturbation that is supported outside the critical fixed point) has nonzero Lyapunov exponents almost everywhere.

7. COEXISTENCE PHENOMENA

The presence of elliptic behavior is a persistent obstruction to nonuniform hyperbolicity. Coexistence of elliptic islands and "chaotic sea" — an open (mod 0) set on which the system has nonzero Lyapunov exponents and is ergodic — is one of the most interesting phenomena in dynamical systems and very few results are known in this direction. It is well-known that elliptic islands can not be destroyed by small perturbations of the system. Let us illustrate this situation by the following statement. It also describes the situation that can occur on the boundary of Anosov maps.

Theorem 10 (see [37, 38]). *Let $f_\alpha(x, y) = (x', y')$ be a one-parameter family of diffeomorphisms of the 2-torus given by*

$$x' = x + y + h_\alpha(x) \pmod{2\pi},$$
$$y' = y + h_\alpha(x) \pmod{2\pi},$$

where $h_\alpha(x) = x - (1+\alpha)\sin x$. Then for $-1 < \alpha < 0$ the map f is Anosov, for $\alpha = 0$ it has nonzero Lyapunov exponents almost everywhere and for sufficiently small positive α it has an elliptic island outside of which f has nonzero Lyapunov exponents almost everywhere.

Another type of coexistence phenomenon is the presence of a Cantor set of codimension one invariant tori of positive volume; on each such torus the diffeomorphism is C^1 conjugate to a diophantine translation; all of the Lyapunov exponents are zero on the invariant tori. This picture is well-known in KAM theory and is shown to be present on any compact smooth manifold of dimension $\geqslant 2$ for an open set of volume-preserving C^r diffeomorphisms of M for any sufficiently large r (see [39–42]). It is however not known whether the Lyapunov exponents outside this set of invariant tori are nonzero almost everywhere. We present a result that shows that this may be the case.

Theorem 11 (see [43]). *There exists a volume-preserving C^∞ diffeomorphism f of a compact smooth Riemannian manifold M, which is arbitrarily close to the identity map, such that*

1. *f is ergodic on an open and dense subset $U \subset M$;*
2. *Lyapunov exponents of $f|U$ are nonzero almost everywhere in U;*
3. *the complement $C = M \setminus U$ has positive volume and is a Cantor set of invariant tori;*
4. *$f = Id$ on C and the Lyapunov exponents of $f|C$ are all zero.*

We outline the proof of this result. Let A be an Anosov automorphism of the 2-torus $X = T^2$. Consider the special flow f^t over A with roof function $H(x) = 1$, acting on the manifold $N = \{(x,t) : x \in X,\ t \in [0,1]\}/\sim$ where "$\sim$" is the identification $(x,1) = (Ax,0)$.

To construct the desired map we choose:

(1) a Cantor set $C \subset Y = T^2$ of positive volume such that the complement $U = Y \setminus C$ is connected;

(2) an open set U_0 such that $\overline{U_0} \subset U$;

(3) a C^∞ function ϕ supported in U_0;

Set $M = N \times Y$ and consider the map $T : M \to M$ given by

$$T((x,t),y) = (f^{\phi(y)}(x,t),y).$$

It is a volume-preserving, nonuniformly partially hyperbolic map with one-dimensional stable, one-dimensional unstable and three-dimensional central directions.

We further choose:

(4) a sequence of open connected subsets $U_n \subset U$ such that $U_0 \subset \overline{U_0} \subset U_1$, $\overline{U_n} \subset U_{n+1}$ and $\bigcup_{n\geqslant 1} U_n = U$;

(5) a sequence of numbers ϵ_n, which tend to zero sufficiently fast.

The proof then goes by constructing a sequence of diffeomorphisms $P_n : M \to M$, $P_0 = T$ for which

(6) $||P_n - P_{n-1}||_{C^n} < \varepsilon_n$;

(7) the set $N \times U_n$ is invariant under P_n and $P_n = T$ outside $N \times U_n$;

(8) each diffeomorphism P_n is nonuniformly partially hyperbolic on U; in particular, it possesses two transversal stable and unstable foliations of $N \times U$; furthermore, it has the accessibility property on $N \times U_n$ via these foliations;

(9) $P_n|U$ is ergodic and has nonzero Lyapunov exponents for almost every $x \in U$.

The two most crucial parts in constructing the diffeomorphisms P_n are: (1) removing three zero Lyapunov exponents of the map T (one in the flow direction and two in the direction of the torus Y) and (2) ensuring accessibility of P_n within the invariant set $N \times U_n$. To achieve this one can modify methods in [30, 32, 44] and adjust them to this situation. One can now show that the map $f = \lim_{n\to\infty} P_n$ has all the desired properties.

8. OPEN SETS OF DIFFEOMORPHISMS OF THE TORUS WITH NONZERO LYAPUNOV EXPONENTS

We present a result that illustrates the above mentioned two conjectures in the particular case of multi-dimensional tori. Let f be an Anosov diffeomorphism of the torus T^n. Consider the direct product map $F = (f, Id) : T^n \times S^1 \to T^n \times S^1$. The map F is uniformly partially hyperbolic and preserves volume but is not ergodic. We consider a small perturbation G of F, which preserves volume. Note that

1. G is uniformly partially hyperbolic with an invariant splitting
$$T_xT^{n+1} = E^s_G(x) \oplus E^c_G(x) \oplus E^u_G(x),$$
where E^c_G is the one-dimensional central distribution;

2. the distributions E^s_G and E^u_G are integrable and their integral manifolds form G-invariant foliations W^s_G and W^u_G of T^{n+1}; these foliations are continuous;

3. the distribution E^c_G is integrable and its integral manifolds form a G-invariant foliation W^c_G of T^{n+1} whose global leaves are diffeomorphic to circles (see [45]).

Theorem 12 (see [44, 46–48]). *There is a open set of perturbations G of F in the C^1 topology of T^{n+1} such that G is of class C^2, preserves volume, ergodic (in fact, it is Bernoulli) and has negative Lyapunov exponents in the central direction E_G almost everywhere.*

Corollary 1. *There is an open set in the space $Diff^2(T^{n+1}, m)$, which consists of non-Anosov ergodic diffeomorphisms with nonzero Lyapunov exponents.*

It is shown in [48] that in the case of the 3-torus the central foliation W_G^c for the map G constructed above has the following "pathological" property: there is a set $E \subset M$ of full measure such that E intersects almost every leaf of the central foliation at one point — the remarkable phenomenon known as "Fubini's nightmare" (see [49] for more information and further references). This phenomenon is robust as it holds for any sufficiently small perturbation of the map G.

9. THE GENERICITY PROBLEM: FAMILIES OF MAPS

We present a different approach to the genericity problem that is inspired by recent study of the Hénon family of maps (see [50–55]). Namely, given a one-parameter family of C^2 diffeomorphisms f_a, $a \in [\alpha, \beta]$ of a compact smooth manifold M there exists a set $A \subset [\alpha, \beta]$ of positive Lebesgue measure such that for every $a \in A$ the diffeomorphism f_a has nonzero Lyapunov exponents almost everywhere (and indeed, is ergodic). Here are the three examples where genericity in this sense is expected:

1) The standard (Chirikov–Taylor) family of maps of the 2-torus given by $T_\alpha(x, y) = (x', y')$ where

$$x' = x + \alpha \sin(2\pi y)(\mathrm{mod}\, 1),$$
$$y' = y + x'(\mathrm{mod}\, 1)$$

(see [56, 57]).

2) A family of maps f_α of the 2-torus such that f_0 is the Katok map.

3) A family of automorphism of real $K3$ surfaces (see [58]).

Each of these three examples represents a wonderful and quite difficult open problem in the modern theory of dynamical systems whose solutions will lead to a deeper understanding of the nature of chaos and of possible mechanisms of appearing chaotic motions in pure deterministic dynamical systems.

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